ON PRODUCTS OF INVERSE SEMIGROUPS

MÁRIA B. SZENDREI

Abstract. A wreath product of inverse semigroups of partial bijections is introduced which generalizes the usual wreath product of permutation groups, and it is proved that an extension of inverse semigroups is embeddable in a λ-semidirect or λ-wreath product, or in Houghton's wreath product if and only if it is embeddable in such a wreath product.

1. Introduction and preliminaries

Semidirect and wreath products are fundamental constructions in group thoery, since by the Kaloujnine–Krasner theorem, each extension of a group K by T is embeddable in the wreath product of K by T. A crucial fact behind this theorem is that a congruence of a group is determined by the subgroup consisting of all elements congruent to the identity element. Inverse semigroups have a similar feature, namely, a congruence of an inverse semigroup is determined by the inverse subsemigroup, called the kernel of the congruence and consisting of all elements congruent to an idempotent element, and by the partition the congruence induces on the idempotents. This allows us to investigate extensions of inverse semigroups. A number of structure theorems and embedding theorems have been proved for inverse semigroups which can be viewed in this way even if the original motivation for them was not this; see [4]. Some of these results have been generalized also for wider classes, see e.g. [5].

If K, T and S are inverse semigroups and θ is a congruence on S such that S/θ is isomorphic to T and the kernel of θ is isomorphic to K then (S, θ) is called an *extension of K by T*. If (S, θ) and (S', θ') are such extensions then an injective homomorphism $\phi\colon S \to S'$ is defined to be an *embedding of (S, θ) into (S', θ')* if θ coincides with the congruence induced by $\phi\theta'^{\natural}$.

Since a semidirect product or a wreath product of inverse semigroups is not necessarily inverse (except when the second factor is a group), alternative versions of these constructions have been introduced in the theory of inverse semigroups.

C. H. Houghton [2] studied a generalization of the Kaloujnine–Krasner theorem for inverse semigroups, and he introduced the following construction. Let K and T be inverse semigroups, and denote by $H_{K,T}$, or simply by H, if it causes no confusion, the set of all maps from a principal left ideal of T into K. The domain of a map $h \in H$ is denoted

---

Research partially supported by the Hungarian National Foundation for Scientific Research grants no. K104251, and K115518.

Mathematical Subject Classification (2010): 20M10, 20M18.
Key words: Inverse semigroup, wreath product, λ-semidirect product.

$\operatorname{dom} h$. Define "pointwise"multiplication on H in the natural way: for every $h, k \in H$, let $\operatorname{dom}(h \oplus k) = \operatorname{dom} h \cap \operatorname{dom} k$, and put $x(h \oplus k) = xh \cdot xk$ for any $x \in \operatorname{dom}(h \oplus k)$. Moreover, introduce an action of T on K as follows: for every $t \in T$ and $h \in H$, let ${}^t h \colon (\operatorname{dom} h)t^{-1} \to K,\ x \mapsto (xt)h$. Finally, consider the set

$$K \operatorname{Wr}^H T = \{(h,t) \in H_{K,T} \times T : \operatorname{dom} h = Tt^{-1}\}$$

with the multiplication

$$(h,t)(k,u) = (h \oplus {}^t k, tu)$$

on it. It turns out that $K \operatorname{Wr}^H T$ is an inverse semigroups which we call *Houghton's wreath product* of K by T.

While investigating the normal extensions of inverse semigroups, Billhardt [1] introduced constructions called λ-*semidirect product* and λ-*wreath product* of inverse semigroups K and T, and denoted $K *^\lambda T$ and $K \operatorname{Wr}^\lambda T$, respectively. For their definitions and properties, the reader is referred to [4].

Notice that Houghton's wreath products, λ-semidirect and λ-wreath products of inverse semigroups can be naturally considered as extensions. If K and T are inverse semigroups and $\star$ stands for any of these products then the second projection of $K \star T$ induces a congruence Θ_2 on $K \star T$, and $(K \star T, \Theta_2)$ is an extension of an inverse semigroup K' by T. We will refer to such an extension as *Houghton's wreath product extension*, a λ-*semidirect product extension* and the λ-*wreath product extension* of K by T, respectively.

