EMBEDDING INTO ALMOST LEFT FACTORIZABLE RESTRICTION SEMIGROUPS

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Abstract. The notion of almost left factorizability and the results on almost left factorizable weakly ample semigroups, due to Gomes and the author, are adapted for restriction semigroups. The main result of the paper is that each restriction semigroup is embeddable into an almost left factorizable restriction semigroup. This generalizes a fundamental result of the structure theory of inverse semigroups.

1. Introduction

Each inverse semigroup can be obtained from a semidirect product of a semilattice by a group in two different ways, namely, by taking (a) an (idempotent separating) homomorphic image of an inverse subsemigroup, and (b) an inverse subsemigroup of an (idempotent separating) homomorphic image of such a semidirect product ([11], [12], [13], [14], [9]). The members of the intermediate classes of these approaches are just the E-unitary and the almost factorizable inverse semigroups, respectively. These subclasses are well studied, and are playing a central role in the structure theory of inverse semigroups ([10]). These results have been generalized, at least partly, in a number of directions, for example, for orthodox, locally inverse, weakly (left) ample semigroups ([15], [8], [1], [16], [2], [4], [3]). The aim of this paper is to generalize approach (b) for restriction semigroups.

The algebraic structures we call in this paper left restriction semigroups have been studied from various points of view, and under different names, since the 1960's. The notion of a restriction semigroup is a two-sided version of a left restriction semigroup. For a historical overview, and for a more complete introduction in the basic properties of these structures than it is provided here, the reader is referred to [3] and [6]. Note that the name '(left) restriction' is fairly recent, such a structure was formerly often called a 'weakly (left) E-ample semigroup'.

A restriction semigroup is a semigroup equipped with two additional unary operations which satisfy certain identites. Thus, from universal algebraic point of view, a restriction semigroup is an algebra of type $(2, 1, 1)$. Among others, the defining identities imply that both unary operations assign an idempotent to any element, and the ranges of the two unary operations coincide. This common range is the set of projections. In particular,

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each inverse semigroup naturally determines a restriction semigroup if we consider the unary operations which assign the idempotents aa^{-1} and $a^{-1}a$, respectively, to any element a. Thus restriction semigroups are non-regular generalizations of inverse semigroups.

Notions of factorizability and almost factorizability, more precisely, left factorizability and almost left factorizability are introduced in [5] for weakly ample semigroups in a way similar to the usual notions of factorizability and almost factorizability, respectively, for inverse semigroups. Moreover, the fundamental results known on the structure of factorizable inverse monoids and almost factorizable inverse semigroups are generalized for the weakly ample case. The construction taking over the role of a semidirect product of a semilattice by a group is a construction from [2] which we call a W-product of a semilattice by a monoid. It turns out that these results can be easily adapted to the more general class of restriction semigroups (Section 3).

The main result of the paper establishes that each restriction semigroup is embeddable in an almost left factorizable restriction semigroup (Section 4). Combining this with the main theorem of Section 3, we deduce that each restriction semigroup can be obtained as a $(2,1,1)$-subsemigroup of a (projection separating) homomorphic image of a W-product of a semilattice by a monoid. This generalizes approach (b) from inverse semigroups to restriction semigroups. Note that we prove in [17] that each restriction semigroup has a proper cover which is $(2,1,1)$-embeddable into a W-product of a semilattice by a monoid, which extends approach (a) to restriction semigroups.

2. Preliminaries

In the first part of this section we provide the notions and basic facts on restriction semigroups we need in the paper. For the proofs of these facts and for more details, the reader is referred to [6]. In the second part we recall the definition of a W-product, and establish its most important properties. In the third part we recall several basic universal algebraic notions. For details, see [7]. Finally, based on [17], we present a model of the free restriction semigroup on a set X together with an embedding into a W-product.

