ON PRODUCTS OF INVERSE SEMIGROUPS

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Abstract. A wreath product of inverse semigroups of partial bijections is introduced which generalizes the usual wreath product of permutation groups, and a product of semidirect type, called I-semidirect product, is defined for arbitrary inverse semigroups where the action involved is by partial automorphisms. It is proved that these products are equivalent to each other, and are equivalent to the λ-semidirect and λ-wreath product and to the Houghton wreath product from the point of view of which extensions of inverse semigroups are embeddable in them.

1. Introduction and preliminaries

Semidirect and wreath products are fundamental constructions in group theory, since by the Kaloujnine–Krasner theorem, each extension of a group K by a group T is embeddable in the standard wreath product of K by T. A crucial fact behind this theorem is that a congruence of a group is determined by the subgroup consisting of all elements congruent to the identity element. Inverse semigroups have a similar feature, namely, a congruence of an inverse semigroup is determined by the following pair: the inverse subsemigroup, called the kernel of the congruence, consisting of all elements congruent to an idempotent element, and the partition induced by the congruence on the idempotents. This allows us to investigate extensions of inverse semigroups similarly to extensions of groups. A number of structure theorems and embedding theorems have been proved for inverse semigroups which can be viewed in this way even if the original motivation for them was not this; see [8]. Some of these results have been generalized also for wider classes, see e.g. [2], [9], [10].

If K, T and S are inverse semigroups and θ is a congruence on S such that S/θ is isomorphic to T and the kernel of θ is isomorphic to K then (S, θ) is called an *extension of K by T*. If (S, θ) and (S', θ') are extensions of inverse semigroups then an injective homomorphism $\phi\colon S \to S'$ is defined to be an *embedding of (S, θ) into (S', θ')* if θ coincides with the congruence induced by $\phi\theta'^{\natural}$.

Since a usual semidirect product or a wreath product of inverse semigroups is not necessarily inverse (except when the second factor is a group), alternative versions of these constructions have been introduced in the theory of inverse semigroups.

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C. H. Houghton [4] studied a generalization of the Kaloujnine–Krasner theorem for inverse semigroups, and he introduced the following construction (see also [7, Definition 11.2.21]). Let K and T be inverse semigroups, and denote by $H_{K,T}$ the set of all maps from principal left ideals of T into K. The domain of a map $h \in H_{K,T}$ is denoted $\operatorname{dom} h$. Define "pointwise" multiplication on $H_{K,T}$ in the natural way: for every $h, k \in H_{K,T}$, let $\operatorname{dom}(h \oplus k) = \operatorname{dom} h \cap \operatorname{dom} k$, and put $x(h \oplus k) = (xh)(xk)$ for any $x \in \operatorname{dom}(h \oplus k)$. Since the intersection of principal ideals of an inverse semigroup is a principal ideal, $H_{K,T}$ forms an inverse semigroup with respect to the operation $\oplus$. Moreover, introduce a left action of T on $H_{K,T}$ by endomorphisms (i.e., a homomorphism $T \to \operatorname{End}^d H_{K,T},\ t \mapsto \varepsilon_t$ where $t \cdot h$ is written for $\varepsilon_t(h)$) as follows: for every $t \in T$ and $h \in H_{K,T}$, let $t \cdot h\colon (\operatorname{dom} h)t^{-1} \to K,\ x \mapsto (xt)h$. Finally, consider the set

$$K \operatorname{Wr}^H T = \{(h,t) \in H_{K,T} \times T : \operatorname{dom} h = Tt^{-1}\}$$

with the multiplication

$$(h,t)(k,u) = (h \oplus (t \cdot k), tu)$$

on it. It turns out that $K \operatorname{Wr}^H T$ is an inverse semigroups which we call the *Houghton wreath product* of K by T.

While investigating the normal extensions of inverse semigroups, Billhardt [1] introduced constructions (see also [8, Section 5.3]), called λ-*semidirect product* and λ-*wreath product* of inverse semigroups K and T, and denoted $K *^\lambda T$ and $K \operatorname{Wr}^\lambda T$, respectively. For their definitions and properties, the reader is referred to [8, Chapter 5].

Notice that Houghton wreath products, λ-semidirect and λ-wreath products of inverse semigroups can be naturally considered as extensions. If K and T are inverse semigroups and $\star$ stands for any of these products then the second projection of $K \star T$ induces a congruence Θ_2 on $K \star T$, and $(K \star T, \Theta_2)$ is an extension of an inverse semigroup K' by T. We will refer to such an extension as a *Houghton wreath product extension*, a λ-*semidirect product extension* and the λ-*wreath product extension* of K by T, respectively.

