On embedding well-separable graphs

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Abstract

Call a simple graph H of order n *well-separable*, if by deleting a separator set of size $o(n)$ the leftover will have components of size at most $o(n)$. We prove, that bounded degree well-separable spanning subgraphs are easy to embed: for every $\gamma > 0$ and positive integer Δ there exists an n_0 such that if $n > n_0$, $\delta(G) \geq (1 - \frac{1}{2(\chi(H)-1)} + \gamma)n$ for a simple graph G of order n, and $\Delta(H) \leq \Delta$ for a well-separable graph H of order n, then $H \subset G$.

1 Notation

In this paper we will consider only simple graphs. We mostly use standard graph theory notation: we denote by $V(G)$ and $E(G)$ the vertex and the edge set of the graph G, respectively. $deg_G(x)$ (or $deg(x)$) is the degree of the vertex $x \in V(G)$, $\delta(G)$ is the minimum degree and $\Delta(G)$ is the maximum degree. Denote $deg_G(v, A)$ the number of neighbors of v in the set A. We write $N_G(x)$ (or $N(x)$) for the neighborhood of the vertex $x \in V(G)$, hence, $deg_G(x) = |N_G(x)|$. $N_G(U) = \cup_{x \in U} N(x)$ for a set $U \subset V(G)$. $N_G(v, A)$ is the set of neighbors of v in A. Set $e(G) = |E(G)|$ and $v(G) = |V(G)|$. When A and B are disjoint subsets of $V(G)$, then we denote by $e(A, B)$ the number of edges with one endpoint in A and the other in B. We write $\chi(G)$ for the chromatic number of G. If A is a subset of the vertices of G, we write $G - A$ for the graph induced by the vertices of $V(G) - A$.

If G has a subgraph isomorphic to H, then we write $H \subset G$. In this case we sometimes call G the *host graph*. We say that G has an H–factor, if there are $\lfloor v(G)/v(H) \rfloor$ disjoint copies of H in G. Throughout the paper we will apply the relation "$\ll$": $a \ll b$, if a is sufficiently smaller, than b.

2 Introduction

In this paper we consider a problem in extremal graph theory. Before getting on the subject of our result let us take a short historical tour in the field.

One of the main results of the area is Turán's Theorem:

Theorem 1 (Turán 1941 [15]) *If G is a graph on n vertices, and*

$$e(G) > \left(1 - \frac{1}{r-1}\right) \frac{n^2}{2},$$

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then $K_r \subset G$.

Another milestone in extremal graph theory is the following theorem:

Theorem 2 (Erdős–Stone–Simonovits 1946/1966 [6, 5]) *For every graph* H *and* $\varepsilon > 0$ *there exists an* $N = N(H, \varepsilon)$ *such that if* G *is a graph on* $n > N$ *vertices, and*

$$e(G) > \left(1 - \frac{1}{\chi(H) - 1} + \varepsilon\right)\frac{n^2}{2},$$

then $H \subset G$.

The deep result of Hajnal and Szemerédi shows that when we are looking for a K_r–factor in a graph, the situation is different.

Theorem 3 (Hajnal–Szemerédi 1969 [7]) *If* G *is a graph of order* n *and* $\delta(G) \geq (1 - 1/r)n$, *then* G *has a* K_r*–factor.*

There are two important changes in the formulation of the above result: first, it is not sufficient to bound the number of edges anymore – we need a lower bound on the minimum degree of the host graph. Second, that $1/(r-1)$ changed to $1/r$. Obviously, it is harder to have a K_r–factor than just a single copy of K_r.

The following results were conjectured by Alon and Yuster ([1, 2]), and proved by Komlós, Sárközy and Szemerédi:

Theorem 4 (Komlós–Sárközi–Szemerédi [12]) *For every graph* H *there is a constant* K *such that if* G *is a graph on* n *vertices, then*

$$\delta(G) > \left(1 - \frac{1}{\chi(H)}\right)n$$

implies that there is a union of vertex disjoint copies of H *covering all but at most* K *vertices of* G.