A *left restriction semigroup* is defined to be an algebra of type $(2,1)$, more precisely, an algebra $S = (S;\cdot,{}^+)$ where $(S;\cdot)$ is a semigroup and ${}^+$ is a unary operation such that the following identities are satisfied:

$$x^+x = x, \quad x^+y^+ = y^+x^+, \quad (x^+y)^+ = x^+y^+, \quad xy^+ = (xy)^+x. \tag{2.1}$$

For our later convenience, we list several consequences of these identities which, together with the defining identities (2.1), are used throughout the paper without further reference:

$$x^+x^+ = x^+, \quad (x^+)^+ = x^+, \quad x^+(xy)^+ = (xy)^+, \tag{2.2}$$

$$(x^+y^+)^+ = x^+y^+, \quad (xy)^+ = (xy^+)^+. \tag{2.3}$$

A *right restriction semigroup* is defined dually, that is, it is an algebra $S = (S;\cdot,{}^*)$ satisfying the duals of the identities (2.1). Finally, if $S = (S;\cdot,{}^+,{}^*)$ is an algebra of type $(2,1,1)$ where $S = (S;\cdot,{}^+)$ is a left restriction semigroup, $S = (S;\cdot,{}^*)$ is a right restriction semigroup and the identities

$$(x^+)^* = x^+, \quad (x^*)^+ = x^* \tag{2.4}$$

hold then it is called a *restriction semigroup*. Notice that the defining properties of a restriction semigroup are left-right dual. Therefore later on the dual definitions and statements will not be explicitely formulated.

Among the restriction semigroups, the notions of a subalgebra, homomorphism, congruence and factor algebra are understood in type $(2,1,1)$. In order to emphasize this, we use the expressions $(2,1,1)$-subsemigroup, $(2,1,1)$-morphism, $(2,1,1)$-congruence and $(2,1,1)$-factor semigroup, respectively.

If a restriction semigroup S has an identity element 1 with respect to the multiplication then it is straightforward to see by (2.1) that

$$1^+ = 1^* = 1. \tag{2.5}$$

Such a restriction semigroup is called a *restriction monoid.*

If S is a restriction semigroup having no identity element with respect to the multiplication then the semigroup S^1 can be easily formed to a restriction monoid by extending the nullary operations by the rule (2.5). This restriction monoid will also be denoted by S^1. As usual, the same notation will be used for S itself if S is a restriction monoid.

The class of restriction semigroups is fairly big. For example, each inverse semigroup $S_{\text{inv}} = (S;\cdot,{}^{-1})$ determines a restriction semigroup $S = (S;\cdot,{}^+,{}^*)$ where the unary operations are defined by the rules

$$a^+ = aa^{-1} \quad \text{and} \quad a^* = a^{-1}a \quad \text{for every } a \in S.$$

By the Wagner–Preston theorem, such a restriction semigroup is, up to $(2,1,1)$-isomorphism, a $(2,1,1)$-subsemigroup of $(I(X);\cdot,{}^+,{}^*)$ for some set X, where $I(X)$ is the set of all partial bijections on X, and

$$\alpha^+ = 1_{\operatorname{dom}\alpha} \quad \text{and} \quad \alpha^* = 1_{\operatorname{im}\alpha} \quad \text{for every } \alpha \in I(X).$$

On the other hand, each monoid M becomes a restriction semigroup by defining $a^+ = a^* = 1$ for any $a \in M$. It is easy to see that these restriction semigroups are just those with both unary operations being constant. Such a restriction semigroup will be called *unary trivial*, and, since it is necessarily a monoid, we also call it a unary trivial restriction monoid. Notice that the congruences, homomorphisms, etc. of monoids and the $(2,1,1)$-congruences, $(2,1,1)$-morphisms of the unary trivial restriction semigroups (monoids) obtained from them coincide. Therefore we often consider unary trivial restriction semigroups just as monoids, and vice versa. Note that, in the literature, unary trivial restriction semigroups (monoids) are sometimes called reduced restriction semigroups (monoids).

Let S be any restriction semigroup. By (2.4), we have $\{x^+ : x \in S\} = \{x^* : x \in S\}$. This set is called *the set of projections of* S, and is denoted by $P(S)$. By (2.1)–(2.3) and their duals, $P(S)$ is a $(2,1,1)$-subsemilattice in S, and both unary operations are identical on it. In particular $P(S)$ consists of idempotent elements only. Notice that a restriction semigroup S is unary trivial if and only if $P(S)$ is a singleton, and, if this is the case then the unique element of $P(S)$ is the identity element of S. If S,T are restriction semigroups, and $\phi\colon S \to T$ is a $(2,1,1)$-morphism then ϕ is said to be *projection separating* if $e\phi = f\phi$ implies $e = f$ for every $e, f \in P(S)$.

Given a restriction semigroup S, we define a relation $\leq$ on S such that, for every $a, b \in S$,

$$a \leq b \quad \text{if and only if} \quad a = a^+b.$$

Observe that the dual of this relation is the same since $a = a^+b$ implies $a = b(a^+b)^* = ba^*$, and the dual implication is also valid. The relation $\leq$ is a compatible partial order on S, and it extends the natural partial order of the semilattice $P(S)$. It is called the *natural partial order on* S.

