1. PRELIMINARIES

Throughout this article, we restrict ourselves to algebras whose base set is finite, has \( n \geq 3 \) elements and is identified with \( \{0, 1, \ldots, n-1\} \).

A well-known theorem of ŚWIERCZKOWSKI [6] says that if the base set of an algebra \( A \) is independent then \( A \) is trivial. This theorem can be reformulated as follows: let \( A \) be an \( n \)-element algebra; whose at most \( n \)-ary term functions are projections; then all term functions of \( A \) are projections. A very close result was proved independently by HARRAU [3] (Lemma 2.2.): let \( C \) be a clone on an \( n \)-element set, whose at most \( n \)-ary functions are essentially at most unary; then every function in \( C \) is essentially at most unary.

Consider a \( k \)-ary function \( f : A^k \rightarrow A \), and a partition \( \pi \) of the set \( K = \{1, \ldots, k\} \). Let \( \varphi \) be a self-map of \( K \) mapping every \( x \in K \) on the least element of the block of \( \pi \) containing \( x \) and let \( \{i_1, \ldots, i_\ell\} \) (with \( i_1 \leq \ldots \leq i_\ell \)) be the image of \( \varphi \). We define the \( \pi \)-ary function \( f^\pi \) by \( f^\pi(x_1, \ldots, x_\ell) = f(x_{\varphi(1)}, \ldots, x_{\varphi(\ell)}) \). It is said to be a factor function (shortly: factor) of \( f \). The factor \( f^\pi \) is proper if \( \pi \) is
not the least partition (i.e. has a nonsingleton block).
We associate with \( f \) and \( \tau \) also the restriction \( f^\tau \)
of \( f \) to the set of all \( k \)-tuples \((a_1, \ldots, a_k) \in A^k\) with
\( a_i = a_j \) whenever \( i \) and \( j \) belong to the same class of \( \tau \).

Clearly, if \( C \) is a clone and \( f \in C \) then any factor \( f^\tau \) of \( f \) also belongs to \( C \). As we shall see,
the converse is not true in general; however, it is easy to see that the theorem of Świerczkowski mentioned above
is equivalent to the assertion that every more than \( n \)-ary function is a projection provided all of its proper
factors are projections. Substituting "projection" by "essentially at most unary function" here, we obtain
Harnau's result. These examples suggest the following definition: a clone \( C \) on an \( n \)-element set inductive
if every more than \( n \)-ary function belongs to \( C \) provided all of its proper factors are members of \( C \).

In this note, we prove the inductivity or the non-inductivity of several clones, especially of those occurring
in Rosenberg's completeness theorem (see, e.g., [5], p. 155), and we give an application of the inductivity
of clones of linear functions.

2. INDUCTIVE CLONES

The following lemma is trivial:

**LEMMA.** Let \( Q \) be a subset of the poset \( P \) of partitions of \([1, \ldots, k]\) \((k \geq 4)\) distinct from the least partition. If

1. \( r \in Q \) whenever \( r > \tau \in Q \)
2. at least two minimal elements of \( P \) belong to \( Q \),

then \( P = Q \).

The above quoted results can be expressed as follows:

**PROPOSITION 1.** The clone of projections \( P \) and the clone \( U \) of essentially at most unary functions
are both inductive.

For reader's convenience we give a short proof based on the lemma. Let \( k > n \) and let \( f \) be a \( k \)-ary
function whose all proper factors belong to \( P \) \((U)\). For each partition \( \tau \) of \([1, \ldots, k]\) denote by \( \tau \)
the block of \( \tau \) such that \( f^\tau(x_1, \ldots, x_k) = g(x_j) \) for some \( j \in \tau \) and \( g \) unary (where \( g \) is the identity if
the clone is \( P \)). It suffices to show that there exists an \( \tau \) \((\simeq k)\) such that \( \tau \in Q \) for every proper \( \tau \).

