On the Bollobás – Eldridge conjecture for bipartite graphs

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Abstract

Let G be a simple graph on n vertices. A conjecture of Bollobás and Eldridge [5] asserts that if $\delta(G) \geq \frac{kn-1}{k+1}$ then G contains any n vertex graph H with $\Delta(H) = k$. We prove a strengthened version of this conjecture for bipartite, bounded degree H, for sufficiently large n. This is the first result on this conjecture for expander graphs of arbitrary (but bounded) degree. An important tool for the proof is a new version of the Blow-up Lemma.

1 Introduction

In this paper we will consider only simple graphs. We mostly use standard notation: we denote by $V(F)$ and $E(F)$ the vertex and the edge set of the graph F, $deg_F(x)$ is the degree of the vertex $x \in V(F)$, $\delta(F)$ is the minimum degree and $\Delta(F)$ is the maximum degree. If $F = F(A, B)$ is a bipartite graph with color classes A and B, then let $\Delta_A = \max_{x \in A} deg(x)$, $\Delta_B = \max_{x \in B} deg(x)$ and $\Delta = \min\{\Delta_A, \Delta_B\}$.

Let G_1 and G_2 be two graphs on n vertices. If there is a bijection $\phi : V(G_1) \to V(G_2)$ such that $(i, j) \in E(G_1)$ implies $(\phi(i), \phi(j)) \notin E(G_2)$, then G_1 and G_2 can be packed. Equivalently (and we will consider this formulation in the paper), G_1 and G_2 can be packed, if $G_2 \subset \overline{G}_1$, i.e., G_2 is a spanning subgraph of $\overline{G}_1$.

Packing of graphs is a heavily studied subject in graph theory. The reader can find a good survey on packing of graphs in [4] and [15]. Packing of graphs has applications in computer science as well, see eg. [5, 10].

In 1978 the following deep conjecture was formulated by Bollobás and Eldridge in [5]:

Conjecture 1 (Bollobás-Eldridge) *If G is a simple graph on n vertices with*

$$\delta(G) \geq \frac{kn-1}{k+1}$$

then G contains any spanning subgraph H with $\Delta(H) = k$.

Perhaps the simplest special case of Conjecture 1 is the case of $\Delta(H) = 1$, which can be solved easily. The case when H is the union of disjoint $(k+1)$-cliques was proved by Hajnal and Corrádi [6] ($k = 2$), and Hajnal and Szemerédi [9] (for arbitrary k). Aigner and Brandt [1] and Alon and Fischer [2] proved the conjecture for the case H is the disjoint union of cycles (this special case was first considered in [16]). Csaba, Shokoufandeh and Szemerédi [8] gave the proof for $\Delta(H) = 3$

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and Csaba [7] proved it for $\Delta(H) = 4$ (if the number of vertices is large enough). However, the conjecture is wide open for most cases.

In this paper we investigate the case when $\Delta(H)$ is bounded and H is bipartite. Let us emphasize, that this problem is different from the problem of *bipartite packing*, when the two graphs to be packed are bipartite (see eg., [11]).

It is easy to see that Conjecture 1 is tight in general: Let H be the disjoint union of $\frac{n}{k+1}$ cliques of size $k+1$ and G be a complete $(k+1)$-partite graph with $k-1$ color classes of size $\frac{n}{k+1}$, one color class of size $\frac{n}{k+1}+1$ and the last of size $\frac{n}{k+1}-1$. Then $\delta(G) = \frac{kn}{k+1} - 1$, but $H \not\subset G$.

Still, in the special case of embedding bipartite graphs we can strengthen the conjecture: we show that the minimum degree requirement of Conjecture 1 is unnecessarily strong for bipartite graphs.

Theorem 2 *Given two integers Δ_1 and Δ_2 $(\Delta_1, \Delta_2 \geq 2)$, there exists a threshold n_0 and $\beta > 0$ real such that for all $n \geq n_0$ the following statement holds: Let $H = H(A, B)$ be a bipartite graph on n vertices, with $\Delta_A = \Delta_1$, $\Delta_B = \Delta_2$ and $\Delta = \min\{\Delta_A, \Delta_B\}$. Then if G is any graph of order n having minimum degree*

$$\delta(G) \geq \frac{\Delta}{\Delta+1}(1-\beta)n,$$

then H is a spanning subgraph of G.

Sometimes we will call G the *host graph*.
Note that in the above theorem $\delta(G)$ depends on $\Delta = \min\{\Delta_A, \Delta_B\}$, not on $\Delta(H)$. Besides, even in case $\Delta_A = \Delta_B$ a smaller minimum degree is sufficient than what is required in the Bollobás–Eldridge conjecture. Note that in contrast to the Bollobás–Eldridge conjecture, some bound on the maximum degree of H is clearly needed in Theorem 2 (to see this consider the graph $H = K_{2,n-2}$ and let G be a complete 3-partite graph on n vertices with equal color class sizes.)

In understanding the proof of the result some familiarity with the Regularity Lemma of Szemerédi [17] will be helpful, although we will give a brief survey on the necessary notions in the second section. As it happens frequently in combinatorics, for proving the main theorem we need a lemma, which is similar to another one which was published several years ago: the Blow-up Lemma of Komlós, Sárközy and Szemerédi, see [13, 14]. For our application we had to make changes in the statement. However, the proof is similar to the proof of [14]. We note, that a special case of this version (for embedding graphs of maximum degree three) appeared in [8]. We will prove this modified Blow-up Lemma in the third section, and then show Theorem 2 in the fourth section. We finish the paper with a section on concluding remarks.

2 Review of Tools for the Proof

Firstly, we will discuss the graph theoretic tools what we will need: Szemerédi's Regularity Lemma [17], and some related results. Secondly, we will consider an inequality from probability theory, which is a generalization of Hoeffding's inequality.

2.1 Graph Theory

Let us introduce some more notation first. For any vertex v of the graph G, $deg_G(v, X)$ is the number of neighbors of v in the set X, and $e(X, Y)$ is the number of edges between the disjoint sets X and Y. $N_G(v)$ is the set of neighbors of v and $N_G(v, X)$ is the set of neighbors of v in X. For a set $U \subset V(G)$, $N: G(U) = \cup_{v \in U} N_G(v)$. Throughout the paper we will apply the relation "$\ll$": $a \ll b$, if a is sufficiently smaller, than b.

The *density* between disjoint sets X and Y is defined as:

$$d(X,Y) = \frac{e(X,Y)}{|X||Y|}.$$

In the proof of Theorem 2, Szemerédi's Regularity Lemma [17, 15] plays a pivotal role. We will need the following definition to state the Regularity Lemma.

Definition 1 (Regularity condition) *Let $\varepsilon > 0$. A pair (A,B) of disjoint vertex-sets in G is ε-regular if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \ |Y| > \varepsilon|B|$$

we have

$$|d(X,Y) - d(A,B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely, the density of any reasonably large subgraph is almost the same as the density of the regular pair.

We will use the following form of the Regularity Lemma:

Lemma 3 (Degree Form) *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V,E)$ is any graph and $d \in [0,1]$ is any real number, then there is a partition of the vertex set V into $\ell+1$ clusters $V_0, V_1, \dots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \le M$,
- $|V_0| \le \varepsilon|V|$,
- *all clusters V_i, $i \ge 1$, are of the same size $m\,(\le \varepsilon|V|)$,*
- $deg_{G'}(v) \ge deg_G(v) - (d+\varepsilon)|V|$ *for all $v \in V$,*
- $G'|_{V_i} = \emptyset$ *(V_i is an independent set in G') for all $i \ge 1$,*
- *all pairs (V_i, V_j), $1 \le i < j \le \ell$, are ε-regular, each with density either 0 or at least d in G'.*

Often we call V_0 the *exceptional cluster*. In the rest of the paper we assume that $0 < \varepsilon \ll d \ll 1$.

Remark 1 *It is clear, that $\frac{n(1-\varepsilon)}{\ell} \le m \le \frac{n}{\ell}$. Recall, that $m \le \varepsilon n$. Therefore, if $\varepsilon \to 0$, then $\ell \to \infty$.*

Definition 2 (Reduced graph) *Apply Lemma 3 to the graph $G = (V,E)$ with parameters ε and d, and denote the clusters of the resulting partition by $V_0, V_1, \dots, V_\ell$, V_0 being the exceptional cluster. We construct a new graph G_r, the reduced graph of G' in the following way: The non-exceptional clusters of G' are the vertices of the reduced graph G_r (hence $|V(G_r)| = \ell$). We connect two vertices of G_r by an edge if the corresponding two clusters form an ε-regular pair with density at least d. Sometimes we will refer to the vertices of G_r as clusters, too.*

The following corollary is immediate:

Corollary 4 *Apply Lemma 3 with parameters ε and d to the graph $G = (V,E)$ satisfying $\delta(G) \ge \gamma n$ ($|V| = n$) for some $\gamma > 0$. Denote G_r the reduced graph of G'. Then $\delta(G_r) \ge (\gamma - \theta)\ell$, where $\theta = 2\varepsilon + d$.*

In our application of Lemma 3 we will assume that all densities "almost equal" to d. We take each regular pair with density exceeding this number, and randomly discard edges with the appropriate probability. As a result we will have ε'-regular pairs having the desired densities, all being very close to d. By applying Chernoff's bound one can see that these densities will get arbitrarily close to d as the number of vertices tends to infinity. ε' will only be slightly bigger than ε, and for simplicity we will call it ε. We will refer to the densities of the regular pairs as if they were actually equal to d, and, as it will be clear later in Section 3, this approximation is good enough for our purposes.

2.2 Probability Theory

There is a standard inequality in probability theory due to Hoeffding [12], we will use a version of it. Let us assume, that we are given an urn with r red and b blue balls. Let $n = r + b$. We conduct the following experiment: we randomly, uniformly draw m balls ($1 \leq m \leq n$) without replacement. Denote the number of chosen red balls by ρ, this is a random variable. It is easy to see, that $\mathrm{E}[\rho] = m\frac{r}{n}$. Hoeffding's inequality states, that for every $0 \leq t \leq n$

$$\Pr[|\rho - \mathrm{E}[\rho]| \geq t] \leq 2e^{-\frac{t^2}{2m}}.$$

Alternatively, one can prove the above inequality by the help of Azuma's inequality [3], applying martingales. For that let us define the random variables $\rho_0, \rho_1, \ldots, \rho_m$: $\rho_0 = \mathrm{E}[\rho]$, and for every $1 \leq i \leq m$ let $\rho_i = \mathrm{E}[\rho|\text{the first } i \text{ balls are known}]$. Then the sequence $\rho_0, \rho_1, \ldots, \rho_m = \rho$ defines a martingale. Moreover, $|\rho_i - \rho_{i-1}| \leq 1$ for every $1 \leq i \leq m$. Hence, we get Hoeffding's inequality by applying Azuma's inequality:

$$\Pr[\rho \geq \mathrm{E}[\rho] + t] \leq e^{-\frac{t^2}{2m}} \text{ and } \Pr[\rho \leq \mathrm{E}[\rho] - t] \leq e^{-\frac{t^2}{2m}}.$$

As we promised, we will need a somewhat different version. Let us assume that we are given a ground set X and the non-empty sets $R, B_1, \ldots, B_m$, where $R \subset X$ and $B_i \subset X$ for $1 \leq i \leq m$. We assume, that $|R| = r$ and $|B_i| \geq b$ for every $1 \leq i \leq m$.

We will conduct another experiment now. At time i we randomly choose an element of $R_i \cup B_i - P_{i-1}$, where $R_i \subset R$ and P_{i-1} is the set of previously chosen elements.

Analogously to ρ, we can define the random variable μ, and the sequence $\mu_0, \mu_1, \ldots, \mu_m$. As before, we will have a martingale process with $|\mu_i - \mu_{i-1}| \leq 1$ for every $1 \leq i \leq m$, and therefore a strong concentration result by applying Azuma's inequality. However, we cannot give a simple formula for $\mathrm{E}[\mu]$ now. Still, it is obvious, that $\mathrm{E}[\mu] \leq mr/(r + b)$. Applying Azuma's inequality, we will have:

$$\Pr[\mu \geq m\frac{r}{r+b} + t] \leq \Pr[\mu \geq \mathrm{E}[\mu] + t] \leq e^{-\frac{t^2}{2m}}.$$

Therefore,

$$\Pr[\mu \geq m\frac{r}{r+b} + t] \leq e^{-\frac{t^2}{2m}}.$$

We will refer to the above inequality as the *modified Hoeffding's bound.*

2.3 A rough outline of the proof

Our goal is to embed H into the host graph G. For achieving this goal first we apply the Regularity Lemma to G. Then we distribute the vertices of H among the non-exceptional clusters of G' – at this point vertices of H are assigned to clusters of G_r, but not mapped to vertices of G. It is important to do this distribution *evenly* and *consistently.* That is, we assign $m + |V_0|/\ell \pm o(n)$ vertices of H to each non-exceptional cluster ("evenness"), and if $(x, y) \in E(H)$ and x is assigned to the cluster V_x and y is assigned to V_y, then $(V_x, V_y) \in E(G_r)$ ("consistency"). Then we map appropriately chosen vertices of H to V_0. After this step we will have m vertices of H assigned to each non-exceptional cluster. For mapping these vertices we will apply the modified Blow-up Lemma.

