[5] I.G. Rosenberg, Completeness properties of multiple-valued logic algebras, in: *Computer Science and Multiple-valued Logic* (edited by D.C. Rine), North-Holland, Amsterdam, 1977; pp. 144-186.

[6] S. Świerczkowski, Algebras which are independently generated by every n elements, *Fund. Math.*, 49(1960), 93-104.

Béla Csákány
József Attila University
Bolyai Institute
Aradi vértanuk tere 1.
Szeged, Hungary, 6720

Tat'jana Gavalcová
Kat.matematiky SF VST
Svermova 5/a
04187 Kosice
Czechoslovakia

COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
28. FINITE ALGEBRA AND MULTIPLE-VALUED LOGIC
SZEGED (HUNGARY), 1979.

INDUCTIVE CLONES

B. CSÁKÁNY - T. GAVALCOVÁ

1. PRELIMINARIES

Throughout this article, we restrict ourselves to algebras whose base set is finite, has $n \geq 3$ elements and is identified with $\{0,1,\dots,n-1\}$.

A well-known theorem of ŚWIERCZKOWSKI [6] says that if the base set of an algebra A is independent then A is trivial. This theorem can be reformulated as follows: let A be an n-element algebra; whose at most n-ary term functions are projections; then all term functions of A are projections. A very close result was proved independently by HARNAU [3] (Lemma 2.2.): let C be a clone on an n-element set, whose at most n-ary functions are essentially at most unary; then every function in C is essentially at most unary.

Consider a k-ary function $f: A^k \to A$, and a partition π of the set $K = \{1,\dots,k\}$. Let φ be a selfmap of K mapping every $x \in K$ on the least element of the block of π containing x and let $\{i_1,\dots,i_t\}$ (with $i_1 \leq \dots \leq i_t$) be the image of φ. We define the t-ary function f^π by $f^\pi(x_1,\dots,x_t) = f(x_{\varphi(1)},\dots$ $\dots,x_{\varphi(k)})$. It is said to be a *factor function* (shortly: factor) of f. The factor f^π is *proper* if π is not the least partition (i.e. has a nonsingleton block). We associate with f and π also the *restriction* f_π of f to the set of all k-tuples $\langle a_1,\dots,a_k\rangle\in A^k$ with $a_i = a_j$ whenever i and j belong to the same class of π.

Clearly, if C is a clone and $f\in C$ then any factor f^π of f also belongs to C. As we shall see, the converse is not true in general; however, it is easy to see that the theorem of Świerczkowski mentioned above is equivalent to the assertion that every more than n-ary function is a projection provided all of its proper factors are projections. Substituting "projection" by "essentially at most unary function" here, we obtain Harnau's result. These examples suggest the following definition: call a clone C on an n-element set *inductive* if every more than n-ary function belongs to C provided all of its proper factors are members of C.

In this note, we prove the inductivity or the non--inductivity of several clones, especially of those occurring in Rosenberg's completeness theorem (see, e.g., [5], p. 155), and we give an application of the inductivity of clones of linear functions.

2. INDUCTIVE CLONES

The following lemma is trivial:

LEMMA. *Let* Q *be a subset of the poset* P *of partitions of* $\{1,\dots,k\}$ $(k \geq 4)$ *distinct from the least partition. If*

(1) $\rho\in Q$ *whenever* $\rho > \pi\in Q$

(2) *at least two minimal elements of* P *belong to* Q, *and*

(3) Q *contains every minimal element of* P *covered /in* P/ *by two elements of* Q,

then $P = Q$.

The above quoted results can be expressed as follows:

PROPOSITION 1. *The clone of projections* P *and the clone* U *of essentially at most unary functions are both inductive.*

For reader's convenience we give a short proof based on the lemma. Let $k > n$ and let f be a k-ary function whose all proper factors belong to P (U). For each partition π of $\{1,\dots,k\}$ denote by $\hat\pi$ the block of π such that $f_\pi(x_1,\dots,x_k) = g(x_j)$ for some $j\in\hat\pi$ and g unary (where g is the identity if the clone is P). It suffices to show that there exists an i $(\leq k)$ such that $i\in\hat\pi$ for every proper π.