In this terminology, [4, Theorem 4] (see also [7, Theorem 11.2.27]) says that if S is an inverse semigroup and θ is a congruence on S such that the idempotents are inversely well ordered in each idempotent θ-class (in particular, this is the case if θ is idempotent separating), then (S, θ) is embeddable in $(K \operatorname{Wr}^H (S/\theta), \Theta_2)$ with K being the kernel of θ. Similarly, the main result of [1] (see also [8, Theorem 5.3.5]) says that if S is an inverse semigroup and θ is a congruence on S such that each idempotent θ-class contains a greatest idempotent (in particular, this is the case if θ is idempotent separating), then (S, θ) is embeddable in $(K \operatorname{Wr}^\lambda (S/\theta), \Theta_2)$ with K being the kernel of θ. For the details, see [8, Chapter 5].

Recently, M. Kambites [5, p. 44] has formulated the following criticism on the λ-semidirect product (but it fits also the Houghton wreath product because of the action by endomorphisms in its definiton):

'We remark that the λ-semidirect product is somewhat unusual, indeed arguably even unnatural, in the context of inverse semigroup theory. Inverse semigroups usually arise as models of "partial symmetry", and by the Wagner–Preston Theorem [8, Theorem 1.5.1] can all be represented as such. It is thus customary (and almost always most natural) to consider them acting by partial bijections, rather than by functions, and an action by endomorphisms thus seems intuitively like a "category error".'

The present note is motivated by this remark. In Section 2 we generalize the usual wreath product of permutation groups for inverse semigroups of partial bijections, and even more generally, for inverse semigroups acting by partial bijections. In Section 3 we introduce a product of semidirect type for arbitrary inverse semigroups, called I-semidirect product, demanded in the above critisism, that is, where the 'action' involved is by partial automorphisms rather than by total endomorphisms. In Section 4 we prove that wreath products introduced in Section 2, I-semidirect products introduced in Section 3, Houghton wreath products, λ-semidirect and λ-wreath products are equivalent to each other in the sense of which extensions of inverse semigroups are embeddable in them. This implies that, in any result where λ-semidirect product has been applied so far in this context, the λ-semidirect product can be replaced by a construction containing no "category error". However, let us mention that it is much easier to carry out calculations in a λ-semidirect or λ-wreath product than in a wreath product or in an I-semidirect product, since maps are total in the former one, and so there is no need for checking domains.

For the undefined notions and notation, the reader is referred to [8].

2. Wreath product of inverse semigroups

We introduce a generalization of the usual wreath product of permutation groups for inverse semigroups of partial bijections, more precisely, for inverse semigroups acting by partial bijections. It turns out that this construction also generalizes the Houghton wreath product. We also show that each wreath product extension embeds into a λ-semidirect product extension.

Let K be an inverse semigroup and Ω a set. Denote by $F_{K,\Omega}$ the set of all partial maps from Ω to K, that is, of all maps from a subset of Ω to K. We denote the domain of a partial map $f \in F_{K,\Omega}$ by $\operatorname{dom} f$. It is routine to see that $F_{K,\Omega}$ forms an inverse semigroup with respect to the pointwise multiplication defined formally as follows: for every $f, g \in F_{K,\Omega}$, let $f \oplus g$ be the partial map from Ω to K such that

$$\operatorname{dom}(f \oplus g) = \operatorname{dom} f \cap \operatorname{dom} g \quad \text{and} \quad \omega(f \oplus g) = (\omega f)(\omega g) \quad (\omega \in \operatorname{dom}(f \oplus g)).$$

As a consequence, we have

$$\operatorname{dom} f^{-1} = \operatorname{dom} f \quad \text{and} \quad \omega f^{-1} = (\omega f)^{-1} \quad (\omega \in \operatorname{dom} f^{-1}).$$

In fact, $F_{K,\Omega}$ is the strong semilattice $(\mathcal{P}(\Omega); \cap)$, with $\mathcal{P}(\Omega)$ standing for the power set of Ω, of the direct powers K^{Ξ} $(\Xi \subseteq \Omega)$ where, for every $\Xi, \Xi' \subseteq \Omega$ with $\Xi \supseteq \Xi'$, the structure homomorphism from K^{Ξ} to $K^{\Xi'}$ maps $f \in K^{\Xi}$ to its restriction to Ξ'. Notice that if T is an inverse semigroup then $H_{K,T}$ is an inverse subsemigroup of $F_{K,T}$, and K^{Ξ} $(\Xi \subseteq \Omega)$ is contained in $H_{K,T}$ if and only if Ξ is a principal left ideal of T. Thus $H_{K,T}$ is a strong semilattice $E(T)$ of the direct powers K^{Te} $(e \in E(T))$.