3 Modified Blow-up Lemma

As it was mentioned above, most of H will be embedded by a similar procedure to that of the Blow-up Lemma. Readers familiar with the lemma may observe that unlike in our setup, the Blow-up Lemma applies for a fixed reduced graph which does not depend on the parameters ε and d, and all the edges of that (fixed) reduced graph are super-regular pairs (this is a stronger notion than ε-regularity). Besides, as we will see, we will have restrictions for the embedding of certain vertices of H. Hence, we need a stronger statement than that of the Blow-up Lemma. It will require several new conditions, and this version below will be more technical. However, the main message has not changed: if certain conditions are satisfied, one can embed bounded degree spanning subgraphs into pseudo-random graphs. In this section we discuss this embedding algorithm, and then prove its correctness. This embedding algorithm and its analysis is not much different from the algorithm of [14] or [8]. In particular, Lemma 5 is a generalization of the embedding lemma of [8] for embedding spanning subgraphs of arbitrary, but bounded degree.

Given H and G our goal is to find a subgraph of G which is isomorphic to H. We assume, that $D = \Delta(H)$ is at least 1, otherwise there is nothing to prove. Let us denote by $I' \subset V(H)$ a set the elements of which are of distance at least 4 from each other, and $|I'| \geq \frac{n}{2D^3}$ - the existence of I' can be shown easily by the help of a greedy algorithm.

Assume that $V(G) = V_0 \cup V_1 \cup \ldots \cup V_\ell$, and $V(H) = L_0 \cup L_1 \cup \ldots \cup L_\ell$ are partitions such that there is a bijective mapping $\varphi : L_0 \to V_0$; hence, $|V_0| = |L_0|$. We also assume that for every $1 \leq i \leq \ell$ $|V_i| = |L_i| = m$. Set $I'_i = L_i \cap I'$.

Let $x \in L_i$; a vertex $v \in V_i$ is called (d, ε)*–good* for x if $y \in N(x) \cap L_j$ implies $deg_G(v, V_j) \geq (d - \varepsilon)m$ for every $1 \leq j \leq \ell$.

Lemma 5 (Modified Blow-up lemma) *For every integer $D \geq 1$ there exists n_0 and $\varepsilon, d > 0$ such that if $n > n_0$, H and G are graphs of order n, $\Delta(H) = D$, and*

$$0 < \varepsilon \ll \varepsilon'' \ll \delta' \ll d \ll 1,$$

for every $1 \leq i < j \leq \ell$ the pair (V_i, V_j) is ε-regular, with density 0 or d, and the conditions listed below hold, then H could be embedded in G by a randomized algorithm.

There exist positive constants K_1, K_2, K_3, c_1 and c_2 which may depend on D but not on any other parameter such that:

C1: $|L_0| = |V_0| \leq K_1 dn$;

C2: $L_0 \subset I'$;

C3: *for every $1 \leq i \leq \ell$, L_i is independent;*

C4: *for every $1 \leq i \leq \ell$, $|N_H(L_0) \cap L_i| \leq K_2 dm$;*

C5: *for every $1 \leq i \leq \ell$ there is $B_i \subset I'_i$ with $|B_i| = \delta' m$, such that for $B = \cup_i B_i$ and every $1 \leq i, j \leq \ell$,*

$$||N_H(B) \cap L_i| - |N_H(B) \cap L_j|| < \varepsilon m;$$

C6: *for every $1 \leq i, j \leq \ell$ if $(x, y) \in E(H)$ and $x \in L_i$, $y \in L_j$, then (V_i, V_j) is an ε-regular pair with density d;*

C7: *if $(x, y) \in E(H)$ and $x \in L_0$, then $y \in L_j$ implies $deg(\varphi(x), V_j) \geq c_1 |V_j| = c_1 m$;*

C8: *for every $1 \leq i \leq \ell$, given any $E_i \subset V_i$ such that $|E_i| \leq \varepsilon'' m$ there exists a bijection*

$$\psi_i : E_i \to F_i \subset (L_i \cap (I' - B))$$

such that for every $v \in E_i$, v is (d, ε)–good for $\psi_i(v)$;

C9: *for $F = \cup F_i$,*

$$|N_H(F) \cap L_i| \leq K_3 \varepsilon'' m.$$

The elements of B will be called *buffer vertices*. The reader may notice that we have not defined which vertices would belong to E_i – this will be determined during the execution of the algorithm.

Let us briefly explain the role of conditions C1 - C9. We want to map the vertices of L_i to vertices of V_i ($0 \le i \le \ell$). First, $x \in L_0$ will be mapped to $\varphi(x) \in V_0$, that is why we need C1 and C2. We have C3 since L_i will be mapped to V_i ($1 \le i \le \ell$). C6 and C7 are so called *consistency conditions*. The meaning of C4 and C5 will be clear later, these are measures for the "evenness" of the distribution of the vertices of H among the clusters of G. We need C8 and C9 since we have to take special care of E_i.

3.1 The embedding algorithm

From now on we suppose that the requirements of Lemma 5 are satisfied. For discussing the embedding algorithm and proving its correctness we will need more constants: $\varepsilon', \varepsilon''', \delta'', \delta'''$ are such, that

$$0 < \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll \delta''' \ll \delta'' \ll \delta' \ll d \ll 1.$$

Having φ at hand, we map $x \in L_0$ to $\varphi(x) \in V_0$ (recall, that L_0 is independent). Let $n' = |V(H - L_0)|$, we order the vertices of $H - L_0$ into a sequence $\mathcal{S} = (x_1, x_2, \ldots, x_{n'})$ which is almost the order in which $V(H - L_0)$ will be mapped. The structure of $\mathcal{S}$ and how it is reordered occasionally plays an important role, we give the details below.

For every $1 \le i \le \ell$, we have a subset B_i of L_i of size $\delta' m$, the set of buffer vertices in L_i. Recall, that $B = \cup_i B_i$. Let $M = |B|$, and denote by $b_1, b_2, \ldots, b_M$ the buffer vertices, these will form the last part of $\mathcal{S}$. The sequence $\mathcal{S}$ begins with the vertices of $N_H(L_0)$, followed by $\{N_H(b_1),\ N_H(b_2), \ldots, N_H(b_M)\}$, the neighbors of the buffer vertices. We let $T_0 = |N_H(L_0)|$ and $T_1 = \sum_{i=1}^{M} |N_H(b_i)|$. Then we add all the other vertices to the sequence, in such a way that the buffer vertices form the tail of $\mathcal{S}$.

For technical reasons we assume that $\mathcal{S}$ is ordered *evenly*. This means that if we divide $\mathcal{S}$ starting from the $(T_0 + T_1 + 1)$th element into $\frac{1}{\delta''}$ consecutive segments of length $\delta'' n'$, then these segments will have about the same number of vertices from every L_i set, no such segment will contain more than $2\delta'' m$ vertices of L_i for every $1 \le i \le \ell$. Observe, that this is possible by using conditions C4 and C5: they imply that the first $T_0 + T_1$ elements of $\mathcal{S}$ contain roughly the same number of vertices from each L_i. Later, during the execution of the embedding algorithm we may place some vertices forward – only a very small proportion, as we will show. If we have to do so, we immediately reorder the remaining unmapped vertices of $\mathcal{S}$ to maintain this property.

Reordering will mean renaming as well, so as to have that the jth vertex in $\mathcal{S}$ is called x_j. We do the reordering with special care to have only buffer vertices in the tail of $\mathcal{S}$. In fact, it is possible to have at least $\frac{\delta'}{2} m$ buffer vertices in the tail of $\mathcal{S}$ from L_i for every $1 \le i \le \ell$ during the execution of the embedding algorithm.

The mapping of the vertices of $H - L_0$ is done in three separate phases. In the first phase we are going to map the vertices of $N_H(L_0)$. In the second phase will come the mapping of the next vertices of $\mathcal{S}$ after each other according to their position in the sequence (some reordering is possible in this phase), until only buffer vertices are left in $\mathcal{S}$. In the third phase, by a matching procedure we map the remaining buffer vertices. The embedding algorithm is a randomized procedure; we will prove, that with probability $1 - o(1)$ H can embedded by the help of it.

If we map $x \in V(H)$ to $v \in V(G)$, then we say v is *covered* by x or that v is the *host vertex* for x. Since no $v \in V(G)$ will be covered by more than one vertex of H, we always map a vertex to an uncovered one of G.

We say that the embedding algorithm *succeeds for t*, if it can find a host vertex for mapping the tth vertex in $\mathcal{S}$. If the algorithm cannot find a host vertex for some $x \in H$, then it *halts with failure*. Hence, if the algorithm succeeds for t, then it has been succesful for finding host vertices for the first

t vertices in $\mathcal{S}$. We say we are at *time* t if we have succesfully mapped the first t vertices of $\mathcal{S}$, and mapping of the $(t+1)$th vertex is next.

In the following subsection we outline our method for the embedding, with the exception of selecting a vertex to be covered. That will be done in a separate subsection.

3.1.1 Outline of the algorithm

For an unmapped vertex $x \in L_i$ we will denote by $H_{t,x}$ its monotonically shrinking *host set* in V_i at time t, i.e., $H_{t,x}$ is a subset of the uncovered vertices at time t which are the candidates for being covered by x. Also, for technical reasons we keep track of another set, $C_{t,x}$. By Z_t we denote the set of covered vertices at time t (note that $Z_0 = V_0$). Similarly, Y_t denotes the set of mapped vertices of H. Obviously, $Y_0 = L_0$. We also maintain a set Bad_t of *exceptional pairs* (or *bad pairs*) in $H - L_0$ (the definition of an exceptional pair will follow later).

At time 0, we set $Bad_0 = \emptyset$, and $C_{0,x} = H_{0,x} = V_i$, where $x \in L_i$, and x does not have any neighbor in L_0. For those vertices having a neighbor in L_0 the setup is different. Let x in L_0 have neighbors $y_1 \in L_{i_1}, y_2 \in L_{i_2}, \ldots, y_D \in L_{i_D}$, and $v = \phi(x)$. By virtue of condition C7 we have ensured that v has at least $c_1 m$ neighbors in $V_{i_1}, V_{i_2}, \ldots, V_{i_D}$. These neighborhoods give $C_{0,y_1} = H_{0,y_1}, C_{0,y_2} = H_{0,y_2}, \ldots, C_{0,y_D} = H_{0,y_D}$, respectively. Note that this makes sense as every vertex of H has at most one neighboring vertex in L_0.

Recall, that $T_0 = |N_H(L_0)|$ and $T_1 = \sum_{i=1}^{M} |N_H(b_i)|$. We let $T_2 = \delta'' n'$. Given the initial host sets, the embedding algorithm will go as follows:

Phase 1. For $1 \le t \le T_0$ repeat the following steps

Step 1.1.

Pick an appropriate vertex v_t for $x_t \in N_H(L_0)$ from H_{t-1,x_t} using the *Selection Algorithm* of Section 3.1.2, then map x_t to v_t.

Update

$$Z_t = Z_{t-1} \cup \{v_t\}, \quad Y_t = Y_{t-1} \cup \{x_t\},$$

and for all unmapped vertices x_i, with $t < i \le n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} - Z_t$$

Step 1.2. If there is an $x \in H - L_0 - Y_t$ such that $|H_{t,x}| \le \delta'' m$, then *halt with failure.*

Step 1.3. Set $t \leftarrow t+1$. If $t \le T_0$, then go back to Step 1.1.

Phase 2. For $t \ge T_0 + 1$ repeat the following steps

Step 2.1. Map the vertex x_t from the sequence $\mathcal{S}$: using the *Selection Algorithm* choose an appropriate vertex v_t from the set H_{t-1,x_t} as x_t's image.