In this terminology, the main result of [2] is that if S is an inverse semigroup and θ is a congruence on S such that the idempotents are inversely well ordered in each idempotent θ-class (in particular, this is the case if θ is idempotent separating), then (S, θ) is embeddable in $(K \operatorname{Wr}^H (S/\theta), \Theta_2)$ with K being the kernel of θ. Similarly, the main result of [1] says that if S is an inverse semigroup and θ is a congruence on S such that each idempotent θ-class contains a greatest idempotent (in particular, this is the case if θ is idempotent separating), then (S, θ) is embeddable in $(K \operatorname{Wr}^\lambda (S/\theta), \Theta_2)$ with K being the kernel of θ. For the details, see [4].

Recently, M. Kambites [3, p. 44] has formulated the following criticism on λ-semidirect product, but it fits also Houghton's wreath product:

'We remark that the λ-semidirect product is somewhat unusual, indeed arguably even unnatural, in the context of inverse semigroup theory. Inverse semigroups usually arise as models of "partial symmetry", and by the Wagner–Preston Theorem [4, Theorem 1.5.1] can all be represented as such. It is thus customary (and almost always most natural) to consider them acting by partial bijections, rather than by functions, and an action by endomorphisms thus seems intuitively like a "category error".'

This short note is motivated by this remark. In Section 2 we generalize the usual wreath product of permutation groups for inverse semigroups of partial bijections. In Section 3 we prove that this wreath product of inverse semigroups of partial bijections, Houghton's wreath product, λ-semidirect and λ-wreath products are equivalent to each other in the sense of which extensions of inverse semigroups are embeddable in them. This implies that, in any result λ-semidirect product has been applied so far within the context of inverse semigroups, it can be replaced by a construction containing no "category error". At the same time, let us mention that it is much easier to carry out calculations in a λ-semidirect or λ-wreath product than in a wreath product of inverse semigroups of partial bijections since maps are total, and so there is no need for checking domains.

2. Wreath product of inverse semigroups of partial bijections

We introduce a generalization of the usual wreath product of permutation groups for inverse semigroups of partial bijections. It turns out that, up to isomorphism, this construction also generalizes Houghton's wreath product.

Let Γ and Ω be any sets, and let $K \leq I(\Gamma)$ and $T \leq I(\Omega)$. Denote by $F_{K,T}$, or simply by F, if it causes no confusion, the set of all partial maps from Ω to K, that is, of all maps from a subset of Ω to K. We denote the domain of a partial map $f \in F$ by $\operatorname{dom} f$. It is routine to see that F forms an inverse semigroup with respect to the pointwise multiplication defined formally as follows: for every $f, g \in F$, let $f \oplus g$ be the partial map from Ω to K such that

$$\operatorname{dom}(f \oplus g) = \operatorname{dom} f \cap \operatorname{dom} g \quad \text{and} \quad \omega(f \oplus g) = \omega f \cdot \omega g \quad (\omega \in \Omega).$$

As a consequence, we have

$$\operatorname{dom} f^{-1} = \operatorname{dom} f \quad \text{and} \quad \omega f^{-1} = (\omega f)^{-1} \quad (\omega \in \Omega).$$

In fact, F is the strong semilattice $(\mathcal{P}(\Omega); \cap)$ of the direct powers K^{Ξ} ($\Xi \subseteq \Omega$) where, for every $\Xi, \Xi' \subseteq \Omega$ with $\Xi \supseteq \Xi'$, the structure homomorphism from K^{Ξ} to $K^{\Xi'}$ maps $f \in K^{\Xi}$ to its restriction to Ξ'.

