Note on regular spanning subgraphs of bipartite graphs of high minimum degree

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Preliminary version

Abstract

Let G be a simple balanced bipartite graph on $2n$ vertices, $\delta = \delta(G)/n$, and $\rho = \frac{\delta+\sqrt{2\delta-1}}{2}$. If $\delta > 1/2$ then it has a ρn-regular spanning subgraph. The statement is tight.

1 Introduction

In this paper we will consider simple graphs. We mostly use standard graph theory notation: $V(G)$ and $E(G)$ will denote the vertex and the edge set of a graph G, respectively. The degree of $x \in V(G)$ is denoted by $deg_G(x)$ (we may omit the subscript), $\delta(G)$ is the minimum degree of G. We call a bipartite graph $G(A, B)$ with color classes A and B *balanced* if $|A| = |B|$. For $X, Y \subset V(G)$ we denote the number of edges of G having one endpoint in X and the other endpoint in Y by $e(X, Y)$. $N_G(x)$ is the set of neighbors of $x \in V(G)$ and $N_G(x, X)$ is the set of neighbors of x in X. For a set $S \subset V(G)$ let $N(S) = \cup_{x\in S} N(x)$. If $T \subset V(G)$ then $G|_T$ denotes the subgraph we get after deleting every vertex of $V - T$ and the edges incident to them. Finally, $K_{r,s}$ is the complete bipartite graph on color classes of size r and s for two positive integers r and s.

The purpose of this note is to prove the following theorem:

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Theorem 1 *Let $G(A, B)$ be a balanced bipartite graph on $2n$ vertices, and assume that*
$\delta = \delta(G)/n > 1/2$. Then
(I) G has a ρn–regular spanning subgraph, with $\rho = \frac{\delta + \sqrt{2\delta - 1}}{2}$;
(II) moreover, for every $\delta > 1/2$ and n large enough there exist a balanced bipartite graph F_δ such that it does not admit a spanning regular subgraph of degree larger than ρn.

The above result plays a crucial role in the proof of some results in extremal graph theory, see [1, 2].

2 Factors of bipartite graphs

Let H be a bipartite graph with color classes A and B. By the well-known König–Hall theorem there is a perfect matching in H if and only if $|N(S)| \geq |S|$ for every $S \subset A$. We are going to need a generalization of the this result, this appeared in a paper of Tutte [4].

If $f : V(H) \to \mathrm{Z}^+$ is a function, then an f-factor is a subgraph H' of H such that $deg_{H'}(x) = f(x)$ for every $x \in V(H)$. Notice, that when $f \equiv r$ for some $r \in \mathrm{Z}^+$, then H' is an r-regular subgraph of H. The result below gives a necessary and sufficient condition for the existence of an f-factor:

Proposition 2 *Let H be a bipartite graph with bipartition $\{A, B\}$, and $f(x) \geq 0$ an integer valued function on $A \cup B$. H has an f–factor if and only if*

$$(i) \sum_{x \in A} f(x) = \sum_{y \in B} f(y)$$

and

$$(ii) \sum_{x \in X} f(x) \leq e(X, Y) + \sum_{y \in B - Y} f(y)$$

for all $X \subset A$ and $Y \subset B$.

One can find the proof in [3] as well.

3 Proof of Theorem 1

We will show the two parts of the theorem in separate subsections.

3.1 Proof of part I

Observe, that since we are looking for a spanning regular subgraph, the f function of Proposition 2 will be identically ρn. We start with some notation. For $X \subset A$ let $\xi = |X|/n$, and for $Y \subset B$ let $\sigma = |Y|/n$. We will normalize $e(X,Y)$: $\eta(X,Y) = e(X,Y)/n^2$. Let

$$\eta_m(\xi,\sigma) = \min\{\eta(X,Y) : X \subset A,\ Y \subset B,\ |X|/n = \xi, |Y|/n = \sigma\}.$$

Since f is identically ρn, condition (i) of Proposition 2 is satisfied. If $\rho(\xi + \sigma - 1) \le \eta_m(\xi,\sigma)$ for some ρ and for every $0 \le \xi,\sigma \le 1$, then (ii) is satisfied, hence, G has a ρn–regular spanning subgraph.

Clearly, $e(X,Y) \ge |X|(\delta n - |B - Y|)$ and $e(X,Y) \ge |Y|(\delta n - |A - X|)$ for arbitrary sets $X \subset A$ and $Y \subset B$. Hence, we have that $\eta_m(\xi,\sigma) \ge \max\left(\xi(\delta + \sigma - 1), \sigma(\delta + \xi - 1)\right)$.

First consider the case $\xi = \sigma$. We are looking for a ρ for which $\rho(2\xi - 1) \le \xi(\delta + \xi - 1)$. In another form, we need that

$$p_\rho(\xi) = \xi^2 + (\delta - 2\rho - 1)\xi + \rho \ge 0.$$

The discriminant of the above polynomial is the polynomial $dcr(\rho) = 4\rho^2 - 4\delta\rho + \delta^2 - 2\delta + 1$. One can directly find the roots of $dcr(\rho)$: $\frac{\delta \pm \sqrt{2\delta - 1}}{2}$. Let $\rho = (\delta + \sqrt{2\delta - 1})/2$, then we have that $p_\rho(\xi) \ge 0$.

Let $g(\xi,\sigma) = \sigma(\delta + \xi - 1) - \rho(\xi + \sigma - 1)$. We will show, that $g(\xi,\sigma) \ge 0$ for $0 \le \sigma \le \xi \le 1$. Notice, that g is bounded in the triangle above, $-2 \le g(\xi,\sigma) \le \eta_m(\xi,\sigma) - \rho(\xi + \sigma - 1)$, and continously differentiable.