The number of minimal partitions of $\{1,\dots,k\}$ is $\binom{k}{2}$, which is greater than k, since $k \geq 4$. Hence there exists an i such that $i\in\hat\pi_1,\hat\pi_2$ for two distinct minimal partitions π_1,π_2. We show that the set Q of proper partitions π satisfying $i\in\hat\pi$ fulfils the conditions of the Lemma. If $\pi < \rho$, then f^ρ is a factor of f^π, whence (1) follows. (2) is true by the choice of i. Now assume that two distinct partitions ρ and σ from Q cover a minimal element π of P. As $\pi = \rho \wedge \sigma$ and $i\in\hat\rho$, $i\in\hat\sigma$, we have $i\in\hat\rho \cap \hat\sigma = \hat\pi$, whence $\pi\in Q$ follows, proving (3).

PROPOSITION 2. *Let* R *be a ring /ring with unit* 1/ *and* A *a faithful* R*-module. Then the clone*

$$C = \{\lambda_1 x_1 + \ldots + \lambda_k x_k + a : k \geq 1;\ \lambda_1, \ldots, \lambda_k \in R,\ a \in A\}$$

$$(C' = \{\lambda_1 x_1 + \ldots + \lambda_k x_k : k \geq 1, \lambda_1, \ldots, \lambda_k \in R,$$
$$\lambda_1 + \ldots + \lambda_k = 1\})$$

is inductive.

We prove this for C only; for C', the same argument works. Suppose that f is a k-ary function $(k > n)$ on A whose all proper factors are in C. It is enough to find $\alpha_1, \ldots, \alpha_k \in R$ and $a \in A$ such that $f_\pi(x_1, \ldots, x_k) = \alpha_1 x_1 + \ldots + \alpha_k x_k + a$ for every π. Assume

$$f(x_1, x_1, x_3, \ldots, x_k) = \beta_1 x_1 + \beta_3 x_3 + \ldots + \beta_k x_k + b,$$
$$f(x_1, x_3, x_3, \ldots, x_k) = \gamma_1 x_1 + \gamma_3 x_3 + \ldots + \gamma_k x_k + c.$$

We show that $\alpha_1 = \gamma_1$, $\alpha_2 = \beta_1 - \gamma_1$, $\alpha_3 = \beta_3, \ldots, \alpha_k = \beta_k$, $a = b$ will do. By the lemma, we have to show that the set Q of proper partitions π satisfying $f_\pi(x_1, \ldots$ $\ldots, x_k) = \gamma_1 x_1 + (\beta_1 - \gamma_1) x_2 + \beta_3 x_3 + \ldots + \beta_k x_k + b$ fulfils the conditions of the lemma. Indeed, $f(x_1, x_1, x_3, \ldots, x_k) = \gamma_1 x_1 + (\beta_1 - \gamma_1) x_1 + \beta_3 x_3 + \ldots + \beta_k x_k + b = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k + a$, and $f(x_1, x_3, x_3, \ldots, x_k) = \gamma_1 x_1 + (\gamma_3 - \beta_3) x_3 + \beta_3 x_3 + \gamma_4 x_4 + \ldots + \gamma_k x_k + c = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_k x_k + a$ since $c = f(0, \ldots, 0) = b$, $(\gamma_3 - \beta_3) x = f(x, x, x, 0, \ldots, 0) - (\beta_3 + \gamma_1) x - c = (\beta_1 - \gamma_1) x$, $\gamma_4 x = f(0,0,0,x,0,\ldots,0) - c = \beta_4 x$, and so on. Thus Q contains two minimal partitions. Clearly, if $\pi_1 < \pi_2$ and $\pi_1 \in Q$ then $\pi_2 \in Q$.