Theorem 5 (Komlós–Sárközi–Szemerédi [12]) *For every graph* H *there is a constant* K *such that if* G *is a graph on* n *vertices, then*

$$\delta(G) > \left(1 - \frac{1}{\chi(H)}\right)n + K$$

implies that G *has an* H*–factor.*

These theorems suggest that the crucial parameter in extremal graph theory is the chromatic number. However, it is easy to come up with examples when the maximum degree turns out to be much more important. We give one possible set of examples for this fact. Let $\{H_d\}_{d>2}$ be a family of random bipartite graphs with equal color classes of size $n/2$: H_d will be the union of d random 1–factors. Let r be an odd positive integer, and consider the graph G of order n having r independent sets of equal size, and all the edges between any two independent sets. By a standard application of the probabilistic method one can prove that for a given r, if d is large enough ($d = \text{constant} \cdot r$ is sufficient), then $H_d \not\subset G$. Since H_d is bipartite for every d, this proves, that the critical parameter for embedding expanders cannot be the chromatic number. (Although, the chromatic number still has a role, see [4].) One may think, that the main reason of this fact is that H_d is an expander graph with large expansion rate.

We show, that if a graph is "far from being an expander", then again, the chromatic number comes into picture. First, let us define what we mean on "non–expander" graphs.

Definition 1 *Let H be a graph of order n. We call H well-separable, if there is a subset $S \subset V(H)$ of size $o(n)$ such that $H - S$ has components of size $o(n)$.*

We call S the separator set, and write $C_1, C_2, \ldots, C_t$ for the components of $H - S$. Note, that if H is an expander graph, then it is not well–separable. For simpler notation let us denote $\chi(H)$ by k, and exclude the trivial case $k = 1$ in the rest of the paper. We will show the following property of well–separable graphs.

Theorem 6 *For every $\gamma > 0$ and positive integer Δ there exists an n_0 such that if $n > n_0$ and $\delta(G) \geq (1 - \frac{1}{2(k-1)} + \gamma)n$ for a simple graph G of order n, and $\Delta(H) \leq \Delta$ for the well-separable graph H of order n, then $H \subset G$.*

Observe, that trees are well–separable graphs. A conjecture of Bollobás [3] (proved by Komlós, Sárközy and Szemerédi [9]) states that trees of bounded degree can be embedded into graphs of minimum degree $(1/2 + \gamma)n$ for $\gamma > 0$. Since every tree is bipartite, this result is a special case of Theorem 6.

Our proof of Theorem 6 uses the Regularity Lemma of Szemerédi [14] (sometimes called Uniformity Lemma). In the next section we will give a brief survey on this powerful tool, and related results. For more information see e.g., [13, 8]. We will prove Theorem 6 in the fourth section.

3 A review of tools for the proof

We introduce some more notation first. The *density* between disjoint sets X and Y is defined as:

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

We need the following definition to state the Regularity Lemma.

Definition 2 (Regularity condition) *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-regular if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \;\; |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

We will employ the fact that if (A, B) is an ε–regular pair as above, and we place *constant* $\cdot\, \varepsilon|A|$ new vertices into A, the resulting pair will remain regular, although with a somewhat larger ε, depending on the constant.

An important property of regular pairs is the following:

Fact 7 *Let (A, B) be an ε–regular pair with density d. Then for any $Y \subset B$, $|Y| > \varepsilon|B|$ we have*

$$|\{x \in A : deg(x, Y) \leq (d - \varepsilon)|Y|\}| \leq \varepsilon|A|.$$

We will use the following form of the Regularity Lemma:

Lemma 8 (Degree Form) *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,

- $|V_0| \le \varepsilon |V|$,
- *all clusters* V_i*,* $i \ge 1$*, are of the same size* $m \left(\le \lfloor \frac{|V|}{\ell} \rfloor < \varepsilon |V|\right)$,
- $deg_{G'}(v) > deg_G(v) - (d+\varepsilon)|V|$ *for all* $v \in V$*,*
- $G'|_{V_i} = \emptyset$ *(*V_i *is an independent set in* G'*) for all* $i \ge 1$*,*
- *all pairs* (V_i, V_j)*,* $1 \le i < j \le \ell$*, are* ε*-regular, each with density either 0 or greater than* d *in* G'*.*

Often we call V_0 the *exceptional cluster.* In the rest of the paper we assume that $0 < \varepsilon \ll d \ll 1$.