Given an inverse semigroup T and a set Ω, we say that T *acts on* Ω *by partial bijections*, or ${}_{[\Omega]}T$ is an *inverse semigroup action* if a homomorphism $T \to I(\Omega),\ t \mapsto \tilde{t}$ is given. If T

is a monoid then this homomorphism is always supposed to be a monoid homomorphism, that is, the image of the identity element of T is supposed to be 1_Ω. For simplicity, we write $\operatorname{dom} t$ for $\operatorname{dom} \tilde{t}$ for any $t \in T$, and we write ωt for $\omega \tilde{t}$ for any $\omega \in \operatorname{dom} t$. If the homomorphism $T \to I(\Omega)$ is injective then we say that the inverse semigroup action ${}_{[\Omega]}T$ is *faithful*, or T acts faithfully on Ω. In particular, each inverse subsemigroup T of $I(\Omega)$ can be considered a faithful inverse semigroup action ${}_{[\Omega]}T$ where the homomorphism $T \to I(\Omega)$ is the inclusion map.

Let K be an arbitrary inverse semigroup and ${}_{[\Omega]}T$ an inverse semigroup action. Define the *wreath product* $K \wr {}_{[\Omega]}T$ *of* K *by* ${}_{[\Omega]}T$ to be the set

$$K \wr {}_{[\Omega]}T = \{(f,t) \in F_{K,\Omega} \times T : \operatorname{dom} t = \operatorname{dom} f\}$$

with the multiplication

$$(f,t)(g,u) = (f \oplus \tilde{t}g, tu), \tag{2.1}$$

where $\tilde{t}g$ is the usual composition of the partial maps $\tilde{t}$ and g. Note that this multiplication is well defined since $\operatorname{dom} g = \operatorname{dom} u$ implies $\operatorname{dom}(\tilde{t}g) = \operatorname{dom}(\tilde{t}\tilde{u}) = \operatorname{dom} \widetilde{tu} = \operatorname{dom}(tu)$, and so $\operatorname{dom} f = \operatorname{dom} t \subseteq \operatorname{dom}(tu)$ implies $\operatorname{dom}(f \oplus \tilde{t}g) = \operatorname{dom}(tu)$.

Before verifying that this structure forms an inverse semigroup, we consider two special cases.

Proposition 2.1. *Consider an inverse semigroup K and an inverse semigroup action ${}_{[\Omega]}T$.*

(1) *If both K and T are groups (and so T necessarily acts on Ω by permutations) then $K \wr {}_{[\Omega]}T$ coincides with the usual wreath product of K by ${}_{[\Omega]}T$.*
(2) *The wreath product $K \wr {}_{[T]}T$, where T acts on T by the Wagner–Preston representation, coincides with the Houghton wreath product $K \operatorname{Wr}^H T$.*

Proof. (1) Since the group T is a monoid, and the identity of T acts identically on Ω, each element of T acts by permutations. The rest easily follows.

(2) Obviously, if $(f,t) \in K \wr (T,T)$ then $f \in H_{K,T}$. Moreover, for any $t \in T$ and $h \in H_{K,T}$, we have $t \cdot h = \tilde{t}h$. For, if $\operatorname{dom} h = Tu^{-1}$ then $\operatorname{dom}(t \cdot h) = (\operatorname{dom} h)t^{-1} = Tu^{-1}t^{-1} = (Tt \cap Tu^{-1})t^{-1} = (\operatorname{im} \tilde{t} \cap \operatorname{dom} h)\tilde{t}^{-1} = \operatorname{dom} \tilde{t}h$, and for every element x from this set, we have $x(t \cdot h) = (xt)h = (x\tilde{t})h = x(\tilde{t}h)$. $\square$

Now we show that a wreath product is an inverse semigroup, and it is isomorphic to a subsemigroup of a λ-semidirect product.

Proposition 2.2. *Let K be an inverse semigroup and ${}_{[\Omega]}T$ an inverse semigroup action.*

(1) *The wreath product $K \wr {}_{[\Omega]}T$ is an inverse semigroup.*
(2) *The second projection from $K \wr {}_{[\Omega]}T$ to T is a surjective homomorphism, and if Θ_2 is the congruence induced by this homomorphism then $(K \wr {}_{[\Omega]}T, \Theta_2)$ is an extension of a semigroup K' by T where K' is a subsemigroup of the direct product of $F_{K,\Omega}$ and the semilattice $E(T)$. If the inverse semigroup action ${}_{[\Omega]}T$ is faithful then K' is isomorphic to a subsemigroup of $F_{K,\Omega}$.*

(3) *Consider the inclusion map* $\iota\colon K \wr_{[\Omega]} T \to F_{K,\Omega} *^{\lambda} T$, *where this* λ*-semidirect product is defined by the left action* $t \cdot f = \tilde{t} f$ $(t \in T,\ f \in F_{K,\Omega})$ *of* T *on* $F_{K,T}$ *by endomorphisms. Then* ι *is an embedding of the extension* $(K \wr_{[\Omega]} T, \Theta_2)$ *into the extension* $(F_{K,\Omega} *^{\lambda} T, \Theta_2)$.