Finally we show that Q satisfies (3), too. Let $\pi = (12)$, $\varkappa = (123)$, $\lambda = (12)(34)$, where the parentheses indicate the at least two-element classes of the partition; the further possible cases may be treated analogously. Suppose

$$f(x_1, x_1, x_1, x_4, \ldots, x_k) =$$
$$= (\alpha_1 + \alpha_2 + \alpha_3) x_1 + \alpha_4 x_4 + \ldots + \alpha_k x_k + a.$$

$$f(x_1, x_1, x_3, x_3, \ldots, x_k) =$$
$$= (\alpha_1 + \alpha_2) x_1 + (\alpha_3 + \alpha_4) x_3 + \ldots + \alpha_k x_k + a,$$

$$f(x_1, x_1, x_3, x_4, \ldots, x_k) =$$
$$= \delta_1 x_1 + \delta_3 x_3 + \delta_4 x_4 + \ldots + \delta_k x_k + d,$$

with suitable $\delta_1, \ldots, \delta_k \in R$, $d \in A$. Then $d = f(0, \ldots, 0) = a$; $\delta_1 x = f(x, x, 0, \ldots, 0) - a = (\alpha_1 + \alpha_2) x$, i.e., $\delta_1 = \alpha_1 + \alpha_2$; $(\delta_1 + \delta_3) x = f(x, x, x, 0, \ldots, 0) - a = (\alpha_1 + \alpha_2 + \alpha_3) x$ whence $\delta_3 = \alpha_3$; considering $f(0,0,x,x,0,\ldots$ $\ldots, 0)$ and $f(0,0,0,0,x,0,\ldots,0)$ we obtain $\delta_4 = \alpha_4$, $\delta_5 = \alpha_5$, etc., showing that $f(x_1, x_1, x_3, x_4, \ldots, x_k) = (\alpha_1 + \alpha_2) x_1 + \alpha_3 x_3 + \alpha_4 x_4 + \ldots + \alpha_k x_k + a$, which was needed.

A special case of C is the clone of all functions of form $\sum \lambda_i x_i + a$, where A is a p-elementary Abelian group considered as an R-module with R the full endomorphism ring of A. (This is the third type of maximal clones in Rosenberg's theorem.)

It is easy to observe that the clone of all functions preserving a given non-trivial equivalence relation on A and the clone of all functions invariant under a given permutation p of A are also inductive.

Indeed, let $f: A^k \to A$ where $|A| < k$ and suppose that, for any proper π, f^π preserves a given non-trivial equivalence $\sim$ on A. Let $a_i, b_i \in A$ and assume $a_i \sim b_i$ $(i = 1, \ldots, k)$. Since $|A| < k$, at least two a_i's - say, a_1 and a_2 - coincide; the same is valid for the b_i's. If $b_1 = b_2$ then $f(a_1, a_2, \ldots, a_k) \sim f(b_1, b_2, \ldots, b_k)$, since $f^{(12)}$ preserves $\sim$. If $b_1 = b_3$ then $f(a_1, a_2, a_3, a_4, \ldots, a_k) \sim f(b_1, b_1, b_1, b_4, \ldots, b_k)$, since $a_2 = a_1 \sim b_1$, $a_3 \sim b_3 = b_1$, and $f^{(12)}$ preserves $\sim$; further, $f(b_1, b_1, b_1, b_4, \ldots, b_k) \sim f(b_1, b_2, b_3, b_4, \ldots, b_k)$, since $b_1 \sim a_1 = a_2 \sim b_2$ and $f^{(13)}$ preserves $\sim$. The remaining case $b_3 = b_4$ can be settled in the same fashion. Thus, f preserves $\sim$.

As for the clone of p-invariant functions, it suffices to remark that $b_i = a_i p$ implies that the pattern of equalities in $\langle a_1, \ldots, a_k \rangle$ is the same as in $\langle b_1, \ldots, b_k \rangle$, whence $(f(a_1, \ldots, a_k))p = f(b_1, \ldots, b_k)$, since every f^π is invariant under p.

3. NON-INDUCTIVE CLONES

Let $\leq$ be a non-trivial partial order on A. Then the clone of all functions preserving $\leq$ is not inductive. We show this by constructing an $(n+1)$-ary non-monotone function on A with monotone factor functions only.

For this aim, denote by $A^{n+1}_{(ik)}$ the set of all $(n+1)$-tuples from A whose ith and kth entries coincide. Then $A^{n+1} = \bigcup_{i \neq k} A^{n+1}_{(ik)}$. Making use of the componentwise partial order on A^{n+1}, we define a new partial order on it as follows: for $a, b \in A^{n+1}$ let $b \leqq a$ iff there exist $c_0 = b, c_1, \ldots, c_t = a$ $(t \geq 0)$ in A^{n+1} such that, for every $j = 1, \ldots, t$, $c_{j-1} \leq c_j$ in A^{n+1}, and there exist $i = i(j)$, $k = k(j)$ such that $c_{j-1}, c_j \in A^{n+1}_{(ik)}$.