Step 2.2. Update

$$Z_t = Z_{t-1} \cup \{v_t\}, \quad Y_t = Y_{t-1} \cup \{x_t\},$$

and for all unmapped vertices x_i, with $t < i \le n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} - Z_t.$$

Step 2.3. Taking care of exceptional vertices in G

1. If $t \neq T_0 + T_1$ go to Step 2.4.

2. If $t = T_0 + T_1$ then for every cluster V_i $(1 \leq i \leq \ell)$ form a set E_i – the *exceptional vertices of* V_i – containing those uncovered vertices satisfying

$$|\{b : b \in B_i, v \in C_{t,b}\}| < \delta''|B_i|.$$

We will cover them right after mapping the neighbors of the buffer vertices. (Later we will see, that this way we eliminate a possible obstruction to map the buffer vertices in *Phase 3.*) We slightly change the ordering of $\mathcal{S}$: From every list L_i we take $|E_i|$ vertices belonging to I' to form the set $\psi(E_i) = F_i$. Let $F = \cup F_i$. We place the vertices of F forward, $x = \psi(v) \in F_i$ will be mapped to $v \in E_i$. The requirements for choosing ψ and F have been formulated in C8 and C9. We will maintain the even ordering of $\mathcal{S}$, i.e., if necessary, we reorder the remaining unmapped vertices of $\mathcal{S}$.

Step 2.4. Taking care of exceptional vertices in $H - L_0$

1. If T_2 does not divide t or $t \leq T_0 + T_1$, then go to Step 2.5.

2. Otherwise, we will find all exceptional unmapped vertices y: here $y \in H - L_0$ is defined to be *exceptional*, if $|H_{t,y}| \leq (\delta')^2 m$. We again slightly change the order of the remaining vertices in $\mathcal{S}$ by bringing these exceptional vertices forward in $\mathcal{S}$, including exceptional buffer vertices. If necessary, we reorder the remaining unmapped vertices of $\mathcal{S}$ so as to maintain the even ordering.

Step 2.5. If the unmapped vertices are all buffer vertices, go to *Phase 3.*, otherwise set $t \leftarrow t + 1$ and go back to *Step 2.1.*

Phase 3. We are at time T now, when there are only buffer vertices left in $\mathcal{S}$. Find a system of distinct representatives of the sets $H_{T,y}$ for all unmapped vertices. If there is no such system, then *halt with failure.*

3.1.2 Selection Algorithm

There can be two possible cases.

Case 1. $x_t \notin F$.
As the image of x_t, we will choose some $v_t \in H_{t-1,x_t}$ such that the following conditions are satisfied for every unmapped vertex y with $(x_t, y) \in E(H)$:

$$(d - \varepsilon)|H_{t-1,y}| \leq deg_G(v_t, H_{t-1,y}) \leq (d + \varepsilon)|H_{t-1,y}|, \quad (3)$$

$$(d - \varepsilon)|C_{t-1,y}| \leq deg_G(v_t, C_{t-1,y}) \leq (d + \varepsilon)|C_{t-1,y}|, \quad (4)$$

and

$$(d - \varepsilon)|C_{t-1,y} \cap C_{t-1,y'}| \leq deg_G(v_t, C_{t-1,y} \cap C_{t-1,y'}) \leq (d + \varepsilon)|C_{t-1,y} \cap C_{t-1,y'}|, \quad (5)$$

for at least $(1 - \varepsilon')$ portion of the unmapped vertices y' such that y and y' are assigned to the same cluster V_i, and $\{y, y'\} \notin Bad_{t-1}$. The set Bad_t will be formed as the union of Bad_{t-1} and those pairs $\{y, y'\}$ which does not satisfy (5) for v_t (recall that $Bad_0 = \emptyset$). Clearly, at most $D\varepsilon' m$ new pairs will be added to Bad_t. If there are more than one $v \in H_{t-1,x_t}$, which satisfies the above conditions, then choose among them randomly. If we cannot find v_t for x_t which satisfies the above conditions, then *halt with failure.*

Case 2. $x_t \in F$.
We will assign $x_t \in L_i$ ($i \neq 0$) to an exceptional $v_t \in E_i$ so that for all unmapped $y \in N_H(x_t)$ ($y \in L_j$, $j \neq 0$) the following is satisfied:

$$deg_G(v_t, C_{t-1,y}) \geq (d-\varepsilon)m \geq (d-\varepsilon)|C_{t-1,y}|, \quad (6)$$

and

$$deg_G(v_t, H_{t-1,y}) \geq \frac{d}{2}m. \quad (7)$$

We will use C8 to try to find such vertices. If we cannot find x_t to cover the exceptional v_t, then *halt with failure.*

3.2 Proof of Lemma 5

Our goal is to show that with positive probability the embedding algorithm will not *halt with failure.*

We start by proving that *Phase 1* of the algorithm succeeds with high probability. First we show that the Selection Algorithm is likely to succeed for $1 \leq t \leq T_0$ in finding v_t.

Lemma 6 *Assuming that Phase 1 succeeds for $t-1$, with $1 \leq t \leq T_0$ and $|H_{t-1,x_t}| > \delta'' m$, then it succeeds for t.*

Proof We only need to consider **Case 1** of the *Selection Algorithm.* The selected vertex $v_t \in H_{t-1,x_t}$ should satisfy conditions (3), (4), and (5). By ε-regularity we will have at most $2\varepsilon m$ vertices in H_{t-1,x_t} which do not satisfy (3), and the same holds for (4). For condition (5) we will define a bipartite graph $BG = (W_1, W_2, E(BG))$. Here $W_1 = H_{t-1,x_t}$, and the elements of W_2 are the sets $C_{t-1,y} \cap C_{t-1,y'}$ for all pairs $\{y, y'\}$ where $(x_t, y) \in E(H)$, y and y' are both assigned to the same cluster, and $\{y, y'\} \notin Bad_{t-1}$. For $v \in W_1$ and $u \in W_2$, we have $(v, u) \in E(BG)$ if (5) does not hold for v and the pairs corresponding to u. If we assume that there are more than $\varepsilon' m$ vertices $v \in W_1$ with $deg_{BG}(v) > \varepsilon'|W_2|$, then there should be a vertex $u \in W_2$ such that

$$deg_{BG}(u) > \varepsilon'^2 m \gg \varepsilon m.$$

Denote $\{y_u, y'_u\}$ the pair which corresponds to u. Let

$$g_t = |N_H(y_u) \cap Y_{t-1}| + |N_H(y'_u) \cap Y_{t-1}|.$$

We will show that

$$|C_{t-1,y_u} \cap C_{t-1,y'_u}| \geq (d-\varepsilon)^{g_t} c_1 m \ \gg \varepsilon m.$$

This and the assumption will contradict with ε-regularity.

We proceed by induction on g_t. First assume that $g_t = 0$. We have that $y \notin N_H(L_0)$, since $\mathcal{S}$ begins with the vertices of $N_H(L_0)$ and $N_H(L_0)$ is an independent subset in H. Clearly, $|C_{t-1,y'_u}| \geq c_1 m$ even if $y'_u \in N_H(L_0)$. Hence, we have that

$$|C_{t-1,y_u} \cap C_{t-1,y'_u}| \geq (d-\varepsilon)^{g_t} c_1 m = c_1 m.$$

Let us assume now, that $g_t \geq 1$, and that the induction hypotheses is true up to $g_t - 1$. Consider the largest $\tilde{t}$ for which $g_{\tilde{t}} = g_t - 1$. Then

$$|C_{\tilde{t}-1,y_u} \cap C_{\tilde{t}-1,y'_u}| \geq (d-\varepsilon)^{g_{\tilde{t}}} c_1 m$$

by the induction hypotheses. Observe, that $\{y_u, y'_u\} \notin Bad_{\tilde{t}}$, since $\{y_u, y'_u\} \notin Bad_{t-1}$, and $\tilde{t} \leq t+1$. Therefore, by Step 2.2 of the embedding algorithm we get that at time $\tilde{t}$, after mapping one more neighbor of y_u or y'_u,

$$|C_{\tilde{t},y_u} \cap C_{\tilde{t},y'_u}| \geq (d-\varepsilon)|C_{\tilde{t}-1,y_u} \cap C_{\tilde{t}-1,y'_u}| \geq (d-\varepsilon)^{g_{\tilde{t}}+1}c_1 m = (d-\varepsilon)^{g_t}c_1 m.$$

This in turn implies that H_{t-1,x_t} can contain at most $4\varepsilon m + \varepsilon' m \ll \delta'' m$ vertices which cannot be used to map x_t, proving the succession of *Phase 1.* □

Remark 2 *Observe, that we have actually proved a somewhat stronger statement: if $|H_{t,x}| = \delta'' m + s$ for some $s > 0$, then we have at least s possibilities for mapping x at time t.*

What is left to show is that for all t, $1 \leq t \leq T_0$, the host sets do not become too small. Actually, we prove this not just for the host sets for the unmapped vertices of $N_H(L_0)$, but for all unmapped vertices of H.

Lemma 7 *If Phase 1 succeeds for t with $t \leq T_0$, then $H_{t,x_{t'}} \geq \delta' m$ for all $t' > t$ with high probability.*

Proof Recall, that for $x \in L_i - N_H(L_0)$ (for $1 \leq i \leq \ell$) we have that $H_{0,x} = V_i$, while $|H_{0,x}| \geq c_1 m$ for every $x \in N_H(L_0)$. The host set $H_{t,x}$ of $x \in L_i$ decreases, when either $z \in N_H(x)$ is mapped, or another vertex, $y \in L_i$ is mapped to some vertex of $H_{t,x}$.

First let us consider the host sets of the vertices of $N_H(L_0)$. Since $t \leq T_0$, and no two vertices in $N_H(L_0)$ are adjacent, the only way the host set of $x \in N_H(L_0)$ decreases is that we cover some vertices of the host set by other vertices of $N_H(L_0)$.

By virtue of condition C4, there are at most $K_2 dm$ vertices in $N_H(L_0)$ which will be mapped to certain vertices of V_i for every $1 \leq i \leq \ell$. Even if all of them are mapped to the same host set $H_{t,x}$ (which is of size at least $c_1 m$), there is plenty of room left: unmapped vertices of $N_H(L_0)$ will have a host set of size at least $(c_1 - K_2 d)m \gg \delta' m$ at time $t \leq T_0$. Notice that here we don't need the randomness in the Selection Algorithm.

Let us consider now any vertex $y \in L_i - N_H(L_0)$ for some i. In the beginning $|H_{0,y}| = |C_{0,y}| = m$. Whenever a neighbor of y is mapped at time $\tilde{t}$, the size of its host set will change. Assume that y has s neighbors in $N_H(L_0)$ mapped by time t. Phase 1 succeeded for t, hence, by the Selection Algorithm we have that $(d-\varepsilon)^s m \leq |C_{t,y}| \leq (d+\varepsilon)^s m$ (this follows from inequality (4)).

Let us first assume, that $|C_{t,y}| \geq 2K_2 dm$. Then $|H_{t,y}| = |C_{t,y} - Z_t| \geq K_2 dm \gg \delta' m$, since, by virtue of condition C4, $|Z_t \cap L_i| \leq K_2 dm$ for every $1 \leq i \leq \ell$.

Now let us assume that $|C_{t,y}| < 2K_2 dm$. At this point the randomness in selecting a vertex by the Selection Algorithm will help us. We will apply the modified Hoeffding's bound. For that let $X = L_i$, $R = C_{t,y}$, and $B_j = H_{j,x_j} - R - S_j$ for $j \in J$ where $J = \{j : x_j \in L_i, 1 \leq j \leq t\}$ and S_j is the set of vertices of H_{j,x_j} which cannot be used to map x_j (see the remark after Lemma 6). We have that $|S_j| \leq \delta'' m$ and $|J| \leq K_2 dm$ (condition C4). Since $|H_{j,x_j}| \geq c_1 m$ for $j \in J$, we get that $|B_j| \geq (c_1 - 2K_2 d - \delta'')m$.

For every $x \in N_H(L_0) \cap L_i$ let ξ_x be a 0-1 random variable: set $\xi_x = 1$ iff x is mapped to a vertex in $C_{t,y}$ and let $\Xi = \sum \xi_x$.

Clearly,

$$\mathrm{E}[\Xi] \leq |J| \frac{|R|}{(c_1 - \delta'')m} \leq 2K_2 d|R|/c_1 \ll |R|/4.$$

Notice, that $|R| \geq (d-\varepsilon)^D m > d^{D+1} m$, hence $|R|^2/(2|J|) \geq d^{2D+1} m/(2K_2)$. By the modified Hoeffding's bound

$$\Pr(\Xi \geq |R|/2) \leq e^{-|R|^2/(2|J|)} \leq e^{-d^{2D+1}m/(2K_2)}.$$

That is, with very high probability $\Xi \leq |C_{t,y}|/2$. Consequently, $|H_{t,y}| \geq |C_{t,y}|/2 \geq (d-\varepsilon)^D m/2 \geq d^{D+1}/2m \gg \delta' m$ with very high probability. Observing that we have linear number of host sets in a cluster, we get that with high probability, $|H_{t,x}| \geq d^{D+1}/2m \gg \delta' m$ for every unmapped vertex x at time t, for $1 \leq t \leq T_0$.