Define the *wreath product* $K \wr T$ *of* K *by* T to be the set

$$K \wr T = \{(f, t) \in F_{K,T} \times T : \operatorname{dom} t = \operatorname{dom} f\}$$

with the multiplication

$$(f, t)(g, u) = (f \oplus tg, tu),$$

where tg is the usual product of the partial maps t and g. Note that this multiplication is well defined since $\operatorname{dom} g = \operatorname{dom} u$ implies $\operatorname{dom}(tg) = \operatorname{dom}(tu)$, and so $\operatorname{dom} f = \operatorname{dom} t \subseteq \operatorname{dom}(tu)$ implies $\operatorname{dom}(f \oplus tg) = \operatorname{dom}(tu)$. A direct checking might show that $K \wr T$ is an inverse semigroup. However, here we establish this fact simultaneously with the statement that $K \wr T$ is isomorphic to an inverse semigroup of partial bijections on the set $\Gamma \times \Omega$.

Before doing so, we notice that this construction generalizes both the usual wreath product of permutation groups and, in some sense, Houghton's wreath product of inverse semigroups. The statement for permutation groups is obvious by definition. For the second observe that, up to the Wagner–Preston isomorphisms $K \to \widetilde{K}$ and $T \to \widetilde{T}$, where $\widetilde{K} \leq I(K)$, $\widetilde{T} \leq I(T)$, both the underlying sets and the multiplications of $K \operatorname{Wr}^H T$ and $\widetilde{K} \wr \widetilde{T}$ coincide.

Generalizing the well-known interpretation of a wreath product of permutation groups as a permutation group, we now give an embedding of $K \wr T$ into $I(\Gamma \times \Omega)$. Consider the map

$$\iota \colon K \wr T \to I(\Gamma \times \Omega), \quad (f, t) \mapsto \overline{(f, t)},$$

where

$$\operatorname{dom}\overline{(f,t)} = \{(\gamma,\omega)\in\Gamma\times\Omega : \omega\in\operatorname{dom}t(=\operatorname{dom}f) \text{ and } \gamma\in\operatorname{dom}(\omega f)\}, \tag{2.1}$$

and

$$(\gamma,\omega)\overline{(f,t)} = (\gamma(\omega f),\omega t) \quad \text{for any} \quad (\gamma,\omega)\in\operatorname{dom}\overline{(f,t)}. \tag{2.2}$$

Lemma 2.1. *For every $K \le I(\Gamma)$ and $T \le I(\Omega)$, the wreath product $K \wr T$ is an inverse semigroup, and the map ι is an embedding of $K \wr T$ into $I(\Gamma\times\Omega)$.*

Proof. Let us start with proving that ι is an embedding. Check first that $\overline{(f,t)} \in I(\Gamma\times\Omega)$, that is, $\overline{(f,t)}$ is injective for every $(f,t)\in K\wr T$. Assume that $(\gamma,\omega),(\gamma',\omega')\in\operatorname{dom}\overline{(f,t)}$ and $(\gamma,\omega)\operatorname{dom}\overline{(f,t)} = (\gamma',\omega')\overline{(f,t)}$. Then the equalities $\omega t = \omega' t$ and $\gamma(\omega f) = \gamma'(\omega' f)$ follow. The first one implies $\omega = \omega'$ since $t\in I(\Omega)$, and then the second one implies $\gamma=\gamma'$ since $\omega f\in I(\Gamma)$.

Now we verify that ι is injective. Suppose that $(f,t),(f',t')\in K\wr T$ such that $\overline{(f,t)} = \overline{(f',t')}$. Thus we see that $\operatorname{dom}\overline{(f,t)} = \operatorname{dom}\overline{(f',t')}$, whence it follows by (2.1) that $\operatorname{dom}t = \operatorname{dom}t'$ and $\operatorname{dom}(\omega f) = \operatorname{dom}(\omega f')$ for any $\omega\in\operatorname{dom}f = \operatorname{dom}t = \operatorname{dom}t' = \operatorname{dom}f'$. Morevoer, we have $(\gamma,\omega)\overline{(f,t)} = (\gamma,\omega)\overline{(f',t')}$ for every $(\gamma,\omega)\in\operatorname{dom}\overline{(f,t)} = \operatorname{dom}\overline{(f',t')}$, and so we obtain by (2.2) that $\omega t = \omega t'$ and $\gamma(\omega f) = \gamma(\omega f')$ for each $\omega\in\operatorname{dom}t = \operatorname{dom}t'$ and $\gamma\in\operatorname{dom}(\omega f) = \operatorname{dom}(\omega f')$. These equalities imply $t = t'$ and $f = f'$ as it was to be verified.