Definition 3 (Reduced graph) *Apply Lemma 8 to the graph* $G = (V, E)$ *with parameters* ε *and* d*, and denote the clusters of the resulting partition by* $V_0, V_1, \ldots, V_\ell$*,* V_0 *being the exceptional cluster. We construct a new graph* G_r*, the reduced graph of* G' *in the following way: The non-exceptional clusters of* G' *are the vertices of the reduced graph (hence* $|V(G_r)| = \ell$*). We connect two vertices of* G_r *by an edge if the corresponding two clusters form an* ε*-regular pair with density at least* d*.*

The following corollary is immediate:

Corollary 9 *Apply Lemma 8 to the graph* $G = (V, E)$ *satisfying* $\delta(G) \ge cn$ *(*$|V| = n$*) for some* $c > 0$ *with parameters* ε *and* d*. Denote* G_r *the reduced graph of* G'*. Then* $\delta(G_r) \ge (c - \theta)\ell$*, where* $\theta = 2\varepsilon + d$*.*

A stronger one-sided property of regular pairs is super-regularity:

Definition 4 (Super-Regularity condition) *Given a graph* G *and two disjoint subsets of its vertices* A *and* B*, the pair* (A, B) *is* (ε, δ)-super-regular*, if it is* ε*-regular and furthermore,*

$$deg(a) > \delta |B|, \text{ for all } a \in A,$$

and

$$deg(b) > \delta |A|, \text{ for all } b \in B.$$

Finally, we formulate another important tool of the area:

Theorem 10 (Blow-up Lemma [10, 11]) *Given a graph* R *of order* r *and positive parameters* δ, Δ*, there exists a positive* $\varepsilon = \varepsilon(\delta, \Delta, r)$ *such that the following holds: Let* $n_1, n_2, \ldots, n_r$ *be arbitrary positive integers and let us replace the vertices* $v_1, v_2, \ldots, v_r$ *of* R *with pairwise disjoint sets* $V_1, V_2, \ldots, V_r$ *of sizes* $n_1, n_2, \ldots, n_r$ *(blowing up). We construct two graphs on the same vertex set* $V = \cup V_i$*. The first graph* F *is obtained by replacing each edge* $\{v_i, v_j\}$ *of* R *with the complete bipartite graph between* V_i *and* V_j*. A sparser graph* G *is constructed by replacing each edge* $\{v_i, v_j\}$ *arbitrarily with an* (ε, δ)*–super–regular pair between* V_i *and* V_j*. If a graph* H *with* $\Delta(H) \le \Delta$ *is embeddable into* F *then it is already embeddable into* G*.*

Remark 1 (Strengthening the Blow-up Lemma) *Assume that* $n_i \le 2n_j$ *for every* $1 \le i, j \le r$*. Then we can strengthen the lemma: Given* $c > 0$ *there are positive numbers* $\varepsilon = \varepsilon(\delta, \Delta, r, c)$ *and* $\alpha = \alpha(\delta, \Delta, r, c)$ *such that the Blow-up Lemma remains true if for every* i *there are certain vertices* x *to be embedded into* V_i *whose images are a priori restricted to certain sets* $T_x \subset V_i$ *provided that*

(i) each T_x *within a* V_i *is of size at least* $c|V_i|$*,*

(ii) the number of such restrictions within a V_i *is not more than* $\alpha |V_i|$*.*

4 Proof of Theorem 6

The proof goes along the following lines:

(1) Find a special structure in G by the help of the Regularity Lemma and the Hajnal–Szemerédi Theorem (Theorem 3).

(2) Map the vertices of H to clusters of G in such a way that if $\{x, y\} \in E(H)$, then x and y are mapped to neighboring clusters; moreover, these clusters will form an (ε, δ)–super–regular pair for all, but at most $o(n)$ edges.

(3) Finish the embedding by the help of the Blow-up Lemma.

4.1 Decomposition of G

In this subsection we will find a useful decomposition of G.

First, we apply the Degree Form of the Regularity Lemma with parameters ε and d, where $0 < \varepsilon \ll d \ll \gamma < 1$. As a result, we have $\ell + 1$ clusters, $V_0, V_1, \ldots, V_\ell$, where V_0 is the exceptional cluster of size at most εn, and all the others have the same size m. We deleted only a small number of edges, and now all the (V_i, V_j) pairs are ε–regular, with density 0 or larger than d. By Lemma 9 we will have that $\delta(G_r) \geq (1 - \frac{1}{2(k-1)} + \gamma')\ell$, where $\gamma' = \gamma - d - 2\varepsilon > 0$.