□

From the above one can conclude:

Corollary 8 *Phase 1 succeeds with probability* $1 - o(1)$.

For $t > T_0$ we will need a more thorough analysis. Suppose, that want to map $x_t \in L_i$ for $T_0 + 1 \le t \le T_0 + T_1$. Let $Q_i \subseteq L_i - Y_{t-1}$ such that $|Q_i| \ge (\delta''')^2 m$. We define a bipartite graph $U_t = (V_i, Q_i, E(U_t))$. Here if $x \in Q_i$ and $v \in V_i$, then $(x, v) \in E(U_t)$ iff $v \in C_{t,x}$.

The following lemma is pivotal for the proof of the correctness of *Phase 2*.

Lemma 9 *Let i, t and Q_i as above. If the embedding algorithm succeeds for $t-1$, then apart from an exceptional set J of size at most $\varepsilon'' m$, the following will hold for every $v \in V_i$:*

$$deg_{U_t}(v) \ge (1-\varepsilon'')d(V_i, Q_i)|Q_i| \qquad (\ge \frac{d^D}{2}|Q_i|).$$

Proof We use the so called "defect form" of the Cauchy-Schwarz inequality, which states: if for some $p \le q$

$$\sum_{i=1}^{p} \alpha_i = \frac{p}{q}\sum_{i=1}^{q} \alpha_i + \beta$$

then

$$\sum_{i=1}^{q} \alpha_i^2 \ge \frac{1}{q}\left(\sum_{i=1}^{q} \alpha_i\right)^2 + \frac{\beta^2 q}{p(q-p)}.$$

Assume to the contrary that the lemma is not true, that is, $|J| > \varepsilon'' m$. Choose $J_0 \subset J$ with $|J_0| = \varepsilon'' m$. Define $\nu(t, x)$ as the number of mapped neighbors of x by time t. Observe that, if x has a neighbor in L_0, then $\nu(0, x) \ge 1$, otherwise it is 0. Since $x \notin N_H(L_0)$, we have that $|C_{0,x}| = m$. Then

$$|E(U_t)| = \sum_{x \in Q_i} |C_{t,x}| \ge \sum_{x \in Q_i} (d-\varepsilon)^{\nu(t,x)} m, \qquad (8)$$

here we used inequality (4).

We also have

$$\begin{aligned}
&\sum_{x \in Q_i}\sum_{x' \in Q_i} |C_{t,x} \cap C_{t,x'}| \\
&\le \sum_{x \in Q_i}\sum_{x' \in Q_i} (d+\varepsilon)^{\nu(t,x)+\nu(t,x')} m + |Q_i|m + D^2|Q_i|m + 2D\varepsilon' m^3 \\
&\le \sum_{x \in Q_i}\sum_{x' \in Q_i} (d+\varepsilon)^{\nu(t,x)+\nu(t,x')} m + 4D\varepsilon' m^3 \qquad (9)
\end{aligned}$$

Indeed, for each pair $\{x, x'\}$, we can upper-bound $|C_{t,x} \cap C_{t,x'}|$ by m. So, the diagonal terms $(x = x')$ result in error $|Q_i|m$. For the non-diagonal terms for which $N_H(x) \cap N_H(x') \ne \emptyset$ we have the term $D^2|Q_i|m$. If $\{x, x'\} \in Bad_t$, by **Case 1** of the Selection Algorithm either x or x' can appear in at most $D\varepsilon' m$ bad pairs. Hence there will be at most $D\varepsilon' m^2$ bad pairs (as at each time step the number of bad pairs increases by at most $D\varepsilon' m$, which was observed in Section 3.1.2) associated with the cluster V_i. In the remaining cases we use (5). Using the Cauchy-Schwarz inequality with $p = \varepsilon'' m$, $q = m$ and the variables $\alpha_k = deg_{U_t}(v_k)$, $1 \le k \le m$ with $v_k \in V_i$ and the first $\varepsilon'' m$ values set to degrees in J_0, we have:

$$\begin{aligned}
|\beta| &= \varepsilon'' \sum_{v \in V_i} deg_{U_t}(v) - \sum_{v \in J_0} deg_{U_t}(v) \\
&\ge \varepsilon'' \sum_{v \in V_i} deg_{U_t}(v) - \varepsilon''(1-\varepsilon'')d(V_i, Q_i)|Q_i|m \\
&= (\varepsilon'')^2 \sum_{v \in V_i} deg_{U_t}(v). \qquad (10)
\end{aligned}$$

Then using (8), (10) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\sum_{x\in Q_i}\sum_{x'\in Q_i}|C_{t,x}\cap C_{t,x'}|\\
&=\sum_{v\in V_i}(deg_{U_t}(v))^2\\
&\geq \tfrac{1}{m}\left(\sum_{v\in V_i}deg_{U_t}(v)\right)^2+(\varepsilon'')^3 d(V_i,Q_i)^2 m|Q_i|^2\\
&\geq \tfrac{1}{m}\left(\sum_{x\in Q_i}(d-\varepsilon)^{\nu(t,x)}m\right)^2+(\varepsilon'')^3(d-\varepsilon)^{2D}m|Q_i|^2\\
&\geq \sum_{x\in Q_i}\sum_{x'\in Q_i}(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}m+(\varepsilon'')^3(d-\varepsilon)^{2D}m|Q_i|^2
\end{aligned}$$

which is a contradiction to (9), since $|Q_i|\geq(\delta''')^2 m$,

$$(\varepsilon'')^3(d-\varepsilon)^{2D}(\delta''')^2\gg 4\varepsilon'\gg 4\varepsilon$$

and

$$(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}-(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}\ll 4\varepsilon.$$

□

As a consequence we will have the following bound on the size of the exceptional sets E_i:

Lemma 10 *In Step 2.3, for every $1\leq i\leq\ell$ we have $|E_i|\leq\varepsilon'' m$.*

Proof Recall *Step 2.3* of the embedding algorithm: we put a vertex of V_i into E_i, if it has only a few buffer neighbors in the graph $U_t=(V_i,B_i,E(U_t))$ with $t=T_0+T_1$. Applying the previous lemma with $t=T_0+T_1$ and $Q_i=B_i$, (therefore, $|Q_i|\geq(\delta''')^2 m$) we will have the following lower bound for the number of neighbours of the vertices of Q_i apart from an exceptional set $E_i\subset V_i$ of size at most $\varepsilon'' m$:

$$(1-\varepsilon'')d(V_i,Q_i)|Q_i|\geq\frac{d^D}{2}|Q_i|>\delta''|Q_i|.$$

□

We are ready to prove that the algorithm will not halt with failure in Case 2 of the *Selection Algorithm* with high probability.

Lemma 11 *For every $1\leq i\leq\ell$ and $v_t\in E_i$ there is an $x_t\in L_i$ such that inequalities* (6) *and* (7) *are satisfied with high probability.*

Proof Inequality (6) is easily seen to be satisfied by virtue of condition C8. Let $y\in N_H(x_t)\cap L_j$. We have to check that (7) is satisfied, that is, $deg_G(v_t,H_{t-1,y})\geq\frac{d}{2}m$ where $T_0+T_1<t\leq T_0+T_1+\varepsilon'' n$.

As in the proof of Lemma 7 we can show that with very high probability at most $\frac{d}{4}m$ vertices of $N_G(v_t,H_{t-1,y})$ are covered by some vertex in $N_H(L_0)$ up to time T_0. From time T_0+1 to T_0+T_1 we map the vertices of $N_H(B)$. By condition C5 we have that this way we cover at most $(D\delta'+\varepsilon)m$ vertices of $N_G(v_t,H_{t-1,y})$. There are at most $\varepsilon'' m$ vertices in E_i, thus the following bound holds with very high probability:

$$deg_G(v_t,H_{t-1,y})\geq(d-\varepsilon)m-\frac{d}{4}m-(D\delta'+\varepsilon)m-\varepsilon'' m\geq\frac{d}{2}m.$$

□

Next we will prove a result similar to Lemma 9 for $t>T_0+T_1$.

Lemma 12 *For every $1 \leq i \leq \ell$ and $T_0 + T_1 < t \leq T$ and any set of unmapped vertices $Q_i \subseteq L_i - Y_{t-1}$, with $|Q_i| \geq (\delta''')^2 m$, if Phase 2 succeeds for $t-1$, then apart from an exceptional set of size at most $\varepsilon''' m$ the following will hold for every $v \in V_i$:*

$$deg_{U_t}(v) \geq (1 - \varepsilon''')d(V_i, Q_i)|Q_i|.$$

Proof The proof follows the same line of argument as Lemma 9 with parameter ε''', except those vertices in the neighborhood of F (recall, that $F = \cup\psi(E_i)$). The inequality in (8) will hold with the same parameters, since for all $x \in N_H(F)$ we have

$$|C_{t,x}| \geq (d - \varepsilon)^{\nu(t,x)} m.$$

Here we used condition C8 and the fact that $\nu(t,x) = 1$ since $x \in I'$.

In (9) we have to take the pairs involving exceptional vertices into account. More precisely, based on *Step 2.3* of the embedding algorithm, there will be an additional error term of $2DK_3\varepsilon'' m^2 |Q_i|$ by condition C9. Using the fact that

$$(\varepsilon''')^3 (d - \varepsilon)^{2D} (\delta''')^2 \gg \varepsilon''$$

we can see that the contradiction still holds. □

The following lemma is an easy consequence of Lemmas 9 and 12.

Lemma 13 *For every $1 \leq i \leq \ell$ and $T_0 < t \leq T$ and any set of unmapped vertices $Q_i \subseteq L_i - Y_{t-1}$, with $|Q_i| \geq \delta''' m$ and a set $A \subset V_i$ with $|A| \geq \delta''' m$, if Phase 2 succeeds for $t-1$ then apart from an exceptional set J of size at most $(\delta''')^2 m$, the following will hold for every $x \in Q_i$:*

$$|A \cap C_{t,x}| \geq \frac{|A|}{2m}|C_{t,x}|.$$

Proof Let us suppose that the lemma is not true, there exists a set $J \subseteq Q_i$ such that $|J| > (\delta''')^2 m$, and for every $x \in J$ the inequality of the statement does not hold. We again consider the bipartite graph $U_t = U_t(J, V_i)$.

$$\sum_{v \in A} deg_{U_t}(v) = \sum_{x \in J} |A \cap C_{t,x}| < \frac{|A|}{2m} d(J, V_i)|J|m.$$

Applying Lemmas 9 or 12 with J, we get

$$\sum_{v \in A} deg_{U_t}(v) \geq (1 - \varepsilon''')d(J, V_i)|J|(|A| - \varepsilon''' m),$$

which is a contradiction. □

In the following lemma we show that the host sets do not become too small.

Lemma 14 *For every $T_0 + 1 \leq t \leq T$ and for every vertex $y \in H - Y_{t-1}$, if Phase 2 succeeds for $t-1$ then the following holds:*

$$|H_{t,y}| > \delta'' m.$$

Proof Let $Q_i = L_i - Y_{t-1}$, the set of all the unmapped vertices in L_i at time $t-1$, and let $A_t = V_i - Z_{t-1}$. Applying Lemma 13 we can see that for all $x \in Q_i$ (except at most $(\delta''')^2 m$ vertices), if $|A_t| \geq \frac{\delta'}{2} m$ then

$$|H_{t,x}| = |A_t \cap C_{t,x}| \geq \frac{|A_t|}{2m}|C_{t,x}| \geq \frac{\delta'}{4}(d - \varepsilon)^D m \gg (\delta')^2 m. \qquad (10)$$

Now we will prove that $|A_t| \geq \frac{\delta'}{2}m$. Let us suppose indirectly that there is a T' such that $T_1 + 1 \leq T' < T$ and

$$|A_{T'}| \geq \frac{\delta'}{2}m \text{ but } |A_{T'+1}| < \frac{\delta'}{2}m.$$

We know that at any time t', where T_2 divides t', there are at most $(\delta''')^2 m$ exceptional unmapped vertices. Thus, up to time T' we can find at most

$$\frac{1}{\delta''}(\delta''')^2 m \ll \delta'' m$$

exceptional vertices. This implies that at time T' there are many more than $(\delta' - \delta'')m$ unmapped buffer vertices, thus, on the contrary, $|A_{T'+1}| \gg (\delta' - \delta'')m$. Note, that we also proved that $T \leq \ell m - \ell\delta' m + \ell\delta'' m$.