Finally, we show that ι is a homomorphism. We have to check that, for any $(f,t),(g.u)\in K\wr T$, we have

$$\operatorname{dom}\big(\overline{(f,t)}\cdot\overline{(g,u)}\big) = \operatorname{dom}\overline{(f\oplus tg, tu)}, \tag{2.3}$$

and the equality

$$(\gamma,\omega)\big(\overline{(f,t)}\cdot\overline{(g,u)}\big) = (\gamma,\omega)\overline{(f\oplus tg, tu)},$$

holds for any element (γ,ω) of the common domain in (2.3). In order to determine $\operatorname{dom}\big(\overline{(f,t)}\cdot\overline{(g,u)}\big)$, observe that

$$\operatorname{im}\overline{(f,t)} = \{(\gamma(\omega f),\omega t) : \omega\in\operatorname{dom}t = \operatorname{dom}f,\ \gamma\in\operatorname{dom}(\omega f)\},$$

and that such an element belongs to $\operatorname{dom}\overline{(g,u)}$ if and only if

$$\omega t\in\operatorname{dom}u = \operatorname{dom}g \quad\text{and}\quad \gamma(\omega f)\in\operatorname{dom}((\omega t)g).$$

The first relation in conjunction with $\omega\in\operatorname{dom}t = \operatorname{dom}f$ is equivalent to requiring that $\omega\in\operatorname{dom}(tu) = \operatorname{dom}(f\oplus tg)$, and the second relation together with $\gamma\in\operatorname{dom}(\omega f)$ is equivalent to the relation

$$\gamma\in\big(\operatorname{im}(\omega f)\cap\operatorname{dom}((\omega t)g)\big)(\omega f)^{-1} = \operatorname{dom}(\omega f\cdot\omega(tg)) = \operatorname{dom}(\omega(f\oplus tg)).$$

Applying (2.1), we obtain that (2.3) holds. If (γ,ω) belongs to the common domain in (2.3) then, by definition, we have

$$\begin{aligned}(\gamma,\omega)\big(\overline{(f,t)}\cdot\overline{(g,u)}\big) &= (\gamma(\omega f),\omega t)\overline{(g,u)} = \big((\gamma(\omega f))((\omega t)g),(\omega t)u\big)\\ &= \big(\gamma(\omega f\cdot\omega(tg)),\omega(tu)\big) = (\gamma,\omega)\overline{(f\oplus tg, tu)},\end{aligned}$$

thus proving that ι is, indeed, an embedding.

This implies that $K \wr T$ is isomorphic to a subsemigroup of $I(\Gamma \times \Omega)$. In order to verify that $K \wr T$ is an inverse subsemigroup, it suffices to check that $K \wr T$ is regular. This can be easily seen as follows. For every $(f,t) \in K \wr T$, we have $\mathrm{dom}(t^{-1}f^{-1}) = \mathrm{dom}(t^{-1}f) = \mathrm{dom}(t^{-1}t)$ whence $(t^{-1}f^{-1}, t^{-1}) \in K \wr T$. Moreover, $(f,t)(t^{-1}f^{-1}, t^{-1})(f,t) = (f,t)$ since $tt^{-1} = 1_{\mathrm{dom}\, t}$. For completeness, note that $(t^{-1}f^{-1}, t^{-1}) = (f,t)^{-1}$ for every $(f,t) \in K \wr T$. □

We summarize the results of the section.

Theorem 2.2. (1) *For every inverse semigroups K and T of partial bijections on sets Γ and Ω, respectively, the wreath product $K \wr T$ is an inverse semigroup isomorphic to an inverse semigroup of partial bijections on the set $\Gamma \times \Omega$.*

(2) *In particular, if K and T are permutation groups on Γ and Ω, respectively, then $K \wr T$ is just the usual wreath product of K by T.*

(3) *For arbitrary inverse semigroups K and T, Houghton's wreath product $K\,\mathrm{Wr}^H T$ is isomorphic to the wreath product of the Wagner–Preston representations of K and T.*

3. Embedding of extensions into products

In this section we present that wreath product, Houghton's wreath product, λ-semidirect and λ-wreath products are equivalent to each other from the point of view of which extensions of inverse semigroups are embeddable in them.