Proof. It is routine to check that the rule $t \cdot f = \tilde{t} f$ $(t \in T,\ f \in F_{K,\Omega})$ in (3) defines an action of T on $F_{K,\Omega}$ by endomorphisms, and so the λ-semidirect product $F_{K,\Omega} *^{\lambda} T$ is defined. By definition, a pair $(f, t) \in F_{K,\Omega} \times T$ belongs to $F_{K,\Omega} *^{\lambda} T$ if and only if $\widetilde{tt^{-1}} f = f$. Since $\widetilde{tt^{-1}} = 1_{\operatorname{dom} \tilde{t}} = 1_{\operatorname{dom} t}$, this equality holds for every $(f, t) \in K \wr_{[\Omega]} T$, and so $K \wr_{[\Omega]} T \subseteq F_{K,\Omega} *^{\lambda} T$. The inclusion map ι, which is obviously injective, is easily seen to be a homomorphism. For, if $(f, t), (g, u) \in K \wr_{[\Omega]} T$ then $\operatorname{dom}(f \oplus \tilde{t} g) = \operatorname{dom} \widetilde{tu} = \operatorname{dom}(1_{\operatorname{dom}(tu(tu)^{-1})} f) = \operatorname{dom}(\tilde{t} g) = \operatorname{dom}(1_{\operatorname{dom}(tu(tu)^{-1})} f \oplus \tilde{t} g)$, and this implies $f \oplus \tilde{t} g = 1_{\operatorname{dom}(tu(tu)^{-1})} f \oplus \tilde{t} g$. Since $1_{\operatorname{dom}(tu(tu)^{-1})} f \oplus \tilde{t} g = \big((tu(tu)^{-1})) \cdot f\big) \oplus (t \cdot g)$ in $F_{K,\Omega}$, thus we have verified that ι is an injective homomorphism.

Since $F_{K,\Omega} *^{\lambda} T$ is an inverse semigroup, in order to show that so is $K \wr_{[\Omega]} T$, it suffices to check that, for every $(f, t) \in K \wr_{[\Omega]} T$, the inverse of (f, t) in $F_{K,\Omega} *^{\lambda} T$, that is, the element $\big((\tilde{t}^{-1} f)^{-1}, t^{-1}\big)$ belongs to $K \wr_{[\Omega]} T$. This follows easily since we have $\operatorname{dom}(\tilde{t}^{-1} f)^{-1} = \operatorname{dom}(\tilde{t}^{-1} f) = \operatorname{dom}(\tilde{t}^{-1} \tilde{t}) = \operatorname{dom}(\tilde{t}^{-1}) = \operatorname{dom}(t^{-1})$. This shows statement (1).

The first assertion of (2) is straightforward by definition. To show the second one, observe that the kernel of Θ_2 is

$$K' = \{(f, i) \in F_{K,\Omega} \times E(T) : \operatorname{dom} f = \operatorname{dom} i\},$$

and $(f, i)(g, j) = (f \oplus g, ij)$ for any $(f, i), (g, j) \in K'$, since $\tilde{i} = 1_{\operatorname{dom} i} = \operatorname{dom} f$. If ${}_{[\Omega]}T$ is faithful then, for every $i, j \in E(T)$, the equality $i = j$ is implied by the equality $\operatorname{dom} i = \operatorname{dom} j$. Therefore $(f, i) = (g, j)$ in K' if and only if $f = g$, which implies the last assertion of (2).

Finally, to complete the proof of (3), all we have to notice is that, for any $(f, t), (g, u) \in K \wr_{[\Omega]} T$, we have $(f, t)\, \Theta_2\, (g, u)$ in $K \wr_{[\Omega]} T$ if and only if $t = u$, and this is equivalent to the relation $(f, t)\, \Theta_2\, (g, u)$ in $F_{K,\Omega} *^{\lambda} T$. $\square$

It is well known that the wreath product of groups acting on sets Γ and Ω by permutations acts naturally on the Cartesian product $\Gamma \times \Omega$ (in particular, the wreath product of permutation groups on sets Γ and Ω is a permutation group on $\Gamma \times \Omega$). Now we generalize this property for the wreath product of inverse semigroups.