Let us consider now an arbitrary $y \in L_i$ unmapped at time $t-1$ $(1 \leq t \leq T)$, and let $k\delta'' n' = kT_2 \leq t < (k+1)T_2$ for some $0 \leq k \leq T/T_2$. There are two cases to discuss:

Case 1. If y was not among the at most $(\delta''')^2$ exceptional vertices of *Step 2.4*, then (using (3) and (10))

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^D (\delta')^2 m - K,$$

where K is the number of vertices covered in V_i during the period between kT_2 and $(k+1)T_2$. Recall that the sequence $\mathcal{S}$ is balanced; hence, $K \leq 2\delta'' m$. Indeed, at time kT_2 we had that $|H_{kT_2,y}| \geq (\delta')^2 m$. These facts imply that in this case the statement of the lemma holds.

Case 2. If y was among the at most $(\delta''')^2$ exceptional vertices of *Step 2.4*, then (similarly to Case 1)

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^D (\delta')^2 m - K',$$

where K' is the number of vertices covered in V_i during the period between $(k-1)T_2$ and $(k+1)T_2$. This time K' can be as large as $(4\delta'' + (\delta''')^2)m$, because at time $(k-1)T_2$ at most $(\delta''')^2 m$ exceptional vertices were placed forward. Again, by observing that at time $(k-1)T_2$ we had that $|H_{(k-1)T_2,y}| \geq (\delta')^2 m$, the proof of the lemma is finished.

$\square$

Now it is easy to show that the *Selection Algorithm* will not halt with failure with high probability during *Phase 2*. We have just proved that the host sets can never get too small. In Lemma 6 we proved that *Phase 1* succeeds for t, whenever it succeeds for all t' with $t' < t \leq T_0$ and the host set is large enough. It is easy to see that exactly the same proof works for *Phase 2* and up to time T:

Lemma 15 *If Phase 2 succeeds for $t-1$ with $T_0 < t \leq T-1$ and $|H_{t-1,x_t}| > \delta'' m$, then it succeeds for t.*

Proof The proof of Lemma 6 works without any change. $\square$

Putting these together, we have that *Phase 2* of the algorithm succeeds with high probability.

To prove that *Phase 3* of the algorithm succeeds, we will show that for all $1 \leq i \leq \ell$ there is a system of distinct representatives between the unmapped buffer vertices of L_i and the remaining vertices of V_i. Let $Q_i \subset L_i$ denote the set of unmapped vertices assigned to the cluster V_i, and $R_i \subset V_i$ be the remaining vertices of the cluster V_i, with $M_i = |Q_i| = |R_i|$. Then by Lemma 14 for every $x \in Q_i$ we will have $H_{T,x} > \delta''' M_i$. Furthermore, for all subsets $S \subset Q_i$, if $|S| \geq \delta''' M_i$ then by repeated applications of Lemma 12

$$\left|\bigcup_{x \in S} H_{T,x}\right| \geq (1 - \delta''')M_i.$$

Finally, for any $v \in R_i$, since v cannot be exceptional in G, by *Step 2.3* there are at least $\delta''' M_i$ host sets $H_{T,x}$ containing v. This implies that for the subsets $S \subset Q_i$ with $|S| \geq (1 - \delta''' M_i)$ we have

$$\left| \bigcup_{x \in S} H_{T,x} \right| = M_i,$$

which in turn implies the existence of the system of distinct representatives. This finishes the proof of Lemma 5.

Remark 3 *From the proof it is clear, that we can decrease d: for a given D we can embed H in G with a smaller density d if we decrease ε appropriately. Another observation is, that if the densities of the regular pairs are not the same but "close" to each other in terms of ε, then the embedding can be finished as well.*

4 Assigning H to clusters of G_r

The process of embedding will go as follows: First, we apply the degree form of the Regularity Lemma for G with parameters ε and d. We assume, that the densities of the regular pairs are as close to each other, as it is needed (recall the remark at the end of Section 3). As a result we will have a partitioning of the vertex set into the clusters $V_0, V_1, V_2, \ldots, V_\ell$. We will assume, that $|V_0| \geq \varepsilon \frac{n}{2}$ – if V_0 is too small, than we discard $\varepsilon \frac{m}{2}$ vertices from every non-exceptional cluster of G', and put them into V_0.

Our goal is to find a partitioning of the vertices of H into $\ell + 1$ clusters $L_0, L_1, \ldots L_\ell$ so as to satisfy conditions C1-C9 of the modified Blow-up Lemma. We will find this partitioning by applying a randomized algorithm.

Let us denote the color classes of H by A and B, and suppose that $\Delta_A \geq \Delta_B$, hence, $\Delta = \Delta_B \geq 2$. It is intuitively clear, that if $|E(H)|$ is small, then it is easier to find an embedding of H in G. Still, it is easier to formulate the embedding algorithm, if the number of edges is not too small, say, $|E(H)| \geq \frac{n}{2\Delta}$.

Notice, that if H has no isolated vertices, then $E(H)$ is large enough. By adding extra edges if necessary, we will achieve that H has no isolated vertices: if $\{x_1, x_2, \ldots, x_s\}$ is the set of isolated vertices of H, we do the following. If s is even, we will add $s/2$ new edges to H which are determined by an arbitry matching between the x_is. If s is odd, connect x_1 to an arbitrarily chosen vertex in A, and then find the matching on $\{x_2, \ldots, x_s\}$ as above. Observe that this new graph is bipartite, and it has such a bicoloration that the maximum degree of one color class is at most Δ – this class is called B–, and the maximum degree of the other class, A increased by at most one.

We will perform another operation: If B has a vertex with degree less than Δ we will add some extra edges so as to achive that every vertex in B will have Δ neighbors in A. Clearly, doing the above carefully no vertex of A will have degree larger than $\Delta_A + 2\Delta_B|B|/|A|$. We will call the resulting new graph H. Obviously, embedding it proves the embeddability of the original graph as well.

We randomly distribute the vertices of A among the non-exceptional clusters. Then we are going to assign the vertices of B to non-exceptional clusters *consistently and evenly.* That is, if $y \in B$ has the neighbors $\{x_1, x_2, \ldots, x_\Delta\}$, and the x_is are assigned to the clusters $V_{j_1}, \ldots, V_{j_\Delta}$, then y will be assigned to a cluster V_s which is connected to $V_{j_1}, \ldots, V_{j_\Delta}$ by edges of G_r. Besides, we require that the number of assigned vertices of A and vertices of B to all non-exceptional clusters are $\frac{|A|}{\ell} \pm o(n)$ and $\frac{|B|}{\ell} \pm o(n)$, respectively. The assignment of the vertices of B will be done by the help of matching.

Still, there is no vertex of H assigned to V_0 (and hence all non-exceptional clusters are oversaturated). For dealing with this problem we first discard some appropriately chosen vertices of B (the *surplus*) from each non-exceptional cluster, these will form L_0, and the vertices of H assigned to V_s give the set L_s for $1 \leq s \leq \ell$.

This may not be the final partitioning of H – for satisfying C7 we may have to interchange some vertices of L_0 with vertices which are assigned to non-exceptional clusters of G_r. When all the requirements of C1–C9 will be satisfied, the actual embedding can be done by the help of the modified Blow-up lemma.

4.1 Assigning A

We start by assigning the vertices of A to the non-exceptional clusters of G_r. For every vertex $x \in A$ choose a non-exceptional cluster randomly and independently. It is easy to see that this procedure will guarantee an almost even distribution of the vertices of A among the clusters of G_r. Let A_i, $1 \le i \le \ell$ denote the set of vertices assigned to V_i after distributing the vertices of A using the above procedure.

Lemma 16 *With high probability* $|A_i| = \frac{|A|}{\ell} \pm o(n)$.

Proof Applying Chebyshev's inequality gives the proof of the lemma. □

Let $B' \subset B$ be a maximal set in which any two vertices are of distance at least 4 from each other. (Note that $|B'|/|B|$ depends on Δ, but not on ε or d.) Let us cut B' randomly into three parts of equal size: $B' = B'_1 \cup B'_2 \cup B'_3$. In Section 4.3.2., a subset of the vertices in B'_1 will be assigned to L_0, i.e., they will be mapped to the exceptional set V_0 of G. We will choose the buffer vertices from B'_3 for satisfying condition C5 of the Blow-up Lemma in Section 4.3.3. The vertices of B'_2 will be used in Section 4.3.4 to satisfy conditions C8 and C9 concerning the exceptional sets E_i in the Blow-up lemma.

Now we will argue that an appropriate distribution of A among the clusters of G_r will facilitate an even assignment of the vertices of B'_1, B'_2, B'_3 and $B - B'$ to the clusters of G_r. Let V_i be a cluster in G_r, we define the *associated list* $Q(V_i)$ as $\{y : y \in B,\ x \in A_i, (x,y) \in E(H)\}$, which is the subset of vertices of B with a neighbor assigned to the cluster V_i. Let $V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}$ be any Δ clusters of G_r. We define the random variables R, R_1, R_2 and R_3: $R_i(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) = |B'_i \cap Q(V_{s_1}) \cap Q(V_{s_2}) \cap \dots \cap Q(V_{s_\Delta})|$ for $i = 1, 2, 3$, and $R(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) = |(B - B') \cap Q(V_{s_1}) \cap Q(V_{s_2}) \cap \dots \cap Q(V_{s_\Delta})|$.

We are going to measure the evenness of the distribution of A in terms of these random variables.

Lemma 17 *For any Δ clusters $V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}$ of G_r the following inequalities hold:*

$$\Pr\left[|R(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) - \mathrm{E}[R(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta})]| = \Omega(n^{\frac{4}{5}})\right] = o(1),$$

and for $i = 1, 2, 3$

$$\Pr\left[|R_i(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) - \mathrm{E}[R_i(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta})]| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

Proof Similar to the proof of Lemma 16, again we omit the details. □

We need the following simple corollary of the above lemmas.

Corollary 18 *For any two Δ-tuples of clusters $V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}$ and $V_{t_1}, V_{t_2}, \dots, V_{t_\Delta}$ in G_r the following inequalities hold:*

$$\Pr\left[|R(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) - R(V_{t_1}, V_{t_2}, \dots, V_{t_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1),$$

and for $i = 1, 2, 3$

$$\Pr\left[|R_i(V_{s_1}, V_{s_2}, \dots, V_{s_\Delta}) - R_i(V_{t_1}, V_{t_2}, \dots, V_{t_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

Let N be a positive integer, depending only on ε. For all r $(1 \leq r \leq \ell)$ we randomly divide $B'_1 \cap Q(V_r)$ into N subsets of equal size resulting $Q_1(V_r), Q_2(V_r), \ldots, Q_N(V_r)$. We define a new set of random variables: $R_1^p(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) = |Q^p(V_{s_1}) \cap Q^p(V_{s_2}) \cap \ldots \cap Q^p(V_{s_\Delta})|$, for all $1 \leq p \leq N$. Then the following is also implied by Lemma 16 and 17:

Corollary 19 *For any two Δ-tuples of clusters $V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}$ and $V_{t_1}, V_{t_2}, \ldots, V_{t_\Delta}$ in G_r and two integers p and q $(1 \leq p, q \leq N)$ the following inequalities hold:*

$$\Pr\left[|R_1^p(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - R_1^q(V_{t_1}, V_{t_2}, \ldots, V_{t_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

4.2 Assigning the vertices of B

In this section we will present a consistent assignment of the vertices in B to the clusters of G_r. As we will see, such assignments can be formulated as special matching problems. (In order to finish the embedding of H in G, some vertices of H should be assigned to the exceptional cluster V_0. This will be carried out in another section.)

We repeat the definitions of [8]. For a bipartite graph $J = (V, T, E(J))$ where $|T| = q|V|$ for some positive integer q, $M \subset E(J)$ is a *proportional matching* if every $v \in V$ is adjacent to exactly q vertices in T and every $u \in T$ is adjacent to exactly one $v \in V$ in M. In order to show that J contains a proportional matching we will check the König–Hall conditions, that is, for every subset U of V, its neighborhood in T should satisfy $|N_J(U, T)| \geq q|U|$. One can easily see this from the construction of an auxiliary graph: substitute every $v \in V$ with q instances $v_1, \ldots, v_q$, and if (v, u) $(u \in T)$ was an edge, then connect the v_is to u for all $1 \leq i \leq q$. This auxiliary graph has a perfect matching if and only if J has a proportional matching.

Besides this kind of matching we are going to need another kind of matching about which we demand that the "loads of the vertices" are distributed more evenly. We say that J *allows a strong proportional matching with respect to* μ $(0 < \mu \ll 1)$ if there is a proportional matching in the following bipartite graph J': Its color classes are V and T'. Set $\ell = |V|$ and for every vertex $u \in T$, add $\frac{\ell}{\mu}$ copies, $u_1, \ldots, u_{\frac{\ell}{\mu}}$, to T'. If $N_J(u) = \{v_1, \ldots, v_s\}$ then we will have the following edges: (u_i, v_i) for $1 \leq i \leq s$, and (u_j, v_i) where $1 \leq i \leq s$ and $s < j \leq \frac{\ell}{\mu}$. In other words, the first s copies of u have degree 1, while the others have the same degree, s.