Applying Theorem 3, we have a K_k–factor in G_r. It is possible, that at most $k - 1$ clusters are left out from this k–clique cover – such clusters are put into V_0. It is easy to transform the ε–regular pairs inside this clique cover into super–regular pairs: given a δ with $\varepsilon \ll \delta \ll d$ we have to discard at most εm vertices from a cluster to make a regular pair (ε, δ)–super–regular. In a clique a cluster has $k - 1$ other adjacent clusters in G_r. Hence, it is enough to discard at most $(k - 1)\varepsilon m$ vertices from every cluster, and arrive to the desired result. Note, that now the pairs are ε'–regular, with $\varepsilon' < 2\varepsilon$; for simplicity, we will use the letter ε in the rest of the paper. We will discard the same number of vertices from every non–exceptional cluster, and get, that all the edges of G_r inside the cliques of the cover are (ε, δ)–super-regular pairs. For simplicity we will still denote the common clustersize by m in G_r. The discarded vertices are placed to V_0; now $|V_0| \leq (2k - 1)\varepsilon n$.

Our next goal is to distribute the vertices of V_0 among the non–exceptional clusters so as to preserve super–regularity within the cliques of the cover. We also require that the resulting clusters should have about the same size.

For a cluster V_i in G_r denote $clq(V_i)$ the set of the clusters of V_i's clique in the K_k–cover, but without V_i itself. Hence, $V_i \notin clq(V_i)$, and $|clq(V_i)| = k - 1$ for every $V_i \in V(G_r)$.

Recall, that every cluster in G_r has the same size, m. We want to distribute the vertices of V_0 evenly among the clusters of G_r: we will achieve that $||V_i| - |V_j|| < 4k\varepsilon m$ for every $1 \leq i, j \leq \ell$ after placing the vertices of V_0 to non–exceptional clusters. Besides, we require that if we put a vertex $v \in V_0$ into $V_i \in V(G_r)$, then $deg(v, V_j) \geq \delta m$ for every $V_j \in clq(V_i)$.

For proving the above claim, let us define an auxiliary bipartite graph $F_1 = F_1(V_0, V(G_r), E(F_1))$. That is, the color classes of F_1 are V_0 and the set of the non–exceptional clusters. We draw a $\{v, V_i\}$ edge for $v \in V_0$ and $V_i \in V(G_r)$, if $deg_G(v, V_j) \geq \delta m$ for every $V_j \in clq(V_i)$.

Set $\gamma'' = k(\gamma - 2(\varepsilon + d))(> 0)$. The following lemma is crucial in distributing V_0.

Lemma 11 *$deg_{F_1}(v) \geq (1/2 + \gamma'')\ell$ for every $v \in V_0$.*

Proof: Consider an arbitrary $v \in V_0$. Then we can partition the set of k–cliques of the disjoint clique cover into $k + 1$ pairwise disjoint sets $A_0, A_1, \ldots, A_k$. A clique Q is in A_j if v has at least δm neighbors in exactly j clusters of Q. Set $a_j = k|A_j|/\ell$ for every $0 \leq j \leq k$, that is, a_j is the proportion of cliques in A_j. Clearly, $\sum_j a_j = 1$. At most δn is the number of edges which connects v to such clusters which are not adjacent to it in F_1. Hence, by the minimum degree condition, $1/k \sum_j j a_j \geq \delta(G_r)/\ell - \delta$. Notice, that if v has large enough degree (that is, at least δm neighbors in G) to at most $k-2$ clusters in a clique, then v is not adjacent to any clusters of that clique in F_1. There are two possibilities left: v has one neighbor in a clique in F_1, or it is connected to all the clusters in F_1, depending on whether it has large enough degree to $k-1$ or k clusters of that clique. Putting these together, the solution of the following linear program is a lower bound for $deg_{F_1}(v)/\ell$:

$$\sum_{j=0}^{k} a_j = 1 \text{ and } \sum_{j=0}^{k} j a_j - z = k(\tfrac{2k-3}{2k-2} + \gamma - 2(\varepsilon + d))$$
$$\text{where } a_j, z \geq 0$$
$$\min\{\tfrac{a_{k-1}}{k} + a_k\}$$

Let A be the coefficient matrix of the two equalities above, i.e.,

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & 1 & 2 & \dots & k-1 & k & -1 \end{pmatrix}.$$

Let $a^T = (a_0, a_1, \dots, a_k, z)$, $b^T = (1, k(2k-3)/(2k-2) + \gamma'')$, and $c^T = (0, 0, \dots, 0, 1/k, 1, 0)$. Then the dual of the linear program above is:

$$A^T u \leq c$$
$$\max\{b^T u\}$$

It is easy to check that $u_1 = 2-k$ and $u_2 = \frac{k-1}{k}$ is a feasible solution (in fact the optimal solution as well), and therefore $\max b^T u \geq 1/2 + \gamma''$. ∎

Applying the lemma above it is easy to distribute the vertices of V_0 evenly, without violating our requirement. For every $v \in V_0$ randomly choose a neighboring cluster in F_1, and put v into that cluster. Since $deg_{F_1}(v) \geq (1/2 + \gamma'')\ell$, with very high probability (use eg., Chernoff's bound) no cluster will get more than $2|V_0|/\ell$ new vertices from V_0. Hence, we have that $||V_i| - |V_j|| < 4k\varepsilon m$ for every $1 \leq i, j \leq \ell$.