Suppose that ${}_{[\Gamma]}K$ and ${}_{[\Omega]}T$ are inverse semigroup actions, and consider the map

$$\kappa\colon K \wr_{[\Omega]} T \to I(\Gamma \times \Omega), \quad (f, t) \mapsto \overline{(f, t)},$$

where

$$\operatorname{dom} \overline{(f, t)} = \{(\gamma, \omega) \in \Gamma \times \Omega : \omega \in \operatorname{dom} t (= \operatorname{dom} f) \text{ and } \gamma \in \operatorname{dom}(\omega f)\}, \tag{2.2}$$

and

$$(\gamma, \omega)\overline{(f, t)} = (\gamma(\omega f), \omega t) \quad \text{for any} \quad (\gamma, \omega) \in \operatorname{dom} \overline{(f, t)}. \tag{2.3}$$

Proposition 2.3. *For any inverse semigroup actions ${}_{[\Gamma]}K$ and ${}_{[\Omega]}T$, the map κ defined above is a homomorphism, and so it defines an inverse semigroup action ${}_{[\Gamma\times\Omega]}(K \wr {}_{[\Omega]}T)$. In particular, if the inverse semigroup actions ${}_{[\Gamma]}K$ and ${}_{[\Omega]}T$ are faithful then so is the inverse semigroup action ${}_{[\Gamma\times\Omega]}(K \wr {}_{[\Omega]}T)$.*

Proof. Let us check first that κ is well defined, that is, $\overline{(f,t)}$ is injective for every $(f,t) \in K \wr {}_{[\Omega]}T$. Assume that $(\gamma,\omega),(\gamma',\omega') \in \operatorname{dom}\overline{(f,t)}$ and $(\gamma,\omega)\overline{(f,t)} = (\gamma',\omega')\overline{(f,t)}$. Then the equalities $\omega t = \omega' t$ and $\gamma(\omega f) = \gamma'(\omega' f)$ follow. The first one implies $\omega = \omega'$ since $\tilde{t} \in I(\Omega)$, and then the second one implies $\gamma = \gamma'$ since $\widetilde{\omega f} \in I(\Gamma)$.

Now we show that κ is a homomorphism. We have to check that, for any $(f,t),(g,u) \in K \wr {}_{[\Omega]}T$, we have

$$\operatorname{dom}\big(\overline{(f,t)}\,\overline{(g,u)}\big) = \operatorname{dom}\overline{(f \oplus \tilde{t}g, tu)}, \tag{2.4}$$

and the equality

$$(\gamma,\omega)\big(\overline{(f,t)}\,\overline{(g,u)}\big) = (\gamma,\omega)\overline{(f \oplus \tilde{t}g, tu)},$$

holds for any element (γ,ω) of the common domain in (2.4). In order to determine $\operatorname{dom}\big(\overline{(f,t)}\,\overline{(g,u)}\big)$, observe that

$$\operatorname{im}\overline{(f,t)} = \{(\gamma(\omega f),\omega t) : \omega \in \operatorname{dom} t = \operatorname{dom} f,\ \gamma \in \operatorname{dom}(\omega f)\},$$

and that such an element belongs to $\operatorname{dom}\overline{(g,u)}$ if and only if

$$\omega t \in \operatorname{dom} u = \operatorname{dom} g \quad \text{and} \quad \gamma(\omega f) \in \operatorname{dom}((\omega t)g) = \operatorname{dom}(\omega(\tilde{t}g)).$$

The first relation in conjunction with $\omega \in \operatorname{dom} t = \operatorname{dom} f$ is equivalent to requiring that $\omega \in \operatorname{dom}(tu) = \operatorname{dom}(f \oplus \tilde{t}g)$, and the second relation together with $\gamma \in \operatorname{dom}(\omega f)$ is equivalent to the relation

$$\gamma \in \big(\operatorname{im}(\omega f) \cap \operatorname{dom}(\omega(\tilde{t}g))\big)(\omega f)^{-1} = \operatorname{dom}\big((\omega f)(\omega(\tilde{t}g))\big) = \operatorname{dom}(\omega(f \oplus \tilde{t}g)).$$

Applying (2.2), we obtain that (2.4) holds. If (γ,ω) belongs to the common domain in (2.4) then, by definition, we have

$$\begin{aligned}(\gamma,\omega)\big(\overline{(f,t)}\,\overline{(g,u)}\big) &= (\gamma(\omega f),\omega t)\overline{(g,u)} = \big((\gamma(\omega f))((\omega t)g),(\omega t)u\big)\\ &= \big(\gamma\big((\omega f)(\omega(\tilde{t}g))\big),\omega(tu)\big) = (\gamma,\omega)\overline{(f \oplus \tilde{t}g, tu)},\end{aligned}$$

thus proving that κ is, indeed, a homomorphism.