Now we have the following conditions: for $U \subset V$ we need $|U|/|V| \leq |N_{J'}(U)|/|T'|$. For proving the existence of a strong proportional matching we will use that $|N_J(U)|(1-\mu)/|T| \leq |N_{J'}(U)|/|T'|$ for $U \subset V$ with $|U| \leq (1-\mu)|V|$. So, for such a set U we require that $|N_J(U)|(1-\mu) \geq q|U|$ – these are the *strong König-Hall conditions*. These conditions and additional ideas will help us proving the existence of a strong proportional matching.

We will assign the vertices of B to clusters of G_r by the help of the above two kind of matchings.

Recall from Corollary 4 that G_r is a graph on ℓ vertices with $\delta(G_r) \geq (1 - \frac{1}{\Delta+1})(1-\theta)(1-\beta)\ell$, where $0 < \theta = 2\varepsilon + d \ll 1$ and $0 < \beta < 1$ are two constants, where β will be chosen to be sufficiently small. We will denote $\delta(G_r)/\ell$ by δ, so

$$\delta \geq \frac{\Delta}{\Delta+1}(1-\theta)(1-\beta).$$

Let us construct a bipartite graph $P = (V(G_r), T, E(P))$. One color class is $V(G_r)$ (the non-exceptional clusters), the other, T is the set of all possible Δ–tuples composed of different clusters of G_r. It is easy to see, that $|T| = \binom{\ell}{\Delta}$ and $q = \binom{\ell}{\Delta}/\ell$. There is an edge between $V_j \in V(G_r)$ and a Δ–tuple $t = (V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta})$ iff $(V_j, V_{s_i}) \in E(G_r)$ for every $1 \leq i \leq \Delta$.

Let $d \ll \mu \ll 1$, and denote $(1-\theta)(1-\beta)(1-\mu)$ by $(1-\nu)$ (here μ is the constant for the strong proportional matching). Observe, that $0 < \nu \ll 1$. This time $|T'| = |T|\ell/\mu$ and $|T'|/|V| = \binom{\ell}{\Delta}/\mu$. Having defined μ, we can construct the graph P' analogous to J' as well.

The existence of a strong proportional matching in P' will be crucial in the proof of Lemma 23. It ensures that if G_r had an edge between x and y, then (some copy of) this edge will be involved in the strong proportional matching.

Lemma 20 *P has a proportional matching and if ν is small enough, then P' has a strong proportional matching with respect to μ.*

For proving Lemma 20 we will need the following statement.

Lemma 21 *For $0 \le i \le \Delta - 2$ if ν is small enough, then $\delta^{\Delta-i}(1-\mu) > (i+1)(1-\delta)$.*

Proof [Proof of Lemma 21]

Notice, that

$$\delta^j(1-\mu) \ge \left(\frac{\Delta}{\Delta+1}\right)^j (1-\theta)^j(1-\beta)^j(1-\mu) > \left(\frac{\Delta}{\Delta+1}\right)^j (1-\nu)^j.$$

First we are going to show, that $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$. We proceed by a backward induction on i. We start with the case $i = \Delta - 2$:

$$\left(\frac{\Delta}{\Delta+1}\right)^2 (1-\nu)^{\Delta} > \frac{\Delta-1}{\Delta+1}(1+\nu\Delta),$$

since by multiplying both sides by $\frac{\Delta+1}{\Delta}$ if ν is small enough we get the true inequality

$$\frac{\Delta}{\Delta+1}(1-\nu)^{\Delta} > \frac{\Delta-1}{\Delta}(1+\nu\Delta).$$

So now we may assume that $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$. Decreasing i by 1 we have to check the inequality below:

$$\left(\frac{\Delta}{\Delta+1}\right)^{\Delta-i+1} (1-\nu)^{\Delta} > \frac{i}{\Delta+1}(1+\nu\Delta).$$

Multiplying both sides by $\frac{\Delta+1}{\Delta}$ we get the inequality

$$\left(\frac{\Delta}{\Delta+1}\right)^{\Delta-i} (1-\nu)^{\Delta} > \frac{i}{\Delta}(1+\nu\Delta).$$

Now since $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$, and the latter is larger than $\frac{i}{\Delta}(1+\nu\Delta)$ for $i < \Delta$, for proving the lemma it is enough to show that $\frac{i+1}{\Delta+1}(1+\nu\Delta) > (i+1)(1-\delta)$.

Dividing by $i+1$ and multiplying by $\Delta+1$ (and recalling that $\delta \ge \frac{\Delta}{\Delta+1}(1-\theta)(1-\beta)$), we get

$$1+\nu\Delta > \Delta+1-\Delta(1-\theta)(1-\beta) = 1+\widehat{\nu}\Delta,$$

where $1-\widehat{\nu} = (1-\theta)(1-\beta) > (1-\theta)(1-\beta)(1-\mu) = (1-\nu)$, hence, $\nu > \widehat{\nu}$. From this the lemma follows. $\square$

Remark 4 *Assume that we have a simple graph on n vertices with minimum degree at least $\frac{s-1}{s}n$, where $s \ge 2$ is an integer. It is easy to see that any t vertices (for $1 \le t \le s$) have at least $\frac{s-t}{s}n$ common neighbors. While this is a trivial observation, it will be very useful in the proof of Lemma 20.*

Let $0 < c < 1$, γ is a positive real number and k be a fixed positive integer, then $\gamma\binom{cn}{k} \to \gamma c^k\binom{n}{k} + o(\binom{n}{k})$ if $n \to \infty$. We introduce the following notation: if n is large enough, then we will write $[\gamma c^k\binom{n}{k}]^-$ instead of $\gamma\binom{cn}{k}$. This will allow us a somewhat shorter exposition of the proof of Lemma 20.

Now we can start proving the lemma.

Proof [Proof of Lemma 20] We will check the strong König–Hall conditions.

- Let $V_i \in V(G_r)$ be an arbitrary cluster. Then $|N_P(V_i)| \geq \binom{\delta\ell}{\Delta}$, and $|N_{P'}(V_i)| \geq (1-\mu)\binom{\delta\ell}{\Delta}|T'|/|T| = [\delta^\Delta(1-\mu)|T'|]^-$, therefore it is larger than $(1-\delta)|T'|$ by Lemma 21.

- Let $U_i \subset V(G_r)$ be a set of size greater than $i(1-\delta)\ell$ for some $1 \leq i \leq \Delta - 2$. From the minimum degree condition of G_r every i vertices will have a common neighbor in U_i. Thus $|N_P(U_i)| \geq [\delta^{\Delta-i}|T|]^-$ and $|N_{P'}(U_i)| \geq [\delta^{\Delta-i}(1-\mu)|T'|]^-$, and by Lemma 21 the latter is at least $(i+1)(1-\delta)|T'|$, therefore $|N_{P'}(U_i)| > (i+1)(1-\delta)|T'|$. Note that the above argument applied for each $i \leq \Delta - 2$ means that the (strong) König–Hall conditions are satisfied for all sets U of size at most $(\Delta-1)(1-\delta)\ell$.

- Assume that $U \subset V(G_r)$ with $|U| = (\Delta-1)(1-\delta)\ell$. Then every $\Delta - 1$ vertices will have a common neighbor in U by the minimum degree condition of G_r. Thus, $|N_P(U)| \geq \delta|T|$ and $|N_{P'}(U)| \geq \delta(1-\mu)|T|$.

- Assume that $U \subset V(G_r)$ with $|U| = \delta(1-\mu)\ell$. Now we want to show, that $|N_P(U)|(1-\mu) > \delta|T|$, implying that $|N_{P'}(U)| \geq \delta|T'|$. We will estimate the number of $(\Delta-1)$–tuples having strictly more than $\frac{1}{\Delta+1}\ell$ neighbors in U. If it is not a negligible proportion, then we will see that we are done, since such a $(\Delta-1)$-tuple with any other vertex gives a Δ-tuple, which is connected to U by the minimum degree condition.

  Denote Z the set of all possible $(\Delta-1)$–tuples composed of different clusters of G_r. It is easy to see that there are at least $\delta(1-\mu)\ell[\delta^{\Delta-1}\binom{\ell}{\Delta-1}]^-$ edges going in between U and Z. We divide Z into two parts, Z_1 and Z_2: in Z_1 all the tuples have at most $(1-\delta)\ell$ neighbors in U, while the tuples in Z_2 have more than $(1-\delta)\ell$ neighbors in U.

  Denote $\frac{|Z_1|}{|Z|}$ by x and consider the following inequality:

$$(x(1-\delta) + (1-x)\delta(1-\mu))\ell\binom{\ell}{\Delta-1} \geq \delta(1-\mu)\ell\left[\delta^{\Delta-1}\binom{\ell}{\Delta-1}\right]^-.$$

  On the right hand side of this inequality we have a lower bound on the number of edges between U and Z, while the left is clearly an upper bound for that. We want to get a good estimation for x, for this reason first we consider a simplified version of the inequality (with ξ instead of x, and assuming that $\nu = 0$):

$$\frac{\xi}{\Delta+1} + (1-\xi)\frac{\Delta}{\Delta+1} \geq \left(\frac{\Delta}{\Delta+1}\right)^\Delta.$$

  From this it follows that

$$\xi \leq \frac{\frac{\Delta}{\Delta+1} - (\frac{\Delta}{\Delta+1})^\Delta}{\frac{\Delta-1}{\Delta+1}}.$$

For $\Delta = 2$ simple calculation gives that $\xi \leq 2/3$. It is easy to see that if we increase Δ the upper bound for ξ will decrease, therefore, $\xi \leq 2/3$ for every $\Delta \geq 2$. Since our assumption was that ν is sufficiently small, we get that $|Z_1|/|Z| = x \leq 0.7$ for every $\Delta \geq 2$.

Call a Δ-tuple τ *bad*, if $\tau \in \overline{N_P(U)}$. Let us consider the following 0-1 matrix Mtx: its rows are indexed by the elements of Z, the columns are indexed by the elements of $V(G_r)$. The (i, j)th entry of Mtx is 1 iff the union of the ith $(\Delta - 1)$-tuple and the jth vertex is a bad Δ-tuple. Clearly, no row of Mtx contains more than $(1 - \delta)\ell$ 1's, and if a $(\Delta - 1)$-tuple is in Z_2, then every entry of the corresponding row is 0.

Therefore, we can give an upper bound on the total number of 1's in Mtx:

$$0.7(1 - \delta)\ell\binom{\ell}{\Delta - 1}.$$

Observe, that if $\tau = \{V_{j_1}, V_{j_2}, \ldots, V_{j_\Delta}\}$ is a bad Δ-tuple, then every entry of Mtx of the form $(\{V_{j_1}, \ldots, V_{j_{i-1}}, V_{j_{i+1}}, \ldots, V_{j_\Delta}\}, V_{j_i})$ should have value 1. Hence,

$$0.7(1 - \delta)\frac{\ell}{\Delta}\binom{\ell}{\Delta - 1} \geq 0.7(1 - \delta)\binom{\ell}{\Delta}$$

is an upper bound for the number of bad Δ-tuples.

This implies that $|N_P(U)|(1 - \mu) > \delta|T|$.

- Assume that $U \subset V(G_r)$ with $|U| > \delta\ell$. Now every Δ-tuple will have a neighbor in U, except those having only one neighbor out of U. Observe, that this is enough for the existence of a proportional matching in P: every Δ-tuple of $|T|$ have at least $\delta\ell$ neighbors in V, therefore, the König –Hall conditions for the proportional matching are satisfied.

- For proving the existence of a strong proportional matching in P' assume that $U \subset V(G_r)$ with $|U| = (1 - \omega)\ell$ $(0 < \omega < \mu)$. Clearly, there are at most $\ell\binom{\ell}{\Delta}$ such Δ-tuples in T', which have degree one. Denote the set of these tuples by T_o, and let $T_m = T' - T_o$. We have, that $|T_m| \geq (\ell/\mu - \ell)\binom{\ell}{\Delta}$.

  Previously we observed, that every Δ-tuple of T has a neighbor in U. This implies, that every tuple of T_m has a neighbor in U. Besides, every $v \in U$ has a neighbor in T_o. Hence,

$$|N_{P'}(U)|/|T'| \geq \frac{|T_m| + |T_o| - \omega\ell\binom{\ell}{\Delta}}{|T'|} = 1 - \frac{\omega\ell\binom{\ell}{\Delta}}{\ell\binom{\ell}{\Delta}/\mu} = 1 - \omega\mu > 1 - \omega.$$

We get that $|N_{P'}(U)|/|T'| \geq |U|/|V|$ for every nonempty $U \subset V$, thus, P allows a strong proportional matching with respect to μ. □

We are ready to present the procedure for assigning the vertices of B to clusters of G_r. We start with the vertices in $B - B'_1$ (recall that B'_1 was defined after Lemma 16). First let $L_i = A_i$ for $1 \leq i \leq \ell$ (L_i is the set of vertices to be mapped to V_i by the help of Lemma 5). Assume that $\mathcal{M}$ denotes the (ordinary) proportional matching provided by Lemma 20 with respect to the graph P, and $\mathcal{M}'$ is the strong proportional matching.