4.2 Assigning the vertices of H

In this subsection we will map the vertices of H to clusters of G_r. We will heavily use the fact that H is k–colorable.

Fix an arbitrary k–coloration of H. For an arbitrary set A, denote $A^1, A^2, \dots, A^k$ the color classes determined by this k–coloration.

We will map S and $C_1, C_2, \dots, C_t$ by the randomized procedure below.

Mapping algorithm

Input: the set A

- Pick a clique $Q = \{Q_1, Q_2, \dots, Q_k\}$ in the cover of G_r randomly, uniformly.
- Pick a permutation π on $\{1, 2, \dots, k\}$ uniformly at random.
- Assign the vertices of A^i to the cluster $Q_{\pi(i)}$ for every $1 \leq i \leq k$.

Repeating this algorithm for S and all the components in $H - S$, we will have, that the number of vertices of H assigned to a cluster are almost the same: with probability tending to 1, the difference between the number of assigned vertices to a cluster and the cluster size m will be at most $o(n)$. This follows easily from Chebyshev's inequality.

Recall, that for applying the Blow–up Lemma, it is necessary to map adjacent vertices in H to adjacent clusters in G_r.

For $x \in V(H)$ let $\kappa(x)$ denote the cluster to which x is assigned. After randomly assigning S and $C_1, C_2, \dots, C_t$, we have that if $\{x, y\} \in H$ and $x, y \in S$ or $x, y \in C_j$ for some $1 \le j \le t$, then $\{\kappa(x), \kappa(y)\} \in E(G_r)$. On the other hand, there is no guarantee that a vertex in S and a vertex in some component of $H - S$ are assigned to adjacent clusters, even when they are adjacent in H.

Therefore, we have to reassign a small subset of $V(H)$. We will see that no vertex which is at distance larger than k from S will change its place, and vertices of S will not be reassigned.

Consider an arbitrary component C_j. Set $B = N(S) \cap C_j$, and $B_p = B \cap C_j^p$ for every $1 \le p \le k$. By the algorithm below we will define B'_p, the subset of C_j^p which will be reassigned.

Step 1. Set $B'_k = B_k$, and $i = 1$

Step 2. Set $B'_{k-i} = B_{k-i} \cup \bigcup_{p=0}^{i-1} (N(B'_{k-p}) \cap C_j^{k-i})$

Step 3. If $i < k - 1$, then set $i \leftarrow i + 1$, and go back to *Step 2.*

Informally, when we determine which vertices to reassign from C_{k-i}, we take into account all the neighbors of B'_p with $p > k - i$, and B_{k-i} itself. It is important, that we proceed backwards, that is, we specify the vertices to be reassigned starting from the last, the kth color class. Note, that the vertices of $\cup_{p=1}^k B'_p$ are at distance at most k from S. Hence, $| \cup_{p=1}^k B'_p| < \Delta^k |S| = o(n)$.

Now we have the sets $\{B'_p\}$. First we will find a new cluster for B'_1: Take an arbitrary cluster W_1 from the set

$$\bigcap_{p=2}^{k} N(\kappa(S^p)) \cap \bigcap_{p=2}^{k} N(\kappa(B'_p)).$$

Then we choose W_2 for B'_2 from the set

$$\bigcap_{p \ne 2} N(\kappa(S^p)) \cap \bigcap_{p=3}^{k} N(\kappa(B'_p)) \cap N(W_1).$$

In general, assume that we have the clusters $W_1, W_2, \dots, W_{i-1}$ for some $i \le k$. Then we choose W_i for B'_i from the set

$$\bigcap_{p \ne i} N(\kappa(S^p)) \cap \bigcap_{p=i+1}^{k} N(\kappa(B'_p)) \cap \bigcap_{p=1}^{i-1} N(W_p).$$

Observe, that this way W_i $(1 \le i \le k)$ is chosen from a non–empty set, since it comes from the common neighborhood of $2k - 2$ clusters, and this neighborhood is of size at least $\gamma' \ell$ by the minimum degree condition of G.