Now $\leqq$ is weaker than $\leq$; still they coincide on each $A^{n+1}_{(ik)}$. Hence it is enough to find a function $f: A^{n+1} \to A$ which is not monotone with respect to $\leq$ and is monotone with respect to $\leqq$. Indeed, an n-ary factor $f^{(ik)}$ of f is monotone exactly when $f_{(ik)}$ is monotone, and the latter one is the restriction of f to $A^{n+1}_{(ik)}$.

Let S and T be arbitrary posets: take an arbitrary $a \in S$, and let $c, d \in T$, $c < d$. Then the function

$$f(x) = \begin{cases} c & \text{if } x \leq a, \\ d & \text{otherwise} \end{cases}$$

is monotone. Apply this fact to the case $S = A^{n+1}$ with $\leqq$, $T = A = \{a_1, \ldots, a_{n-2}, 0, 1\}$ where 1 covers 0, $c = 0$, $d = 1$, $a = \langle 0, 1, 1, a_1, \ldots, a_{n-2} \rangle$. Then f is monotone with respect to $\leqq$. Put $b = \langle 0, 0, 1, a_1, \ldots, a_{n-2} \rangle$. Then $b \leqq a$ fails, because there is no $c \in A^{n+1}$ with $b < c < a$, and there are no i, k with $a, b \in A^{n+1}_{(ik)}$. Hence it follows $0 = f(a) < f(b) = 1$, though $b < a$, showing that f is not monotone with respect to $\leq$.

The inductivity of the remaining clones appearing in Rosenberg's theorem will be refuted by a common counter-example. Define $f: A^{n+1} \to A$ as follows:

$$f(a_1,\dots,a_{n+1}) = \begin{cases} 0 & \text{if } |\{a_1,\dots,a_{n+1}\}| \le n-1, \\ r-1 & \text{if } \{a_1,\dots,a_{n+1}\} = A, \\ & a_r = a_s,\ r < s. \end{cases}$$

Then (C_i)

$$f(1,2,\dots,i,0,0,i+1,i+2,\dots,n-1) = i$$

holds for each $i \in A$.

Now, let ρ be an h-ary $(2 \le h < n)$ central relation on A; suppose that 0 belongs to the centre of ρ and $\langle 1,2,\dots,h\rangle \notin \rho$. Then $(C_1),\dots,(C_h)$ together show that f does not preserve ρ. On the other hand, every factor of f takes on at most two values one of which is always 0; thus, ρ is preserved by every factor of f. This means that the clone of all functions preserving ρ is not inductive. In contrast with this, the clone of all functions preserving a unary central relation is inductive.

Finally, let T be a h-regular family of equivalences on the set $A = \{0,1,\dots,n-1\}$ $(2 < h \le n)$, and denote by λ_T the relation determined by T. We can assume without loss of generality that $\{0,1,\dots,h-1\}$ is a full system of representatives for some equivalence in T (i.e., $\langle 0,1,\dots,h-1\rangle \notin \lambda_T$). Then $(C_0),\dots,(C_{h-1})$ together show that f does not preserve λ_T. As in the preceding case, we can check that each factor of f preserves λ_T. Thus, the clone of all functions preserving λ_T is not inductive.

4. AN APPLICATION

Consider the three-element set $A = \{0,1,2\}$. We write $r_3(x,y)$ for $2x + 2y \pmod 3$, d for the dual discriminator, and l_3 for the ternary near-projection on A (see, e.g. [1]). We prove that in the lattice of clones on A the clone $[r_3]$ is covered by $[r_3,d]$. This was discovered by MARCENKOV [4], who determined the lattice of clones of homogeneous functions on the three--element set. We give another proof here, using the inductivity of the clone $[r_3]$ which is the same as the clone of functions of form $\sum \lambda_i x_i$ with $\sum \lambda_i = 1$, where A is the base set of GF(3) and $\lambda_i \in$ GF(3) (clone C' in Proposition 2). Taking into account that $[r_3,d]$ is the clone of all homogeneous functions on A (see Theorem 2 in [2]), our assertion can be formulated in the following more symmetric way: on the three element set an arbitrary nontrivial linear homogeneous function together with an arbitrary non-linear homogeneous function generates the clone of all homogeneous functions.