For a cluster V_t, let $\{V_{i_1}, \ldots, V_{i_\Delta}\}$ be one of the Δ-tuples matched to it in $\mathcal{M}$. We will assign the vertices of $(B - B'_1) \cap Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})$ to the cluster V_t by adding them to the set L_t. We will repeat this for all the Δ–tuples which are matched to V_t, and carry this out for every $1 \leq t \leq \ell$.

By the virtue of Lemma 18 and its corollaries, $|(B - B'_1) \cap Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})|$ is almost the same for all choices of Δ-tuples, which in turn implies that the set L_t for all $V_t \in V(G_r)$ will have almost the same size after the distribution of $B - B'_1$. Also, note that the construction of P and the structure of the proportional matching $\mathcal{M}$ implies that if $x \in B - B'_1$ is assigned to L_t then the vertices in $N_H(x)$ are assigned to neighboring clusters of V_t.

The vertices of B'_1 will be mapped by the help of the strong proportional matching $\mathcal{M}'$, in the same way as we did for $B - B'_1$. The only difference is that since every Δ-tuple $V_{i_1}, \ldots, V_{i_\Delta}$ has $\frac{\ell}{\mu}$ copies, the elements of $Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})$ will be distributed randomly among these copies. Now assume that for some cluster V_s the rth copy of the Δ-tuple $\{V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta}\}$ is matched to it in $\mathcal{M}'$. We will assign the vertices of $B'_1 \cap Q_r(V_{i_1}) \cap Q_r(V_{i_1}) \cap \ldots \cap Q_r(V_{i_\Delta})$ to V_s by adding them to the set L_s. As above, adjacent vertices in H are assigned to adjacent clusters in G_r. It is also easy to see that the strong proportional matching assigns the vertices of B'_1 evenly - we refer to Corollary 19.

Observe that there are other cases to consider. Since the vertices of A were distributed randomly among the clusters of G_r, some vertices in B can have all their neighbors assigned to $\Delta - 1$ (or less) clusters. In fact, as one can easily calculate we will have about

$$\left(1 - \frac{\ell(\ell-1)\ldots(\ell-\Delta+1)}{\ell^\Delta}\right)|B|$$

vertices of B with the above property. Hence, there are other cases of matchings to consider. When the clusters in $V(G_r)$ has to be matched to $(\Delta - i)$-tuples for $1 \le i \le \Delta - 1$, then we construct the corresponding bipartite graph P_i, and then look for the proportional matching. Here $P_i = (V(G_r), T_i, E(P_i))$. One color class is $V(G_r)$, the other, T_i is the set of all possible $(\Delta - i)$–tuples composed of different clusters of G_r. It is easy to see, that $|T_i| = \binom{\ell}{\Delta - i}$. There is an edge between $V_j \in V(G_r)$ and a $(\Delta - i)$–tuple $\tau = (V_{s_1}, V_{s_2}, \ldots, V_{s_{\Delta-i}})$ iff $(V_j, V_{s_i}) \in E(G_r)$ for every $1 \le i \le \Delta - i$.

It is easy to see that if there is a proportional matching in P, then we can find the proportional matching in P_i for $1 \le i \le \Delta - 1$: one can check that the proof of Lemma 20 works for these graphs as well. Then mapping such vertices in B can be done in a similar way as we did for those which have their neghbors assigned to Δ different clusters.

4.3 Finishing the assignment

Now we have to make sure that all conditions of Lemma 5 are satisfied. Obviously, some of them are violated at this moment. E.g. C1, since so far we have not mapped any vertex to V_0 (L_0 is empty). We will take care of these problems in separate subsections.

4.3.1 Bad vertices in G

The edges in G_r are ε-regular pairs of G', hence, in a cluster of such a pair some vertices may have just a small number of neighbors in the other cluster (this number can be even zero). To avoid problems which can be caused by this, we are going to discard some vertices from the clusters and put them into V_0. With this procedure below we prepare for satisfying C8 and C9.

Let $\mathcal{M}$ be the matching provided by Lemma 20. For a cluster $V_i \in V(G_r)$ let $\mathcal{T}_i$ denote the set of Δ-tuples matched to V_i in $\mathcal{M}$ for every $1 \le i \le \ell$. We say that $v \in V(G) - V_0$ has η-small degree to a Δ-tuple, if v has less than $(d - \eta)m$ neighbors in one of the clusters composing that tuple. Let us call a vertex $v \in V_i$ *η-bad*, if v has η-small degree to at least half of the Δ-tuples in $\mathcal{T}_i$.

Lemma 22 *By removing $2\Delta\varepsilon m$ vertices from every non-exceptional cluster of G' we can achieve that no $(3\Delta\varepsilon)$-bad vertices will remain in them.*

Proof First we show, that no cluster can contain more than $2\Delta\varepsilon m$ vertices which are ε-bad. For an arbitrary cluster $V_i \in V(G_r)$ which is matched to the Δ-tuples of $\mathcal{T}_i$, let $\{v_1, \ldots, v_s\}$ denote the set of ε-bad vertices of V_i. If $s > 2\Delta\varepsilon m$ then there should be a tuple $\tau \in \mathcal{T}_i$ to which more than $\Delta\varepsilon m$ vertices of V_i have ε-small degree. Thus for one of the clusters of this tuple there are more than εm vertices with degree less than $(d - \varepsilon)m$, which contradicts the ε-regularity condition.

Applying the above, by removing $2\Delta\varepsilon m$ vertices from every non-exceptional cluster (including all the ε-bad vertices), we can guarantee that all remaining vertices of the non-exceptional clusters have big degrees –at least $(d - 3\Delta\varepsilon)m$– to at least half of the matched tuples in $\mathcal{M}$, and overall $2\Delta\varepsilon n$ bad vertices are added to V_0. □

Remark 5 *After finishing the above procedure the edges of G_r represent $(3\Delta\varepsilon)$-regular pairs with density $d - 4\Delta^2\varepsilon^2$, and $|V_0| \leq 3\Delta\varepsilon n$.*

4.3.2 Selecting the vertices of L_0

At this point every cluster has a surplus, that is, more vertices of H are assigned to them than the clustersize m: $m = \frac{n-|V_0|}{\ell} < \frac{n}{\ell} \pm o(n)$.

We will form L_0 by removing a subset of vertices of B_1' from the L_i sets, achieving that $|L_i| = m$ for $1 \leq i \leq \ell$. This subset is chosen randomly for every $1 \leq i \leq \ell$, this random choice guarantees, that $|N_H(L_0) \cap L_i| \leq 2\Delta|V_0|/\ell$ with very high probability (we refer to Chernoff's bound), that is, condition C4 is satisfied.

Let $\varphi : L_0 \to V_0$ be any bijective mapping. We need to ensure that the assignment of L_0 to V_0 is consistent with $E(H)$; that is, for any $x \in L_0$, with $(x, y_1), (x, y_2), \ldots, (x, y_\Delta) \in E(H)$, if $v = \varphi(x)$, $y_1 \in L_{i_1}, y_2 \in L_{i_2}, \ldots, y_\Delta \in L_{i_\Delta}$ then $deg_G(v, V_{i_1}), deg_G(v, V_{i_2}), \ldots, deg_G(v, V_{i_\Delta})$ are all at least $c_1 m$ (condition C7).

If this condition does not hold for a pair (x, v), a *switching* will be performed. In the switching operation we first randomly and uniformly pick a cluster V_j among those, which are adjacent to $V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta}$ in G_r. Then locate a vertex x' in L_j such that if $(x', y_1'), (x', y_2'), \ldots, (x', y_\Delta') \in E(H)$ with $y_1' \in L_{i_1'}, y_2' \in L_{i_2'}, \ldots, y_\Delta' \in L_{i_\Delta'}$ then $deg_G(v, V_{i_1'}), deg_G(v, V_{i_2'}), \ldots, deg_G(v, V_{i_\Delta'})$ are all at least $c_1 m$. We will switch the roles of x and x', that is, we let $L_j = L_j + x - x'$, $L_0 = L_0 - x + x'$ and $\varphi(x') = v$. We will call x' the *switched vertex*.

We will see that such x' can always be found among those vertices assigned by the strong proportional matching. Moreover, even after performing all the necessary switching operations we will still have condition C4 satisfied.

Lemma 23 *For every $x \in L_0$ there exists an x' as required above.*

Proof It is easy to see that any $v \in V_0$ has degree less than $c_1 m$ to at most $\frac{1-\delta}{1-c_1}\ell$ clusters. Let V_j be as above. We will estimate the number of clusters $S_{j,v} \subset V(G_r)$ where $S_{j,v}$ contains those clusters of $N_{G_r}(V_j)$ in which v has at least $c_1 m$ neighbors. Clearly, $|S_{j,v}| \geq (\delta - \frac{1-\delta}{1-c_1})\ell$.

Recall, that $\delta \geq (1 - \theta)(1 - \beta)\Delta/(\Delta + 1)$. Let $c_1 = 1/\Delta^4$ and assume that $\beta \leq 1/(2\Delta^4)$. Simple calculation shows that by this choice of c_1 and β we will have the following bound for the size of $S_{j,v}$:

$$|S_{j,v}| \geq \left(1 - \frac{1}{\Delta}\right)\frac{\Delta - 1}{\Delta + 1}\ell.$$

This inequality implies that the number of Δ-tuples spanned by the clusters of $S_{j,v}$ is $\binom{|S_{j,v}|}{\Delta} \geq \binom{\ell}{\Delta}/40$ for $\Delta \geq 2$.

Recall, that P' is a bipartite graph with color classes $V(G_r)$ and T'. Since T' contains $\frac{\ell}{\mu}$ copies of every Δ-tuple, $|T'| = \frac{\ell}{\mu}\binom{\ell}{\Delta}$. Denote by ω the number of vertices which are allocated by a copy of a Δ-tuple $\tau \in T'$ in the strong proportional matching, i.e., $\omega = |B_1'|/(\frac{\ell}{\mu}\binom{\ell}{\Delta})$. If the clusters of some $\tau \in T'$ are adjacent to V_j in G_r then $(V_j, \tau) \in \mathcal{M}'$ ($\mathcal{M}'$ is the strong proportional matching with respect to μ). This implies that the number of vertices of H assigned to V_j (by $\mathcal{M}'$) by Δ-tuples of $S_{j,v}$ is at least $\frac{\mu}{40\ell}|B_1'|$ (even if we consider only Δ-tuples of degree one).

When looking for x' we first randomly pick a Δ-tuple $\tau = (V_{i_1'}, V_{i_2'}, \ldots, V_{i_\Delta'})$ such that all clusters of τ are in $S_{j,v}$ and $(\tau, V_j) \in \mathcal{M}'$. Then randomly pick a vertex $x' \in B_1' \cap L_j$ among those which were assigned to L_j by $\mathcal{M}'$ such that the neighbors of x' are assigned to the clusters of τ. Clearly, x' can be switched by x: by the above choice of x' it is mapped to a vertex v which has at least $c_1 m$ neighbors in all the clusters of τ.

Since the common neighborhood of $V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta}$ contains at least $(1-\delta)\ell$ clusters, the number of vertices assigned to them by the strong proportional matching is at least $(1-\delta)\ell\frac{\mu}{40\ell}|B_1'|$ even with the restriction that we consider only those Δ-tuples for which the corresponding vertex of V_0 has degree at least $c_1 m$. This is by far larger than $|V_0|$ (since $|B|$ is large enough and $\varepsilon \ll \mu$). Hence we can find an appropriate x' for any x easily. □

For the satisfaction of condition C4 it should be pointed out that we perform this switching procedure in such a way that the neighbors of the switched x's are scattered almost evenly in a constant proportion of the Δ-tuples, and so in a constant proportion of the clusters:

Lemma 24 *For every $1 \le s \le \ell$ we will have $|L_s \cap N_H(L_0)| \le 22\Delta^3(\Delta+1)^2 dm$ with very high probability after performing all the necessary switchings.*

Proof Observe that for a given $x \in L_0$ we randomly choose the cluster V_j out of a set of size at least Λ, where $\Lambda = (1-\delta)\Delta\ell \ge (\frac{1}{\Delta+1} - \frac{\Delta^2}{\Delta+1}(\beta+\theta) + \frac{\Delta}{\Delta+1}\beta\theta)\ell$. Therefore, with very high probability we choose V_j at most $2|V_0|/\Lambda$ times through the whole process of switching (Chernoff's bound).