We have seen in Section 1 that both Houghton's wreath products and λ-semidirect (λ-wreath) products can be considered extensions of inverse semigroups in a natural way. The same idea applies for wreath products introduced in the previous section. Consider the wreath product $K \wr T$ of inverse semigroups K and T. Notice that the kernel of the congruence Θ_2 induced by the second projection is isomorphic to an inverse subsemigroup K' of $F_{K,T}$. We call $(K \wr T, \Theta_2)$ the *wreath product extension* of K' by T.

First we observe that, in some sense, the kernel of the congruence induced by the second projection of each product considered so far is close to the first factor.

Proposition 3.1. *Consider arbitrary inverse semigroups K, T and a variety $\mathbf{V}$ of inverse semigroups such that $K \in \mathbf{V}$, and $\mathbf{V}$ contains a nontrivial semilattice provided T is not a group. Let $K \star T$ denote any of the wreath product, Houghton's wreath product, a λ-semidirect and the λ-wreath products of K by T. Then $(K \star T, \Theta_2)$ is an extension of an inverse semigroup belonging to $\mathbf{V}$ by T.*

Proof. The statement for the wreath product follows from the description of F in the previous section and from the relation $K' \leq F$. This implies the statement for Houghton's wreath product by Theorem 2.2(3). Finally, the statements for λ-semidirect and λ-wreath products are direct consequences of the following facts. The λ-wreath product $K\,\mathrm{Wr}^\lambda T$ is a λ-semidirect product of the direct power K^T by T, and a λ-semidirect product extension of K by T is an extension of K' by T where K' is an inverse subsemigroup of a direct product of K and the semilattice $E(T)$. □

Now we establish that wreath product extensions are embeddable in λ-semidirect product extensions, and λ-semidirect product extensions in Houghton's wreath product extensions.

Consider the wreath product $K \wr T$ of inverse semigroups $K \leq I(\Gamma)$ and $T \leq I(\Omega)$. It is routine to check that the usual multiplication of elements of $F = F_{K,T}$ on the left by elements of T defines an action of T on F. This defines the λ-semidirect product $F *^\lambda T$ which is an inverse semigroup. By definition, a pair $(f,t) \in F \times T$ belongs to $F *^\lambda T$ if and only if $tt^{-1}f = f$. Since $tt^{-1} = 1_{\operatorname{dom} t}$, this equality holds for every $(f,t) \in K \wr T$, and so $K \wr T$ is a subsemigroup in $F *^\lambda T$. This verifies the following statement.

Proposition 3.2. *For every inverse semigroups K and T, the wreath product $K \wr T$ is an inverse subsemigroup of the λ-semidirect product $F_{K,T} *^\lambda T$ where the action of T on F is defined by multiplication on the left.*

Conversely, we embed any λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ of inverse semigroups K and T into Houghton's wreath product extension $(K \operatorname{Wr}^H T, \Theta_2)$. For any $(a,t) \in K *^\lambda T$, define $h_{(a,t)} \in H$ such that $\operatorname{dom} h_{(a,t)} = Tt^{-1}$ and $xh_{(a,t)} = {}^x a$ for every $x \in Tt^{-1}$. Consider the map $\varphi\colon K *^\lambda T \to K \operatorname{Wr}^H T,\ (a,t) \mapsto \big(h_{(a,t)}, t\big)$. In order to check that φ is injective, let $(a,t),(b,u) \in K *^\lambda T$ such that $\big(h_{(a,t)}, t\big) = \big(h_{(b,u)}, u\big)$. Then $t = u$ and ${}^x a = {}^x b$ for every $x \in Tt^{-1}$. In particular, if $x = tt^{-1}$ then the latter equality implies $a = {}^{tt^{-1}}a = {}^{tt^{-1}}b = b$. Hence φ is, indeed, injective. Now we show that φ is a homomorphism. We have to check that, for any $(a,t),(b,u) \in K *^\lambda T$, we have $h_{(a \cdot {}^t b, tu)} = h_{(a,t)} \oplus {}^t h_{(b,u)}$. By definition, we have $\operatorname{dom} h_{(a\cdot {}^t b, tu)} = T(tu)^{-1}$, $\operatorname{dom} h_{(a,t)} = Tt^{-1}$ and $\operatorname{dom} {}^t h_{(b,u)} = \big(\operatorname{dom} h_{(b,u)}\big)t^{-1} = Tu^{-1}t^{-1} = T(tu)^{-1}$, and so $\operatorname{dom} h_{(a\cdot {}^t b, tu)} = \operatorname{dom} \big(h_{(a,t)} \oplus {}^t h_{(b,u)}\big)$. If x is an element of this common domain then we have