The number of minimal partitions of \([1, \ldots, k]\) is \( \binom{k}{2} \), which is greater than \( k \), since \( k \geq 4 \). Hence
there exists an \( \tau \) such that \( \tau \in Q \), \( \tau \neq 1 \), \( \tau \neq 2 \) for two distinct minimal partitions \( \tau_1, \tau_2 \). We show that the set \( Q \) of proper partitions \( \tau \) satisfying \( \tau \in Q \) fulfills the conditions of the Lemma. If \( \tau < \rho \), then \( \rho^\tau \) is a factor
of \( f^\tau \), whence (1) follows. (2) is true by the choice of \( \tau \). Now assume that two distinct partitions \( \rho \) and \( \sigma \) from \( Q \) cover a minimal element \( \tau \) of \( P \). As \( \tau = \rho \wedge \sigma \) and \( \tau \in Q \), we have \( \tau \in \rho \cap \sigma = \tau \), whence
\( \tau \in Q \) follows, proving (3).
PROPOSITION 2. Let \( R \) be a ring with unit 1/ and \( A \) a faithful \( R \)-module. Then the clone
\[
C = \{ \lambda_1 x_1 + \ldots + \lambda_k x_k : a_1, \ldots, a_k \in A \}
\]
\((C' = \{ \lambda_1 x_1 + \ldots + \lambda_k x_k : a_1, \ldots, a_k \in A \}, \lambda_1 + \ldots + \lambda_k = 1)\)

is inductive.

We prove this for \( C \) only; for \( C' \), the same argument works. Suppose that \( f \) is a \( k \)-ary function \((k > n)\) on \( A \) whose all proper factors are in \( C \). It is enough to find \( a_1, \ldots, a_k \in R \) and \( a \in A \) such that \( f_n(a_1, \ldots, a_k) = a_1 x_1 + \ldots + a_k x_k + a \) for every \( \tau \). Assume
\[
f(x_1, x_2, \ldots, x_k) = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + b,
\]
\[
f(x_1, x_3, \ldots, x_k) = \gamma_1 x_1 + \gamma_2 x_3 + \ldots + \gamma_k x_k + c.
\]

We show that \( a_1 = \gamma_1, a_2 = \beta_1 - \gamma_1, a_3 = \beta_2, \ldots, a_k = \beta_k, a = b \) will do. By the lemma, we have to show that the set \( G \) of proper partitions \( \tau \) satisfying \( f_n(a_1, \ldots, a_k) \) fulfills the conditions of the lemma. Indeed, \( f(x_1, x_2, \ldots, x_k) = a_1 x_1 + a_2 x_2 + \ldots + a_k x_k + a \), and \( f(x_1, x_3, \ldots, x_k) = \gamma_1 x_1 + + \gamma_2 x_3 + \ldots + \gamma_k x_k + c \) since \( \sigma = f(0, \ldots, 0) = b \), \((\gamma_1 - \gamma_2)x = f(x, x, 0, \ldots, 0) = (\beta_1 - \gamma_1)x, \gamma_3 x = f(0, 0, 0, x, \ldots, 0) = \beta_2 x, \gamma_3 x = f(0, 0, 0, 0, x, 0, \ldots, 0) = \beta_3 x, \) and so on. Thus \( G \) contains two minimal partitions. Clearly, if \( \tau_1 < \tau_2 \) and \( \tau_1 \in G \) then \( \tau_2 \in G \).

Finally we show that \( Q \) satisfies (3), too. Let \( \pi = (12), \kappa = (123), \lambda = (12)(34) \), where the parentheses indicate the at least two-element classes of the partition; the further possible cases may be treated analogously. Suppose
\[
f(x_1, x_2, x_3, x_4, \ldots, x_k) = (a_1 + a_2 + a_3)x_1 + a_4 x_4 + \ldots + a_k x_k + a.
\]
\[
f(x_1, x_2, x_3, x_4, \ldots, x_k) = (a_1 + a_2)x_1 + (a_3 + a_4)x_3 + \ldots + a_k x_k + a,
\]
\[
f(x_1, x_2, x_3, x_4, \ldots, x_k) = \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4 + \ldots + \delta_k x_k + d,
\]
with suitable \( \delta_1, \ldots, \delta_k \in R, d \in A \). Then \( d = f(0, \ldots, 0) = a, \delta_1 x = f(x, x, 0, \ldots, 0) = c = (a_1 + a_2)x, \ldots, \delta_k x = f(x, x, x, \ldots, 0) = c = (a_1 + a_2 + a_3)x \) whence \( \delta_3 = a_3 \); considering \( f(0, 0, x, x, 0, \ldots, 0) \) and \( f(0, 0, 0, x, 0, \ldots, 0) \) we obtain \( \delta_4 = a_4, \delta_5 = a_5 \), etc., showing that \( f(x_1, x_2, x_3, x_4, \ldots, x_k) = (a_1 + a_2)x_1 + a_3 x_3 + a_4 x_4 + \ldots + a_k x_k + a \), which was needed.

A special case of \( C \) is the clone of all functions of form \( \Sigma x \), where \( A \) is a \( p \)-elementary Abelian group considered as an \( R \)-module with \( R \) the full endomorphism ring of \( A \). (This is the third type of maximal clones in Rosenberg's theorem.)