We also consider a relation on S, denoted by σ_S, or simply σ: for any $a, b \in S$, let

$$a\sigma b \quad \text{if and only if} \quad ea = eb \quad \text{for some} \quad e \in P(S).$$

Again notice that if there exists $e \in P(S)$ with $ea = eb$ then there exists also $f \in P(S)$ with $af = bf$, and conversely. Therefore the relation defined dually to σ coincides with σ. The relation σ is the least congruence on $S = (S; \cdot)$ where $P(S)$ is in a congruence class, which we denote by $P(S)\sigma$. Consequently, it is the least $(2,1,1)$-congruence ρ on $S = (S; \cdot, {}^+, {}^*)$ such that the $(2,1,1)$-factor semigroup S/ρ is unary trivial. Therefore we call σ *the least unary trivial* $(2,1,1)$*-congruence on* S. Obviously, $P(S)\sigma$ is the identity element of S/σ. The unary trivial restriction monoid S/σ is often considered just as a monoid $S/\sigma = (S/\sigma; \cdot, P(S)\sigma)$. In particular, S is a *weakly ample semigroup* if it is a restriction semigroup and $P(S) = E(S)$, the set of all idempotents of S. Consequently, if S is a weakly ample semigroup then S/σ is a *unipotent monoid*, that is a monoid where the identity element is the unique idempotent. In fact, the relation σ is the *least unipotent monoid congruence* on S.

Now we give an alternative definition for (left) restriction semigroups. Let S be a semigroup, and let E be a subsemilattice in S, whence clearly $E \subseteq E(S)$. The relation $\widetilde{\mathcal{R}}_E$ is defined on S by the rule that, for any $a, b \in S$,

$$a \;\widetilde{\mathcal{R}}_E\; b \quad \text{if and only if} \quad ea = a \Longleftrightarrow eb = b \text{ for any } e \in E.$$

The semigroup S is said to be *weakly left E-ample* if the following conditions are fulfilled: the relation $\widetilde{\mathcal{R}}_E$ is a left congruence on S; for each $a \in S$, the $\widetilde{\mathcal{R}}_E$-class $\widetilde{R}^E_a(S)$ of S containing a has an idempotent (necessarily unique) denoted by a^+; and for all $a \in S$ and $e \in E$, we have $ae = (ae)^+a$. A *weakly right E-ample semigroup* is defined dually, in which case, the relation taking over the role of $\widetilde{\mathcal{R}}_E$ is denoted by $\widetilde{\mathcal{L}}_E$, and the unique idempotent in the $\widetilde{\mathcal{L}}_E$-class $\widetilde{L}^E_a(S)$ of an element a by a^*. A *weakly E-ample semigroup S* is a semigroup that is both weakly left and weakly right E-ample (with the same semilattice E).

It is well known that a semigroup S is weakly (left) E-ample with respect to the semilattice E in S if and only if $(S; \cdot, {}^+, {}^*)$ is a (left) restriction semigroup with $P(S) = E$. Later on, when considering the relations $\widetilde{\mathcal{R}}_{P(S)}, \widetilde{\mathcal{L}}_{P(S)}$ on a restriction semigroup S, we simply write $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$, and analogously, $\widetilde{R}_a(S)$ and $\widetilde{L}_a(S)$. In the literature, the latter notation is used only for weakly ample semigroups, but this will not cause any confusion. Note that

$a \mathrel{\widetilde{\mathcal{R}}} b$ if and only if $a^+ = b^+$, and, dually $a \mathrel{\widetilde{\mathcal{L}}} b$ if and only if $a^* = b^*$. It is natural to introduce the relation $\widetilde{\mathcal{H}} = \widetilde{\mathcal{R}} \cap \widetilde{\mathcal{L}}$.