Finally, assume that ${}_{[\Gamma]}K$ and ${}_{[\Omega]}T$ are faithful, and show the same property for ${}_{[\Gamma\times\Omega]}(K \wr {}_{[\Omega]}T)$. We have to verify that κ is injective. Let $(f,t),(f',t') \in K \wr {}_{[\Omega]}T$ such that $\overline{(f,t)} = \overline{(f',t')}$. Thus $\operatorname{dom}\overline{(f,t)} = \operatorname{dom}\overline{(f',t')}$, whence it follows by (2.2) that $\operatorname{dom} t = \operatorname{dom} t'$ and $\operatorname{dom}(\omega f) = \operatorname{dom}(\omega f')$ for any $\omega \in \operatorname{dom} f = \operatorname{dom} t = \operatorname{dom} t' = \operatorname{dom} f'$. Morevoer, we have $(\gamma,\omega)\overline{(f,t)} = (\gamma,\omega)\overline{(f',t')}$ for every $(\gamma,\omega) \in \operatorname{dom}\overline{(f,t)} = \operatorname{dom}\overline{(f',t')}$, and so we obtain by (2.3) that $\omega t = \omega t'$ and $\gamma(\omega f) = \gamma(\omega f')$ for each $\omega \in \operatorname{dom} t = \operatorname{dom} t'$ and $\gamma \in \operatorname{dom}(\omega f) = \operatorname{dom}(\omega f')$. These equalities imply $t = t'$ and $f = f'$ as it was to be verified. $\square$

A direct consequence of this proposition is that wreath product can be considered as a construction producing an inverse semigroup of partial bijections (on $\Gamma \times \Omega$) from inverse semigroups of partial bijections (on Γ and Ω), and as such, it generalizes the classical wreath product of permutation groups.

We conclude this section by sketching an idea how wreath product might be interpreted as a construction contaning no "category error".

Consider the wreath product $K \wr {}_{[\Omega]}T$ where K is an inverse semigroup and ${}_{[\Omega]}T$ is an inverse semigroup action. It is easy to see that

$$I_t = \{f \in F_{K,\Omega} : \operatorname{dom} f \subseteq \operatorname{dom} t\} \quad (t \in T)$$

is an ideal of $F_{K,\Omega}$. Restrict the action of T on $F_{K,\Omega}$ (on the left by endomorphisms) as follows: for any $t \in T$ and $f \in F_{K,\Omega}$, let us define $t \circ f$ if and only if $f \in I_{t^{-1}}$, and in this case, put $t \circ f = \tilde{t}f$. One can see that $\alpha_t \colon I_{t^{-1}} \to I_t,\ f \mapsto t \circ f$ is an isomorphism, and $\alpha_t^{-1} = \alpha_{t^{-1}}$. Thus α_t can be viewed as a partial automorphism of $F_{K,\Omega}$. Moreover, the map $\alpha\colon T \to I^d(F_{K,\Omega}),\ t \mapsto \alpha_t$ is a homomorphism. To check this, it suffices to show that $\operatorname{dom}(\alpha_t\alpha_u) = \operatorname{dom}\alpha_{tu}$ for every $t, u \in T$. Since $\operatorname{dom}(\alpha_t\alpha_u) = \alpha_u^{-1}(\operatorname{dom}\alpha_t \cap \operatorname{im}\alpha_u)$, we have $f \in \operatorname{dom}(\alpha_t\alpha_u)$ if and only if $f \in \alpha_u^{-1}(I_u \cap I_{t^{-1}})$, which is equivalent to requiring that $\tilde{u}f \in I_u \cap I_{t^{-1}}$. The latter relation holds if and only if $\operatorname{dom}(\tilde{u}f) \subseteq \operatorname{dom} u \cap \operatorname{im} t = (\operatorname{im}(tu))\tilde{u}^{-1}$. Since any of these properties implies $f \in I_{u^{-1}}$, that is, $\operatorname{dom} f \subseteq \operatorname{im} u$, we obtain that $\operatorname{dom}(\tilde{u}f) = (\operatorname{im} u \cap \operatorname{dom} f)\tilde{u}^{-1} = (\operatorname{dom} f)\tilde{u}^{-1}$. This implies that $f \in \operatorname{dom}(\alpha_t\alpha_u)$ if and only if $\operatorname{dom} f \subseteq \operatorname{im}(tu) = \operatorname{dom}(tu)^{-1}$, and so if and only if $f \in \operatorname{dom}\alpha_{tu}$ which was to be shown.