It is easy to observe that the clone of all functions preserving a given non-trivial equivalence relation on \( A \) and the clone of all functions invariant un-
under a given permutation $p$ of $A$ are also inductive.

Indeed, let $f : A^k \rightarrow A$ where $|A| < k$ and suppose that, for any proper $x$, $f^x$ preserves a given non-trivial equivalence $\sim$ on $A$. Let $a_1, b_1 \in A$ and assume $a_1 \sim b_1$ ($i = 1, \ldots, k$). Since $|A| < k$, at least two $a_i$'s -- say, $a_1$ and $a_2$ -- coincide; the same is valid for the $b_i$'s. If $b_1 = b_2$ then $f(a_1, a_2, \ldots, a_k) \sim f(b_1, b_2, \ldots, b_k)$, since $f^x$ preserves $\sim$. If $b_1 = b_3$ then $f(a_1, a_2, a_3, a_4, \ldots, a_k) \sim f(b_1, b_3, b_1, b_4, \ldots, b_k)$, since $a_2 = a_1 \sim b_1$, $a_3 \sim b_3 = b_1$, and $f^x$ preserves $\sim$; further, $f(b_1, b_1, b_1, b_4, \ldots, b_k) \sim f(b_1, b_2, b_3, b_4, \ldots, b_k)$, since $b_1 \sim a_1 = a_2 \sim b_2$ and $f^x$ preserves $\sim$. The remaining case $b_3 = b_4$ can be settled in the same fashion. Thus, $f$ preserves $\sim$.

As for the clone of $p$-invariant functions, it suffices to remark that $b_1 = a_1 \sim p$ implies that the pattern of equalities in $(a_1, \ldots, a_k)$ is the same as in $(b_1, \ldots, b_k)$, whence $(f(a_1, \ldots, a_k))p = f(b_1, \ldots, b_k)$, since every $f^x$ is invariant under $p$.

3. NON-INDUCTIVE CLONES

Let $\leq$ be a non-trivial partial order on $A$. Then the clone of all functions preserving $\leq$ is not inductive. We show this by constructing an $(n+1)$-ary non-monotone function on $A$ with monotone factor functions only.

For this aim, denote by $A^{n+1}_{(\leq)}$ the set of all $(n+1)$-tuples from $A$ whose $i$th and $k$th entries coincide. Then $A^{n+1} = \bigcup_{\neq k} A^{n+1}_{(\leq)}$. Making use of the componentwise partial order on $A^{n+1}$, we define a new partial order on it as follows: for $a, b \in A^{n+1}$ let $b \leq a$ if and only if there exist $a_0 = b, a_1, \ldots, a_n = b$ ($i \geq 0$) in $A^{n+1}$ such that, for every $j = 1, \ldots, k$, $a_{j-1} \leq a_j$ in $A^{n+1}$, and there exist $i = i(j), k = k(j)$ such that $a_{j-1}, a_j \in A^{n+1}_{(i\leq j)}$.

Now $\leq$ is weaker than $\leq$; still they coincide on each $A^{n+1}_{(\leq)}$. Hence it is enough to find a function $f : A^{n+1}_{(\leq)} \rightarrow A$ which is not monotone with respect to $\leq$ and is monotone with respect to $\leq$. Indeed, an $n$-ary factor $f_{(i\leq j)}$ of $f$ is monotone exactly when $f_{(i\leq j)}$ is monotone, and the latter one is the restriction of $f$ to $A^{n+1}_{(i\leq j)}$.

Let $S$ and $T$ be arbitrary posets; take an arbitrary $a \in S$, and let $s, d \in T$, $s < d$. Then the function

$$ f(x) = \begin{cases} s & \text{if } x \leq s, \\ d & \text{otherwise} \end{cases} $$

is monotone. Apply this fact to the case $S = A^{n+1}_{(\leq)}$ with $\leq$, $T = A = \{a_1, \ldots, a_{n+1}, 0, 1\}$ where $1$ covers $0$, $s = a = 0$, $d = 1$, $a = (0, 1, a_1, \ldots, a_{n+1})$. Then $f$ is monotone with respect to $\leq$. Put $b = (0, 0, 1, a_1, \ldots, a_{n+1})$. Then $b \leq a$ fails, because there is no $a \in A^{n+1}_{(\leq)}$ with $b < a < c$, and there are no $i, k$ with $a, b \in A^{n+1}_{(i \leq k)}$. Hence it follows $0 = f(a) < f(b) = 1$, though $b < a$, showing that $f$ is not monotone with respect to $\leq$.