Let us check the sign of g on the border of the triangle. Since $\rho = (\delta + \sqrt{2\delta - 1})/2$, we have that $g(\xi,\xi) \ge 0$. $g(\xi,0) = -\rho(\xi - 1) \ge 0$, and $g(1,\sigma) = \sigma(\delta - \rho) \ge 0$, because $\delta \ge (\delta + \sqrt{2\delta - 1})/2$. Let us check the partial derivatives of g:

$$\frac{\partial g}{\partial \xi} = \sigma - \rho,$$

and

$$\frac{\partial g}{\partial \sigma} = \delta + \xi - 1 - \rho.$$

Assuming that g achieves its minimum inside the triangle at the point (ξ_0,σ_0) the partial derivatives of g have to diminish at (ξ_0,σ_0). It would then follow that $\sigma_0 = \rho$ and $\xi_0 = 1 + \rho - \delta$, therefore, $g(\xi_0,\sigma_0) = \rho^2 - \rho(2\rho - \delta) = \delta\rho - \rho^2$. That is, g is non-negative in the whole closed triangle. The same reasoning works for the triangle $0 \le \xi \le \sigma \le 1$, this follows easily by symmetry.

3.2 Proof of part II

For proving part II of the theorem we construct a graph F for a given $\delta > 1/2$: $F(A,B)$ is a balanced bipartite graph on $2n$ vertices such that $A = A_e \cup A_l$,

$B = B_e \cup B_l$, and $B_e \cap B_l = A_e \cap A_l = \emptyset$. We have that $|A_l| = |B_l| = \gamma n$ and $|A_e| = |B_e| = (1-\gamma)n$, where $\gamma = \frac{1-\sqrt{2\delta-1}}{2}$. We assume that $e(A_l, B_l) = 0$, and that the subgraphs $F|_{A_l \cup B_e}$ and $F|_{B_l \cup A_e}$ are isomorphic to $K_{\gamma n,(1-\gamma)n}$, therefore, every vertex in $A_l \cup B_l$ has degree $(1-\gamma)n$. Every vertex in $A_e \cup B_e$ has degree δn, hence, $F|_{A_e \cup B_e}$ is a $(\delta-\gamma)n$-regular graph. Observe, that $\gamma < \delta < 1-\gamma$, thus, $\delta(F) = \delta$.

First we investigate a simple method for edge removal from F: for $0 < p < 1$ discard $p(1-\gamma)n$ incident edges for every vertex in $A_l \cup B_l$, and no edge from $F|_{A_e \cup B_e}$. Then a vertex in $A_l \cup B_l$ will have degree $(1-p)(1-\gamma)n$, and the average degree of the vertices in $A_e \cup B_e$ will be $\gamma(1-p)n + (\delta-\gamma)n$. Choose p_0 to be the solution of the following equation:

$$(1-p)(1-\gamma)n = \gamma(1-p)n + (\delta-\gamma)n. \tag{1}$$

Notice, that if $p < p_0$ then there is a vertex $x \in A_e \cup B_e$ such that every vertex of $A_l \cup B_l$ will have degree larger than $deg(x)$. That is, for finding a regular subgraph more edges have to be discarded among those which are incident to the vertices of $A_l \cup B_l$. On the other hand, if $p > p_0$ then there will be a vertex $y \in A_e \cup B_e$ which has degree larger than that of the vertices in $A_l \cup B_l$. Hence, edges incident to the vertices of $A_e \cup B_e$ have to be discarded, otherwise the resulting subgraph is not regular.

It is clear that if we look for a spanning regular subgraph of F we have to discard edges incident to the vertices of $A_l \cup B_l$. We have just learned that if $p \neq p_0$ then we cannot stop, more edges have to be removed.

For finishing the proof we prove that by choosing $p = p_0$ and performing the above edge removal process every vertex of $A_l \cup B_l$ will have degree $\frac{\delta+\sqrt{2\delta-1}}{2}n$ and the average degree of the vertices in $A_e \cup B_e$ will be this number, too. Since part I of the theorem shows that F has a spanning regular subgraph of this degree, we are done - a carefully performed edge removal will result in a spanning subgraph in which every vertex has degree $\frac{\delta+\sqrt{2\delta-1}}{2}n$.

It is easy to see that $p_0 = \frac{\delta+\gamma-1}{2\gamma-1}$ is the solution of (1). Then the degree of an arbitrary vertex in $A_l \cup B_l$ is

$$(1-p_0)(1-\gamma)n = \left(1 - \frac{\delta+\gamma-1}{2\gamma-1}\right)(1-\gamma) = \frac{\gamma-\delta}{2\gamma-1}(1-\gamma).$$

Substituting $\gamma = \frac{1-\sqrt{2\delta-1}}{2}$ we get

$$\frac{\delta - \frac{1-\sqrt{2\delta-1}}{2}}{\sqrt{2\delta-1}}(1 - \frac{1-\sqrt{2\delta-1}}{2}) = \frac{2\delta-1+\sqrt{2\delta-1}}{\sqrt{2\delta-1}}\frac{1+\sqrt{2\delta-1}}{2} =$$

$$\frac{1+\sqrt{2\delta-1}}{2}\frac{1+\sqrt{2\delta-1}}{2} = \frac{\delta+\sqrt{2\delta-1}}{2},$$

and this is what we promised to show.

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References

[1] B. Csaba, M. Mydlarz, *Approximate multipartite version of the Hajnal–Szemerédi theorem*, submitted.

[2] B. Csaba, *Approximating the Bollobás–Eldridge conjecture for bounded degree graphs*, manuscript.

[3] L. Lovász, *Combinatorial Problems and Exercises*, Akadémiai Kiadó, Budapest, 1993.

[4] W. T. Tutte, *Graph factors*, Combinatorica (1981) no. 1., 79–97.