If $V_s \in S_{j,v}$ then the probability that V_s will be among the clusters of the randomly chosen τ is

$$\Pr(V_s \in \tau) = \frac{\binom{|S_{j,v}|-1}{\Delta-1}}{\binom{|S_{j,v}|}{\Delta}} = \frac{\Delta}{|S_{j,v}|} \le \frac{\Delta^2(\Delta+1)}{(\Delta-1)^2\ell}.$$

Hence, for a given j the switched vertices from L_j will have at most $\frac{2|V_0|}{\Lambda}\frac{2\Delta^2(\Delta+1)}{(\Delta-1)^2\ell}$ neighbors in L_s with very high probability (again, apply Chernoff's bound). Summing this up for all j ($j = 1, 2, \ldots, \ell$) we get that

$$|L_s \cap N_H(L_0)| \le \frac{2|V_0|}{\ell} + \frac{5|V_0|\Delta^2(\Delta+1)^2}{(\Delta-1)^2\ell}.$$

Here $\frac{2|V_0|}{\ell}$ is the bound before starting the switching, and we substituted the lower bound for Λ. This is at most $\frac{7|V_0|}{\ell}\Delta^2(\Delta+1)^2 \le 22\Delta^3(\Delta+1)^2\varepsilon m$, because $|V_0| \le 3\Delta\varepsilon n$. Since $\varepsilon \ll d$, we get the required bound. □

4.3.3 Selecting the buffer vertices: condition C5

In this section we will determine $Bf \subset B_3'$, the set of buffer vertices so as to satisfy condition C5 of the Blow-up lemma. First we discard those vertices from B_3' which do not have their neighbors in Δ different L_i-sets. Recall, that we estimated the number of such vertices after the proof of Lemma 20.

For a Δ-tuple $(V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta})$ consider the set $B_3' \cap Q(V_{i_1}) \cap Q(V_{i_2}) \cap \ldots \cap Q(V_{i_\Delta})$. We pick

$$\xi = \delta' m \frac{\ell}{\binom{\ell}{\Delta}}$$

vertices from this set and place them to Bf. We perform the above procedure for every Δ-tuple. Recall, that the degree of a cluster of $V(G_r)$ is $\binom{\ell}{\Delta}/\ell$ in the proportional matching $\mathcal{M}$. Hence, $Bf_i = Bf \cap L_i$ will have size $\delta' m$. Moreover,

$$|N_H(Bf) \cap L_i| = \Delta\delta' m$$

for every $1 \le i \le \ell$, since each cluster of $V(G_r)$ appears in $\Delta\binom{\ell}{\Delta}/\ell$ different Δ-tuples. This shows that by the above choice of the buffer vertices we can satisfy condition C5 of the Blow-up Lemma.

4.3.4 Satisfying conditions C8 and C9

In the rest of the proof we are going to show that we will be able to find vertices from H according to C8 and C9 so as to cover the exceptional vertices of G in Step 2.3 of the embedding algorithm. First we need a simple lemma on special subgraphs of bipartite graphs.

Lemma 25 *Let $U = U(S, T, E(S, T))$ be a bipartite graph with $r = |T| = 2|S|$. Furthermore, assume, that $deg(s) \geq \frac{r}{2}$ for every $s \in S$. Then we can find $\frac{r}{2}$ independent edges in U.*

Proof Trivial. $\square$

In what follows $\tilde{\varepsilon}$ will denote $3\Delta\varepsilon$. We find the F_i sets of conditions C8 and C9:

Lemma 26 *Given arbitrary sets $E_i \subset V_i$ such that $|E_i| \leq \varepsilon'' m$ we can find the sets $F_i \subset L_i \cap B_2'$ and bijective mappings $\psi_i : F_i \to E_i$ for every $1 \leq i \leq \ell$ such that the following holds:*
(1) if $(x, y) \in E(H)$ with $x = \psi_i^{-1}(v)$ and $y \in L_j$ then $deg_G(v, V_j) \geq (d - \tilde{\varepsilon})m$,
(2) for $F = \cup F_i$ we will have $|N_H(F) \cap L_i| \leq 2\Delta\varepsilon'' m$.

Proof As before, $\mathcal{T}_i$ denotes the set of Δ-tuples matched to V_i in $\mathcal{M}$ for every $1 \leq i \leq \ell$. Recall that we removed the $3\Delta\varepsilon$-bad vertices from every cluster (Lemma 22) and put them into V_0. Hence, by the definition of "bad" all the vertices of a non-exceptional cluster V_i have degree more than $(d - \tilde{\varepsilon})m$ to at least half of the tuples in $\mathcal{T}_i$.

Denoting the exceptional vertices of G in the ith cluster ($1 \leq i \leq \ell$) by E_i, we are looking for the sets $F_i \subset (L_i \cap B_2')$ and a mapping $\psi_i : F_i \to E_i$ so as to satisfy the conditions of the lemma. We will present a simple algorithm for finding $F_1, F_2, \ldots, F_\ell$ and $\psi_1, \psi_2, \ldots, \psi_\ell$.

First we *mark* the vertices of B_2'. As before, let us denote $\frac{\binom{\ell}{\Delta}}{\ell}$ by q, for simplicity we assume that q is an even integer. Observe, that $|\mathcal{T}_i| = q$ for every $1 \leq i \leq \ell$. Since we can handle every F_i and ψ_i in the same way for every $1 \leq i \leq \ell$, we give the details only for the case of F_1 and ψ_1.

We begin with a partitioning of E_1: $E_1 = E_{1,1} \cup E_{1,2} \cup \ldots E_{1,t-1} \cup E_{1,t}$, here $t = \lceil 2|E_1|/q \rceil$. These sets are disjoint, and we require that $|E_{1,1}| = |E_{1,2}| = \ldots = |E_{1,t-1}| = q/2$, and for the last set, $|E_{1,t}| \leq q/2$.

We define a set of auxiliary bipartite graphs $\{U_j\}_{j=1}^t$ as follows. $U_j = U_j(E_{1,j}, \mathcal{T}_1, E(E_{1,j}, \mathcal{T}_1))$, and $(v, \tau) \in E(E_{1,j}, \mathcal{T}_1)$, if v has degree at least $(d - \tilde{\varepsilon})m$ to every cluster of τ. By Lemma 25, we can find $q/2$ independent edges in every U_j.

Now we discuss the algorithm for finding F_1 and ψ_1. Let $E_{1,1} = \{v_1, v_2, \ldots, v_{q/2}\}$, and assume, that $\{(v_1, \tau_1), (v_2, \tau_2), \ldots, (v_{q/2}, \tau_{q/2})\}$ is the set of the $q/2$ independent edges of U_1. Consider $\tau_1 = (V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta})$. We will pick an arbitrary marked vertex $x \in L_1 \cap B_2' \cap Q(V_{s_1}) \cap Q(V_{s_2}) \cap \ldots \cap Q(V_{s_\Delta})$. Put x in F_1, and let $\psi_1(x) = v_1$. Unmark x, and continue this process with the other vertices of $E_{1,1}$. By the time we have finished with the vertices of $E_{1,1}$, we have found $q/2$ vertices of F_1, and the neighbors of these vertices of F_1 can be found in $q/2$ different Δ-tuples of $\mathcal{T}_1$. Then we go on this way with the vertices of the rest of E_1, and finally achieve, that if τ is a Δ-tuple in $\mathcal{T}_1$, then we use τ at most $\lceil 2|E_1|/q \rceil$ times when determining F_1.

We repeat this algorithm with the vertices of $E_2, \ldots, E_\ell$ as well. At the end we have found $F = \cup_{i=1}^{\ell} F_i$ such that (1) and (2) of the lemma are satisfied.

We will prove, that by the help of this process F is such, that $|N_H(F) \cap L_i| \leq 2\Delta\varepsilon'' m$ for every $1 \leq i \leq \ell$. Pick an arbitrary non-exceptional cluster V_j. Overall it is contained in $q\Delta$ $(= \binom{\ell-1}{\Delta-1})$ different Δ-tuples. Set $s_i = |\{\tau \in \mathcal{T}_i \text{ and } V_j \in \tau\}|$ for every $1 \leq i \leq \ell$. Clearly, $\sum s_i = \binom{\ell}{\Delta-1} = q\Delta$. We calculate the number of neighbors of F which are assigned to V_j: this is at most

$$\sum_{i=1}^{\ell} s_i \left\lceil \frac{2|E_i|}{q} \right\rceil \leq \frac{q\Delta 2\varepsilon'' m}{q} = 2\Delta\varepsilon'' m.$$

V_j was arbitrary, therefore the above bound is valid for all non-exceptional clusters.

$\square$

Proof [Proof of Theorem 2] We start with an ε as small that even $\tilde{\varepsilon} \ll \varepsilon'$ is true. Since by the help of Lemmas 16, 17 and Corollaries 18, 19 we can distribute the vertices of A, and then by Lemmas 20, 22, 23, 24 and 26 we can provide that all conditions of Lemma 5 are satisfied with high probability, we can embed H in G, and thus Theorem 2 is proved. $\square$

5 Concluding remarks

The case $\Delta(H) = 1$ of Conjecture 1 is easily seen to be tight. On the other hand, it is interesting that there is a bipartite graph H with $\Delta = 1$ (recall, that Δ is the minimum of the maximum degrees of the two color classes) and $\Delta(H) > 1$ which is harder to embed: A simple example shows that for containing such a graph, $\delta(G) = \frac{n-1}{2}$ is not sufficient. Let $n = 2q$ be an even number, and $G = K_{q,q}$. Let H be the collection of $q - 2$ independent edges, and a $K_{1,3}$. Trivially, $H \not\subset G$, although, $\delta(G) = \frac{n}{2}$ and $\Delta = 1$. Theorem 2 shows, that such examples exist only in case $\Delta = 1$.

Given an arbitrary fixed $0 < \delta < 1$ it is possible to come up with bipartite graphs which cannot be embedded in a certain graph G with $\frac{\delta(G)}{|V(G)|} \geq \delta$. We sketch a (standard probabilistic) proof of this fact. Let $H = H(A, B)$ be a random bipartite graph with $|A| = |B| = m(= n/2)$, which is the disjoint union of k randomly and independently chosen 1-factors, after leaving only one copy of the parallel edges. Then $\Delta = \Delta(H) \leq k$. Let $A' \subset A$ and $B' \subset B$ with $|A'| = am$ and $|B'| = bm$, then the probability that H has no edge going in between A' and B' is at most $(1 - a)^{bmk}$.

Let r be an *odd* positive integer, such that $\frac{r-1}{r} > \delta$, and G be the complete r-partite graph on n vertices with equal color classes (for simplicity we assume, that n is divisible by r). Clearly, $\delta(G) = \frac{r-1}{r}n$. It is a routine exercise to show that whenever we want to embed H in G, there will be at least one color class of G having at least $\frac{m}{r(r+1)}$ vertices from both color classes of H, because r is odd. Let $a = b = \frac{1}{r(r+1)}$. Then if k is a large enough constant (it depends only on r), we will have that $(1 - a)^{bmk} \ll 2^{-n}$. That is, with positive probability, there will be an edge between any $A' \subset B$ and $B' \subset B$, if $|A'|, |B'| \geq \frac{m}{r(r+1)}$. Hence, $H \not\subset G$.

We have made no attempt to determine the function $\beta = \beta(\Delta)$. Simple but tedious calculation shows that $\beta = \frac{1}{\Delta^5}$ is small enough to guarantee the existence of the matchings in Lemma 20, and thus H can be embedded. We don't think this is best possible, but it is clear, that $\beta = \beta(\Delta) \to 0$, if $\Delta \to \infty$: as we just proved above, $\delta(G)$ has to be a monotone increasing function of Δ which converges to 1, therefore, β cannot have a positive lower bound.

We presented a proof of the Bollobás–Eldridge conjecture for bipartite graphs of bounded degree. We heavily used the fact that the graph to be embedded is bipartite. The conjecture for non-bipartite graphs seems to be much harder, there are only partial results (see [1], [2], [7] and [8]).

Theorem 2 suggests that the chromatic number is an important parameter even if we embed expander graphs. We propose the following conjecture: Let H and G be two simple graphs of order n. If $\chi(H) \leq \Delta(H)$, then there exists $\beta = \beta(\chi(H), \Delta(H)) > 0$ such that $\delta(G) \geq \frac{\Delta(H)}{\Delta(H)+1}(1 - \beta)n$ is sufficiently large to guarantee $H \subset G$. We could prove this for $\chi(H),\ \Delta(H) \leq 4$, but these proofs are technically much more difficult than the present one for bipartite graphs.

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