Graph Theory – lecture 1.

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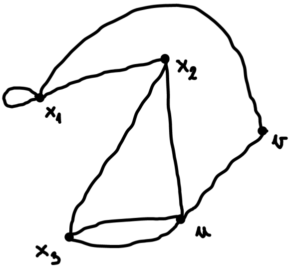
What is a (finite & undirected) graph?

Informally, a graph contains vertices (or points or nodes), and curves such that each curve connects two (not necessarily distinct) vertices. In graph theory these curves are called edges.



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One may say that a picture is just a visualization of this abstract notion (rigorous definition will follow soon), however, keep in mind that a good picture usually can tell a lot about graphs.

Rigorous definition

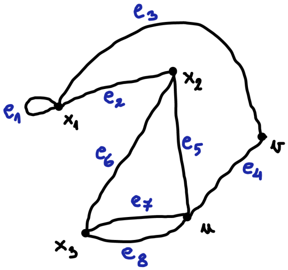
Definition

A multigraph $G = (V; E; \psi)$, where V and E are finite sets, and $\psi : E \longrightarrow \mathcal{P}_2(V)$ is a function. Here $\mathcal{P}_2(V)$ denotes the set of one or two element subsets of V. The set V is called the vertex set of G, and the set E is the edge set of G. We may write $V(G)$ and $E(G)$ if we want to indicate G in the notation and may also write $\psi(G)$. The elements of V are called the vertices, the elements of E are the edges.

Definition (continued)

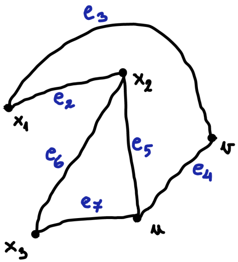
The number of vertices of G will be denoted by $v(G)$, the number of edges of G will be denoted by $e(G)$. If for some $x, y \in V$ and $e \in E$ we have that $\psi(e) = \{x, y\}$, then we say that x and y are adjacent, or that x is a neighbor of y. We also say that they are connected by e, and that x, y are incident to the edge e.

For the graph G on the picture we have $V(G) = \{x_1, x_2, x_3, u, v\}$, $E = \{e_1, e_2, \ldots, e_8\}$, and for example $\psi(e_1) = x_1$, $\psi(e_3) = \{x_1, v\}$ and $\psi(e_7) = \psi(e_8) = \{x_3, u\}$. We call e_1 a loop edge and e_7, e_8 are parallel (or multiple) edges.



We say that a graph G is simple if G has no loops or parallel edges. In such a case ψ maps every edge onto a 2-element subset of V, furthermore, no two edges are mapped onto the same 2-element subset.

Sometimes it is convenient to use the following notation in simple graphs: edge $e = uv$, where u and v are the two endpoints of e.



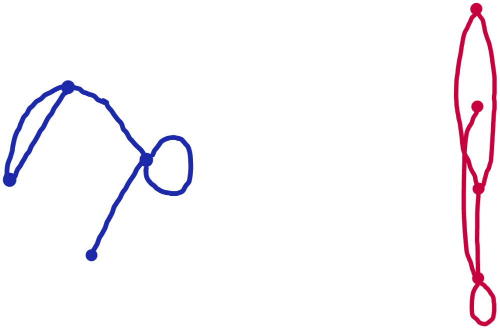
Graphs are ubiquitous in real life (and has many applications in other areas of mathematics, too). The electricity network or the road network of a country can be modelled by graphs. In chemistry, one can consider a molecule as a graph: connect two atoms in the molecule by an edge, if there is a bond between them. The water pipe system is also a graph. There are plenty of other possibilities, graphs can be used even in the social sciences (sociology, economy), etc.

Definition

Let G and H be simple graphs. We say that G and H are isomorphic, if there exist a bijection $\phi : V(G) \longrightarrow V(H)$ such that any two vertices u and v are adjacent in G if and only if the vertices $\phi(u)$ and $\phi(v)$ are adjacent in H.

One way to look at this definition: two graphs are the same (i.e., they are isomorphic) if we can obtain the second one by renaming the vertices and edges of the first one. Similar applies to the drawing of a graph.