The inductivity of the remaining clones appearing in Rosenberg's theorem will be refuted by a common counter-example. Define $f : A^{n+1} \rightarrow A$ as follows:

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4. AN APPLICATION

Consider the three-element set $A = \{0, 1, 2\}$. We write $r_3(x, y)$ for $2x + 2y \pmod{3}$, $d$ for the dual discriminator, and $I_3$ for the ternary near-projection on $A$ (see, e.g. [1]). We prove that in the lattice of clones on $A$ the clone $[r_3]$ is covered by $[r_3, d]$. This was discovered by MARCENKOV [4], who determined the lattice of clones of homogeneous functions on the three-element set. We give another proof here, using the inductivity of the clone $[r_3]$ which is the same as the clone of functions of form $\sum \lambda_\xi x_\xi$ with $\sum \lambda_\xi = 1$, where $A$ is the base set of GF(3) and $\lambda_\xi \in \text{GF}(3)$ (clone $C'$ in Proposition 2). Taking into account that $[r_3, d]$ is the clone of all homogeneous functions on $A$ (see Theorem 2 in [2]), our assertion can be formulated in the following more symmetric way: on the three-element set an arbitrary nontrivial linear homogeneous function together with an arbitrary non-linear homogeneous function generates the clone of all homogeneous functions.

We have to show that

(*) \[ [r_3, f] \supseteq [r_3, d] \]

whenever $f$ is homogeneous and $f \notin [r_3]$. First, let $f$ be a non-trivial pattern function. Then, as the minimal clones of pattern functions on $A$ are exactly $[I_3]$ and $[d]$ (see [2], Theorem 1), $I_3 \in [f]$ or $d \in [f]$ holds. The latter case implies (*) immediately, while the former case can be settled using the identity
(**)  \( d(x,y,z) = \ell_3(\ell_2(y,x,z), r_3(x,y), \ell_3(z,y,z)) \).

Secondly, let \( f \) be a non-pattern function. Then \( f \) is at least ternary. It suffices, however, to consider only the case when \( f \) is exactly ternary. Indeed, if \( n > 3 \) is the minimal integer with the property that there exists an \( n \)-ary homogeneous non-linear \( f \) such that \([r_3,f] \) does not contain \([r_3,d] \), then, for each proper factor \( f^x, [r_3,f^x] \subseteq [r_3,f] \) does not contain \([r_3,d] \) a fortiori, whence \( f^x \in \{r_3\} \). As \([r_3]\) is inductive, this implies \( f \in \{r_3\} \), a contradiction.

Thus, let \( f \) be a ternary homogeneous non-pattern function on \( A \) with \( f \notin \{r_3\} \). Such a function is determined uniquely by the values \( f(0,1,2), f(0,0,1), f(0,1,0), \) and \( f(0,1,1) \). A trivial computation shows that there exist 31 such functions and each of them can be obtained by a permutation of variables from exactly one of the following ten functions:

\[
\begin{array}{cccccccccc}
  f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & f_9 & f_{10} \\
  f_1(0,1,2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  f_1(0,0,1) & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  f_1(0,1,0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  f_1(0,1,1) & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

We have to prove that, for any \( i \), either \( d \in \{f_i\} \) or \( \ell_3 \in \{f_i\} \) holds (the latter one also suffices in view of (**) ). But this is true, as the following identities show:

\[
\begin{align*}
f_1(f_1(x,y,z), z, z) &= d(x,y,z), \\
f_4(\ell_3(x,y,z), \ell_3(y,z,x), f_4(y,x,z)) &= f_2(x,y,z), \\
f_2(x,y, f_2(y,z,x)) &= \ell_3(x,y,z), \\
f_7(x, f_7(x,y,z), y) &= f_6(x,y,z), \\
f_8(z, x, f_8(x,y,z)) &= f_2(x,y,z), \\
f_3(x, z, f_3(y,x,z)) &= f_10(x,y,z), \\
f_{10}(x,y, f_{10}(x,y,z)) &= p(x,y,z)
\end{align*}
\]

where \( p \) is Pixley's ternary discriminator (it is well-known that \( d \in \{p\} \)).

\[
\begin{align*}
f_5(x, f_5(x,y,z), z) &= f_6(x,y,z), \\
f_6(f_6(x,y,z), f_6(x,y,x), f_6(y,x,z)) &= \ell_3(x,y,z), \\
f_9(x, y, f_9(y,x,z)) &= p(x,y,z).
\end{align*}
\]

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