Thus α defines a left action of the inverse semigroup T on the inverse semigroup $F_{K,\Omega}$ by partial automorphisms where the domains of the partial automorphisms involved are ideals of $F_{K,\Omega}$. This allows us to modify the rule (2.1) of multiplication of $K \wr {}_{[\Omega]}T$ as follows:

$$(f,t)(g,u) = \big(t \circ ((t^{-1} \circ f) \oplus g), tu\big). \tag{2.5}$$

Indeed, $\operatorname{dom} f = \operatorname{dom} t$ implies $f \in I_t$, whence $t^{-1} \circ f$ is defined and $t^{-1} \circ f \in I_{t^{-1}}$. Since $I_{t^{-1}}$ is an ideal, $t \circ ((t^{-1} \circ f) \oplus g)$ is also defined, and we have $t \circ ((t^{-1} \circ f) \oplus g) = \tilde{t}(\tilde{t}^{-1}f \oplus g) = (\tilde{t}\tilde{t}^{-1}f) \oplus \tilde{t}g = f \oplus \tilde{t}g$.

We complete this idea in the next section.

3. A new product of inverse semigroups

Motivated by the idea at the end of the previous section, here we introduce a construction, called I-semidirect product, for arbitrary inverse semigroups where the action involved is a strong partial action by partial automorphisms, and investigate its connection to wreath product and ???.

First we recall the notion of a partial action. Partial group actions are studied in [6], and the results are generalized for partial monoid actions in [3].

If S and T are inverse semigroup then a map $\psi\colon T \to S$ is called a *dual premorphism* (unfortunately, terminology is not consistent in the literature; in [8, p. 80] such a map is called a dual prehomomorphism, but [6] and [3] uses simply the term premorphism) if

$$t^{-1}\psi = (t\psi)^{-1} \quad \text{and} \quad (t\psi)(u\psi) \le (tu)\psi \tag{3.1}$$

for every $t, u \in T$. If S and T are monoids, and the dual premorphism $\psi\colon T \to S$ has the property that $1\psi = 1$ then ψ is called *unital.* Obviously, each homomorphism is a dual premorphism, and each monoid homomorphism is a unital dual premorphism.

Given an inverse monoid T and a set Ω, by a *left partial action* $T_{(\Omega)}$, or of T on Ω we mean that a dual premorphism $\alpha\colon T \to I^d(\Omega),\ t \mapsto \alpha_t$ is given. If, instead of the second relation in (3.1), α has the property that

$$(t\psi)(u\psi)(u^{-1}\psi) = (tu)\psi(u^{-1}\psi) \tag{3.2}$$

for every $t, u \in T$, then $T_{(\Omega)}$ is called a *strong* left partial action (see [3]). Notice that (3.2) together with the equality in (3.1) implies the inequality in (3.1). If T is an inverse monoid and α is unital them we say that $T_{(\Omega)}$ is *unital.*

We write $\exists t \circ \omega$ if $\omega \in \alpha_t$, and in this case, $t \circ \omega$ stands for the element $\alpha_t(\omega)$. In this notation, the following two conditions define a strong left partial action: for every $t, u \in T$ and $\omega \in \Omega$,

(SPA1) $\exists t \circ \omega$ implies that $\exists t^{-1} \circ (t \circ \omega)$ and $t^{-1} \circ (t \circ \omega) = \omega$,

(SPA2) $\exists t \circ (u \circ (u^{-1} \circ \omega))$ implies that $\exists (tu) \circ (u^{-1} \circ \omega)$ and $t \circ (u \circ (u^{-1} \circ \omega)) = (tu) \circ (u^{-1} \circ \omega)$.

4. Embedding of extensions into products

In this section we present that wreath product, the Houghton wreath product, λ-semidirect and λ-wreath products are equivalent to each other from the point of view of which extensions of inverse semigroups are embeddable in them.

We have seen in Section 1 that both the Houghton wreath products and λ-semidirect (λ-wreath) products can be considered extensions of inverse semigroups in a natural way. The same idea applies for wreath products introduced in the previous section. Consider the wreath product $K \wr T$ of inverse semigroups K and T. Notice that the kernel of the congruence Θ_2 induced by the second projection is isomorphic to an inverse subsemigroup K' of $F_{K,T}$. We call $(K \wr T, \Theta_2)$ the *wreath product extension* of K' by T.

First we observe that, in some sense, the kernel of the congruence induced by the second projection of each product considered so far is close to the first factor.

Proposition 4.1. *Consider arbitrary inverse semigroups K, T and a variety $\mathbf{V}$ of inverse semigroups such that $K \in \mathbf{V}$, and $\mathbf{V}$ contains a nontrivial semilattice provided T is not a group. Let $K \star T$ denote any of the wreath product, the Houghton wreath product, a λ-semidirect and the λ-wreath products of K by T. Then $(K \star T, \Theta_2)$ is an extension of an inverse semigroup belonging to $\mathbf{V}$ by T.*

Proof. The statement for the wreath product follows from the description of F in the previous section and from the relation $K' \leq F$. This implies the statement for the Houghton wreath product by Proposition 2. Finally, the statements for λ-semidirect and λ-wreath products are direct consequences of the following facts. The λ-wreath product $K\,\mathrm{Wr}^{\lambda}T$ is a λ-semidirect product of the direct power K^T by T, and a λ-semidirect product extension of K by T is an extension of K' by T where K' is an inverse subsemigroup of a direct product of K and the semilattice $E(T)$. □

Now we establish that wreath product extensions are embeddable in λ-semidirect product extensions, and λ-semidirect product extensions in the Houghton wreath product extensions.