We have to show that

$$(*) \qquad [r_3,f] \supseteq [r_3,d]$$

whenever f is homogeneous and $f \notin [r_3]$. First, let f be a non-trivial pattern function. Then, as the minimal clones of pattern functions on A are exactly $[l_3]$ and $[d]$ (see [2], Theorem 1), $l_3 \in [f]$ or $d \in [f]$ holds. The latter case implies $(*)$ immediately, while the former can be settled using the identity

(**) $d(x,y,z) = l_3(l_3(y,x,z),\ r_3(x,y),\ l_3(z,y,x))$.

Secondly, let f be a non-pattern function. Then f is at least ternary. It suffices, however, to consider only the case when f is exactly ternary. Indeed, if $n > 3$ is the minimal integer with the property that there exists an n-ary homogeneous non-linear f such that $[r_3,f]$ does not contain $[r_3,d]$, then, for each proper factor f^π, $[r_3,f^\pi]$ $(\subseteq [r_3,f])$ does not contain $[r_3,d]$ a fortiori, whence $f^\pi \in [r_3]$. As $[r_3]$ is inductive, this implies $f \in [r_3]$, a contradiction.

Thus, let f be a ternary homogeneous non-pattern function on A with $f \notin [r_3]$. Such a function is determined uniquely by the values $f(0,1,2)$, $f(0,0,1)$, $f(0,1,0)$, and $f(0,1,1)$. A trivial computation shows that there exist 51 such functions and each of them can be obtained by a permutation of variables from exactly one of the following ten functions:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
$f_i(0,1,2)$	0	0	0	0	0	0	0	0	0	0
$f_i(0,0,1)$	0	0	1	1	1	1	2	2	2	2
$f_i(0,1,0)$	0	2	0	2	2	2	0	0	2	2
$f_i(0,1,1)$	2	2	2	0	1	2	0	1	0	2

We have to prove that, for any i, either $d \in [f_i]$ or $l_3 \in [f_i]$ holds (the latter one also suffices in view of (**)). But this is true, as the following identities show:

$$f_1(f_1(z,y,x),z,z) = d(x,y,z),$$
$$f_4(f_4(x,y,z),f_4(y,z,x),f_4(z,x,y)) = f_2(x,y,z),$$
$$f_2(z,y,f_2(y,z,x)) = l_3(x,y,z),$$
$$f_7(z,f_7(z,x,y),y) = f_8(x,y,z),$$
$$f_8(x,x,f_8(x,y,z)) = f_3(x,y,z),$$
$$f_3(x,z,f_3(y,x,z)) = f_{10}(x,y,z),$$
$$f_{10}(z,y,f_{10}(z,y,x)) = p(x,y,z)$$

where p is Pixley's ternary discriminator (it is well-known that $d \in [p]$).

$$f_5(x,f_5(x,z,y),x) = f_6(x,y,z),$$
$$f_6(f_6(x,z,y),f_6(z,y,x),f_6(y,x,z)) = l_3(x,y,z),$$
$$f_9(x,y,f_9(y,x,z)) = p(x,y,z).$$

REFERENCES

[1] B. Csákány, Homogeneous algebras are functionally complete, *Algebra Universalis*, 11 (1980), 149-158.

[2] B. Csákány - T. Gavalcová, Finite homogeneous algebras I., *Acta Sci. Math.* (Szeged), 42 (1980), 57-65.

[3] W. Harnau, Eine Verallgemeinerung der Vertauschbarkeit in der k-wertigen Logik, *Elektronische Informationsverarbeit. Kybernetik*, 12 (1976), 33-43.

[4] S.S. Marčenkov, On closed classes of self-dual functions of many-valued logic (in Russian), *Problemy Kibernet.*, 36 (1979), 5-22.