4.3 Achieving $|V_i| - |L_i| = 0$

We have, that if $\{x, y\} \in E(H)$, then $\{\kappa(x), \kappa(y)\} \in E(G_r)$. Moreover, the $\{\kappa(x), \kappa(y)\}$ edges are super–regular pairs for all, but at most $o(n)$ edges in $E(H)$.

Still, we cannot apply the Blow–up Lemma, since $|V_i| = |L_i|$ is not necessarily true for every $1 \le i \le \ell$. What we know for sure is that $||V_i| - |L_i|| < 5\varepsilon m$, because these differences were at most $o(n)$ in the beginning, distributing the vertices of V_0 had contribution at most $4k\varepsilon m$ for every $1 \le i \le \ell$, and we relocated $o(n)$ vertices in the previous subsection.

We will partition the clusters of G_r into three disjoint sets: $V_<, V_=$ and $V_>$. If $|V_i| < |L_i|$, then $V_i \in V_<$; if $|V_j| = |L_j|$, then $V_j \in V_=$, and we put V_p into $V_>$, if $|V_p| > |L_p|$. Clearly, it is enough to replace at most $5k\varepsilon n$ vertices of G so as to achieve $|V_i| = |L_i|$ for every $1 \le i \le \ell$, while preserving regularity for the edges of G_r. But we need super–regular pairs for the edges of the k–cliques of the cover, hence, a straightforward relocation of some vertices of G is not helpful. Instead, we will apply an idea similar to what we used for distributing the vertices of V_0.

First, we define a directed graph F_2: the vertices of F_2 are the clusters of G_r, and $(V_i, V_j) \in E(F_2)$, if $(V_i, V_p) \in E(G_r)$ for every $V_p \in clq(V_j)$. We will have that the out–degree of every cluster is at least $(1/2 + \gamma'')\ell$ by considering the linear program of Subsection 4.1. Since $\delta(G_r) \ge (\frac{2k-3}{2k-2} + \gamma')\ell$, it is easy to see that any $k-1$ clusters have at least $(1/2 + \gamma')\ell$ common neighbors. That is, the in–degree of F_2 is at least $(1/2 + \gamma')\ell$. Therefore, there is a large number – at least $(\gamma' + \gamma'')\ell$ – of directed paths of length at most two between any two clusters in F_2.

Let $V_i \in V_<$ and $V_j \in V_>$ be arbitrary clusters. If $(V_j, V_i) \in E(F_2)$, then we can directly place a vertex from V_j into V_i which has at least δm neighbors in V_i (and most of the vertices has actually at least dm neighbors, since d is the lower bound for the density of regular pairs). If there is no such edge, then there are several different directed paths of length two from V_j to V_i. These paths differ in their "center" cluster. Assume that V_p is such a cluster, i.e., (V_j, V_p) and (V_p, V_i) are edges in F_2. It is useful to choose V_p randomly, uniformly among the possible "center" clusters.

Take any vertex $v \in V_j$ which has at least δm neighbors in V_p, and put it into V_p. Then choose any vertex from V_p which has at least δm neighbors in V_i, and put it into V_i. As a result, we decreased $||V_j| - |L_j||$ and $||V_i| - |L_i||$, while $||V_p| - |L_p||$ did not change. Now, by the remark after the definition of a regular pair it is clear that if we make all $||V_i| - |L_i|| = 0$ this way, we will preserve regularity and super–regularity as well.

4.4 Finishing the proof

Now we are prepared to prove Theorem 6.

We have to check if the conditions of the Blow–up Lemma are satisfied. There are $o(n)$ edges of $E(H)$ which are problematic: those edges having their endpoints in clusters which do not constitute a super–regular pair. Denote the set of these edges by E'. Suppose that x is a vertex which occurs in some edges of E'. It can have neighbors assigned to at most $2k-2$ clusters $V_{x_1}, V_{x_2}, \ldots, V_{x_{2k-2}}$. Since $(\kappa(x), V_{x_i})$ is a regular pair for every $1 \le i \le 2k-2$, there is a set $T_x \subset \kappa(x)$ of size at least $(1-(2k-2)\varepsilon)m$ (by Fact 7 and applying induction), all the vertices of which has at least $(d-\varepsilon)^{2k-2}m > \delta m$ neighbors in V_{x_i} for every $1 \le i \le 2k-2$. T_x will be the set to which x is restricted. Since $|E'| = o(n)$, the number of restricted vertices is small enough, and therefore we can apply the strengthened version of the Blow-up Lemma. ∎

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