$$\begin{aligned} x\big(h_{(a,t)} \oplus {}^t h_{(b,u)}\big) &= xh_{(a,t)} \cdot (xt)h_{(b,u)} = (x(tu)(tu)^{-1})h_{(a,t)} \cdot (xt)h_{(b,u)} \\ &= {}^{x(tu)(tu)^{-1}}a \cdot {}^{xt}b = {}^{x}\big({}^{(tu)(tu)^{-1}}a \cdot {}^t b\big) = xh_{(a\cdot {}^t b, tu)}. \end{aligned}$$

Thus we have established that φ is an embedding of $(K *^\lambda T, \Theta_2)$ into $(K \operatorname{Wr}^H T, \Theta_2)$ which implies the following proposition.

Proposition 3.3. *For every inverse semigroups K and T where T acts on K by endomorphisms, the λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ is embeddable in the Houghton's wreath product extension $(K \operatorname{Wr}^H T, \Theta_2)$.*

A λ-wreath product $K \operatorname{Wr}^\lambda T$ is, by definition, a λ-semidirect product $F *^\lambda T$, and conversely, an argument similar to but easier than the previous one verifies that each λ-semidirect product extension $(K *^\lambda T, \Theta_2)$ is embeddable also in the λ-wreath product extension $(K \operatorname{Wr}^\lambda T, \Theta_2)$.

Combining this observation with Theorem 2.2(3) and Propositions 3.1, 3.2, 3.3, we deduce the main result of this section.

Theorem 3.4. *Let S be an inverse semigroup and θ a congruence on S. Consider a variety $\mathbf{V}$ of inverse semigroups such that the kernel of θ belongs to $\mathbf{V}$, and $\mathbf{V}$ contains a nontrivial semilattice provided S/θ is not a group. The following statements are equivalent.*

(1) *The extension (S,θ) is embeddable in a wreath product extension of a member of* $\mathbf{V}$ *by an inverse semigroup of partial bijections isomorphic to S/θ.*
(2) *The extension (S,θ) is embeddable in Houghton's wreath product extension of a member of* $\mathbf{V}$ *by S/θ.*
(3) *The extension (S,θ) is embeddable in a λ-wreath product extension of a member of* $\mathbf{V}$ *by S/θ.*
(4) *The extension (S,θ) is embeddable in a λ-semidirect product extension of a member of* $\mathbf{V}$ *by S/θ.*

Note that the equivalence of statements Theorem 3.4(2) and (4) is mentioned without proof in [5, p. 239].

References

[1] B. Billhardt, On a wreath product embedding and idempotent pure congruences on inverse semigroups, *Semigroup Forum* **45** (1992) 45–54.
[2] C. H. Houghton, Embedding inverse semigroups in wreath products, *Glasgow Math. J.* **17** (1976) 77–82.
[3] M. Kambites, Anisimov's Theorem for inverse semigroups, *Internat. J. Algebra Comput.* **26** (2015), 41–49.
[4] M. V. Lawson, *Inverse semigroups: The Theory of Partial Symmetries*, World Scientific, Singapore, 1998.
[5] M. B. Szendrei, Regular semigroups and semidirect products, *Semigroups, Automata and Languages (Proc. Conf. Porto, 1994)*, World Scientific, Singapore, 1996; 233–246.

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, Hungary, H-6720
E-mail address: m.szendrei@math.u-szeged.hu