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Conversely, we embed any λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ of inverse semigroups K and T into the Houghton wreath product extension $(K\,\mathrm{Wr}^H T, \Theta_2)$. For any $(a,t) \in K *^\lambda T$, define $h_{(a,t)} \in H_{K,T}$ such that $\operatorname{dom} h_{(a,t)} = Tt^{-1}$ and $xh_{(a,t)} = {}^x a$ for every $x \in Tt^{-1}$. Consider the map $\varphi\colon K *^\lambda T \to K\,\mathrm{Wr}^H T,\ (a,t) \mapsto \big(h_{(a,t)}, t\big)$. In order to check that φ is injective, let $(a,t),(b,u) \in K *^\lambda T$ such that $\big(h_{(a,t)}, t\big) = \big(h_{(b,u)}, u\big)$. Then $t = u$ and ${}^x a = {}^x b$ for every $x \in Tt^{-1}$. In particular, if $x = tt^{-1}$ then the latter equality implies $a = {}^{tt^{-1}}a = {}^{tt^{-1}}b = b$. Hence φ is, indeed, injective. Now we show that φ is a homomorphism. We have to check that, for any $(a,t),(b,u) \in K *^\lambda T$, we have $h_{(a(t\cdot b),tu)} = h_{(a,t)} \oplus {}^t h_{(b,u)}$. By definition, we have $\operatorname{dom} h_{(a(t\cdot b),tu)} = T(tu)^{-1}$, $\operatorname{dom} h_{(a,t)} = Tt^{-1}$ and $\operatorname{dom} {}^t h_{(b,u)} = \big(\operatorname{dom} h_{(b,u)}\big)t^{-1} = Tu^{-1}t^{-1} = T(tu)^{-1}$, and so $\operatorname{dom} h_{(a(t\cdot b),tu)} = \operatorname{dom} \big(h_{(a,t)} \oplus {}^t h_{(b,u)}\big)$. If x is an element of this common domain then we have

$$\begin{aligned} x\big(h_{(a,t)} \oplus {}^t h_{(b,u)}\big) &= \big(xh_{(a,t)}\big)\big((xt)h_{(b,u)}\big) = \big((x(tu)(tu)^{-1})h_{(a,t)}\big)\big((xt)h_{(b,u)}\big) \\ &= (x(tu)(tu)^{-1} \cdot a)(xt \cdot b) = x \cdot \big(((tu)(tu)^{-1} \cdot a)(t \cdot b)\big) = xh_{(a(t\cdot b),tu)}. \end{aligned}$$

Thus we have established that φ is an embedding of $(K *^\lambda T, \Theta_2)$ into $(K\,\mathrm{Wr}^H T, \Theta_2)$ which implies the following proposition.

Proposition 4.2. *For every inverse semigroups K and T where T acts on K by endomorphisms, the λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ is embeddable in the Houghton wreath product extension $(K\,\mathrm{Wr}^H T, \Theta_2)$.*

A λ-wreath product $K\,\mathrm{Wr}^\lambda T$ is, by definition, a λ-semidirect product $F *^\lambda T$, and conversely, an argument similar to but easier than the previous one verifies that each λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ is embeddable also in the λ-wreath product extension $(K\,\mathrm{Wr}^\lambda T, \Theta_2)$.

Combining this observation with Propositions 2, 4.1, 2.2, 4.2, we deduce the main result of this section.

Theorem 4.3. *Let S be an inverse semigroup and θ a congruence on S. Consider a variety $\mathbf{V}$ of inverse semigroups such that the kernel of θ belongs to $\mathbf{V}$, and $\mathbf{V}$ contains a nontrivial semilattice provided S/θ is not a group. The following statements are equivalent.*

(1) *The extension (S, θ) is embeddable in a wreath product extension of a member of $\mathbf{V}$ by an inverse semigroup of partial bijections isomorphic to S/θ.*
(2) *The extension (S, θ) is embeddable in a Houghton wreath product extension of a member of $\mathbf{V}$ by S/θ.*
(3) *The extension (S, θ) is embeddable in a λ-wreath product extension of a member of $\mathbf{V}$ by S/θ.*
(4) *The extension (S, θ) is embeddable in a λ-semidirect product extension of a member of $\mathbf{V}$ by S/θ.*

Note that the equivalence of statements Theorem 4.3(2) and (4) is mentioned without proof in [10, p. 239].

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