An example for isomorphic graphs



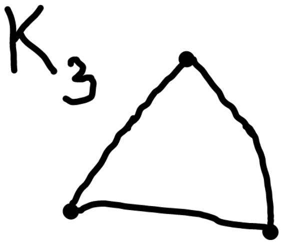
Important graphs

A simple graph is a complete graph, if any two of its vertices are adjacent. A complete graph on 2 vertices contains precisely an edge. A complete graph of 3 vertices is also called a triangle. A complete graph on n vertices is denoted by K_n.

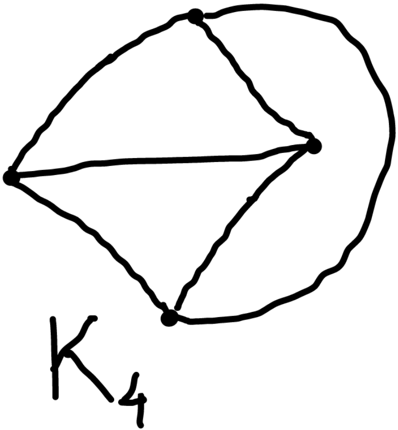
An empty graph contains only vertices and no edges.

Complete graph on 2 vertices

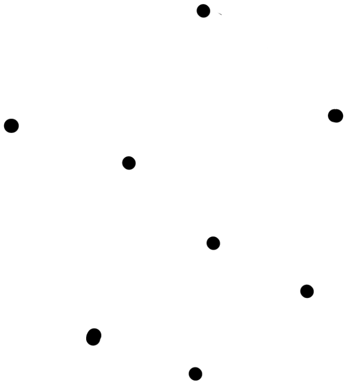
Complete graph on 3 vertices



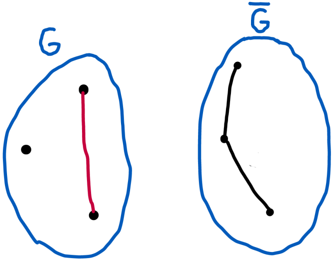
Complete graph on 4 vertices



Empty graph on 8 vertices



Given a simple graph G on n vertices its complement is the graph $\overline{G}$ with vertex set $V(G)$ which contains precisely those edges of K_n that do not belong to G.



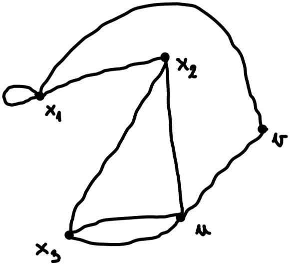
Hence, the empty graph on n vertices is just $\overline{K_n}$, the complement of the complete graph on n vertices.

Degree

Definition

Given a graph G, the degree of a vertex $x \in V(G)$ is the number of edges that are incident to x. Here loops are counted twice. The degree of x is denoted by $d_G(x)$, or just $d(x)$.

Note that in the above definition we do not assume that the graph is simple.



$d(x_1) = 4,\ d(x_2) = 3,\ d(x_3) = 3,\ d(v) = 2,\ d(u) = 4$

The "first theorem" of graph theory

Theorem

Let G be any graph. Then

$$\sum_{x \in V} d(x) = 2e(G),$$

that is, the sum of the degrees in a graph is equal to twice the number of edges.

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Proof: Sum up all the edge endpoints at every vertex of the graph. This is exactly the sum of the degrees in G. Note that every edge is counted twice in the above sum as edges have two endpoints. From this the theorem follows.

Corollary

The sum of the degrees in every graph is an even number. Hence, the number of vertices having odd degrees must be even.

Definition

A vertex x is called isolated, if $d(x) = 0$, in other words, if x is not adjacent to any other vertex. Hence, the empty graph has only isolated vertices.

Definition

A walk in a graph G is sequence

$$\mathcal{W} = v_0 e_1 v_1 e_2 v_2 \dots v_{\ell-1} e_\ell v_\ell,$$

where $v_0, \dots, v_\ell \in V(G)$, $e_1, \dots, e_\ell \in E(G)$, and for every $1 \leq i \leq \ell$ edge e_i connects the vertices v_{i-1} and v_i.
The length of the walk is ℓ, that is, the number of edges in it. The walk is closed if $v_0 = v_\ell$.

A walk in a graph

Definition

If the walk contains ℓ distinct edges, we call it a trail (or tour). If it only contains distinct vertices, then it is a path. If $v_0, \ldots, v_{\ell-1}$ are distinct vertices and $v_0 = v_\ell$, then the walk is a cycle. When $\ell = 2$ then we must have $e_1 \neq e_2$ in order to get a cycle of length 2. A loop is a cycle of length 1.

Definition

Let G be a graph. We call G a connected graph, if for every $x, y \in V(G)$ there is a walk $\mathcal{W}_{x,y}$ that begins at x and ends at y. If G is not connected, we call it disconnected.

That is, G is connected if and only if one can walk from any vertex of G to any other vertex along the edges of G.

Theorem

Graph G is connected if and only if there is a path between any two vertices of G.

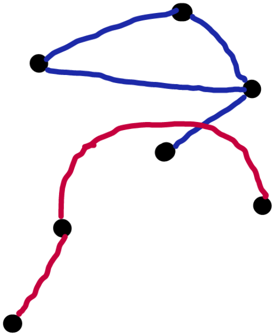
Proof sketch: Clearly, if there is a path between any two vertices, then G must be connected (since paths are special walks). For the other direction, observe, that if an xy-walk contains a vertex u twice, then one can "cut out" the part between the two occurrencies of u (together with one copy of u) obtaining a shorter walk that still connects x and y. Repeating this process will result in an xy-path in a finite number of steps.

Definition

A subset $V' \subset V(G)$ is called a connected component or component of G, if G restricted to V' is connected, furthermore, there is no path between any $x \in V'$ and $y \in V(G) - V'$.

It is easy to see that the vertex set of every graph decomposes into connected components. The empty graph on n vertices has precisely n connected components, while K_n has one component.

A graph with two connected components



Connectedness is a very important notion. One class of connected graphs is especially useful.

Definition

The graph T on at least 2 vertices is called a tree if it is connected, but deleting any edge from it would make it disconnected.

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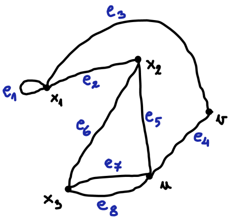
Observe, that the definition immediately implies that trees are simple graphs. If one deletes every loop from a connected graph, it will remain connected, similarly, one may remove all copies of parallel edges but one, the resulting graph is connected, if it was connected.



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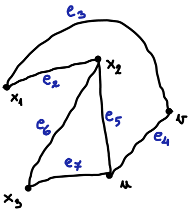

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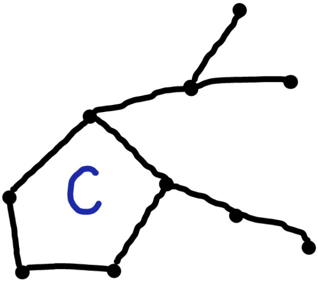
There are several equivalent definitions for trees.

Theorem

A connected graph T is a tree if and only if T does not contain a cycle.

Proof sketch: If T has a cycle, then one can delete any edge on that cycle so that the resulting graph is still connected. Hence, in this case T cannot be a tree. Assume now that T is a cycle-free connected graph. If leaving out an edge e would not make it disconnected, than putting back that edge T had at least two distinct paths between two of its vertices (the endpoints of e). This means that we could find a cycle in T, which proves the theorem.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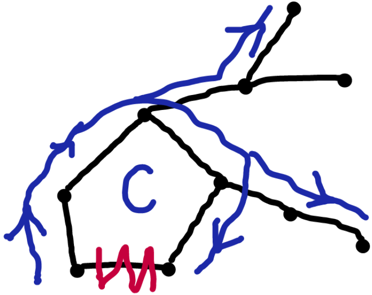
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Theorem

Every connected graph contains a tree on the same number of vertices, that is, a spanning tree.

Proof sketch: Let G be a connected graph. If it does not contain a cycle, then we are done, G is already a tree. If it has a cycle C than we may delete an arbitrary edge of C, the resulting graph will remain connected. Repeating this procedure we can get rid of every cycle of G so that at every step the graph is connected. Hence, at the end we obtain a cycle-free graph which is also connected – a tree.

Let T be a tree. A leaf of T is a vertex having only one neighbor (note: every vertex of T must have at least one neighbor).

Theorem

Every tree has a leaf.

Proof sketch: Assume on the contrary that T is a tree without any leaves. Hence, every vertex of T must have degree at least 2. Then we can find a cycle in T as follows: Start a walk at any vertex. Whenever we enter a vertex x on edge e, we must leave x on another edge – this is possible as $d(x) \geq 2$, until we repeat a vertex in the walk, and then we stop. This procedure must stop in a finite number of steps, as T has finitely many vertices. But if there is a repated vertex, that means a closed walk, which must contain a cycle.

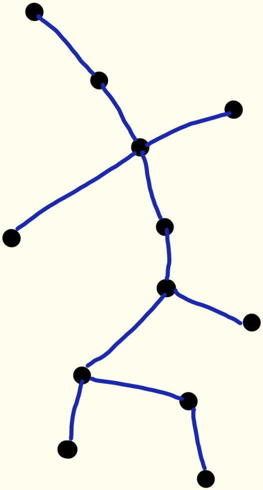
Corollary

Every tree has at least two leaves.

Proof sketch: We may assume that T has at least 3 vertices, since there is only one tree on 2 vertices (up to isomorphism), an edge together with its endpoints. Those endpoints have degree 1, so they are leaves.

We apply induction, and assume that every tree on at most $n-1 \geq 2$ vertices has at least 2 leaves. Let T be a tree on n vertices. It has a leaf x, delete it. together with the edge incident to that leaf together.We obtain a connected graph this way, which must also be cycle-free. Hence, it is a tree on $n-1$ vertices, therefore has at least 2 leaves, u and v. Let us insert back the deleted leaf x. If it is adjacent to u, then x and v will be leaves, analogous holds for the case when x is adjacent to v.

If x is adjacent to a vertex that is not a leaf, then the number of leaves will be at least 3. This finishes the proof.

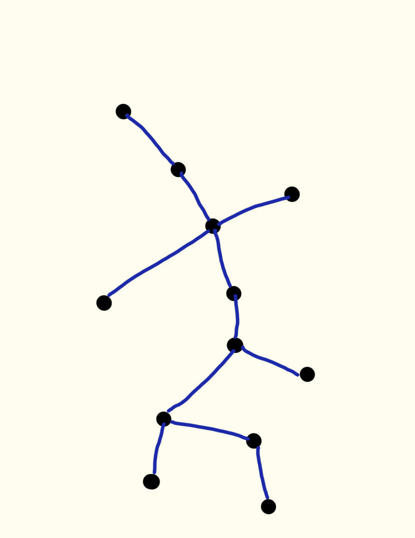


Theorem

If T is a tree on $n \geq 2$ vertices, then the number of edges of T is exactly $n - 1$.

Proof sketch: We apply induction on n. For $n = 2$ it is easy to see that T must have 1 edge. For $n = 3$ we can easily conclude that the only tree on 3 vertices must have 2 edges. Assume now that th etheorem holds for every $n \leq N - 1$, and let T be a tree on N vertices. T has a leaf x. Delete x together with the edge incident to x. This way we obtain a graph T' that does not contain any cycle, and is connected. Hence, T' must be a tree. By the induction hypothesis we get that T' has $N - 2$ edges, since it has $N - 1$ vertices. If we add back the deleted leaf and edge, we obtain T again – a tree having N vertices and $e(T') + 1 = N - 2 + 1 = N - 1$ edges.

A tree on 12 vertices having 11 edges



Some problems for the practice class

(1) Is it possible that a degree sequence of a graph is $3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6$? Prove or disprove!

(2) Let G be a simple graph. Show that it must have two distinct vertices, x and y such that $d(x) = d(y)$. What if G is not simple?

(3) Can the following numbers be the degrees of a simple graph on five vertices? $\mathcal{S} = 3, 3, 4, 4, 6$

Some problems for the practice class, continued

(4) Let G be a graph (not necessarily simple). Assume that it has exactly two vertices, x and y with odd degree, every other vertex has even degree. Show that there is a path between x and y – note that G may be disconnected!

(5) Let G be a connected graph. Prove the following: if G is not a tree, then one can remove an edge from it so that the resulting graph is connected.

(6) Let G be a connected graph. Prove the following: one can remove a *vertex* from it so that the resulting graph is connected.