A *left ample semigroup* is defined to be a restriction semigroup S where $\widetilde{\mathcal{R}}$ coincides with the relation defined, for any $a, b \in S$, by

$$a \mathrel{\mathcal{R}^*} b \quad \text{if and only if} \quad xa = ya \iff xb = yb \text{ for all } x, y \in S^1.$$

A *right ample semigroup* is defined dually, and by an *ample semigroup* we mean a restriction semigroup that is both left and right ample. On a (left) ample semigroup, the relation σ is the *least (right) cancellative congruence.*

A restriction semigroup S is said to be *proper* if the following condition and its dual are fulfilled:

$$a^+ = b^+ \text{ and } a \mathrel{\sigma} b \quad \text{imply} \quad a = b \quad \text{for every } a, b \in S.$$

It is worth mentioning that $\leq$ generalizes the natural partial order of an inverse semigroup, the relation σ generalizes the least group congruence on an inverse semigroup, and the notion of a proper restriction semigroup generalizes that of an E-unitary inverse semigroup. For, if a restriction semigroup S is obtained form an inverse semigroup $S_{\text{inv}} = (S; \cdot, ^{-1})$ as above then the natural partial orders of S and S_{inv} coincide, σ_S is the least group congruence on S_{inv}, and S is proper if and only if S_{inv} is E-unitary.

The construction of $W(T, Y)$ with Y being a semilattice and T a right cancellative monoid was introduced in [2] as a construction of a left ample semigroup. In [5], it was generalized for any unipotent monoid T, and it was noticed that there is a natural unary operation * on $W(T, Y)$, so that it becomes a weakly ample semigroup. If we drop the requirement for T being unipotent, then $W(T, Y)$ is immediately seen to be a restriction semigroup. Since unipotency of T is applied only for deducing that $P(W(T, Y))$ contains all the idempotents of $W(T, Y)$, the main properties presented in [5] remain valid in this more general setting.

Let T be a monoid and Y a semilattice. We say that T acts on Y on the right [left] if a monoid homomorphism is given from T into the endomorphism monoid $\operatorname{End} Y$ of Y [into the dual $\operatorname{End}^d Y$ of the endomorphism monoid of Y]. For brevity, a^t [$^t a$] is used to denote the image of the element $a \in Y$ under the endomorphism assigned to the element $t \in T$. It is well known that the fact that T acts on Y on the right is equivalent to requiring that the equalities

$$(ab)^t = a^t b^t, \quad (a^t)^u = a^{tu}, \quad a^1 = a \tag{2.6}$$

are valid for every $a, b \in Y$ and $t, u \in T$. Suppose that T acts on the right on Y by injective endomorphisms such that the range of each endomorphism corresponding to an element of T forms an order ideal in Y. Equivalently, suppose that, additionally to (2.6), we have

$$a^t = b^t \quad \text{implies} \quad a = b, \tag{2.7}$$

and

$$a \leq b^t \quad \text{implies} \quad a = c^t \text{ for some } c \in Y \tag{2.8}$$

for every $a, b \in Y$ and $t \in T$. Consider the set

$$W(T, Y) = \{(t, a^t) \in T \times Y : a \in Y,\ t \in T\},$$

and define a multiplication and two unary operations on it by the following rules: for any $(t, a^t), (u, b^u) \in W(T, Y)$, put

$$(2.9)\qquad \begin{aligned} (t, a^t)(u, b^u) &= (tu, a^{tu} \cdot b^u), \\ (t, a^t)^+ &= (1, a), \\ (t, a^t)^* &= (1, a^t). \end{aligned}$$

It is straightforward to see that $W(T, Y)$ is a subsemigroup in the reverse semidirect product $T \ltimes Y$. Moreover, this construction has the following basic properties.

Result 2.1. *Let Y be a semilattice and T a monoid acting on Y on the right, so that conditions (2.7), (2.8) are fulfilled.*

1. *$W(T, Y) = (W(T, Y); \cdot, {}^+, {}^*)$ is a restriction semigroup, and its set of projections is $P(W(T, Y)) = \{(1, a) : a \in Y\}$, which is isomorphic to Y.*
2. *The first projection $\pi \colon W(T, Y) \to T$ is a surjective homomorphism whose kernel is σ. Consequently, $W(T, Y)/\sigma$ is isomorphic to T.*
3. *$W(T, Y)$ is proper.*
4. *$W(T, Y)$ is a monoid if and only if Y has an identity.*

Remark 2.2. In particular, if Y is a semilattice and T a monoid acting on Y on the right by automorphisms then conditions (2.7), (2.8) are satisfied, and $W(T, Y) = T \ltimes Y$.

If Y is a semilattice and T a monoid acting on Y on the right, so that conditions (2.7), (2.8) are satisfied, then the restriction semigroup $W(T, Y)$ is called a *W-product of Y by T*. If $W(T, Y)$ has no identity, that is, by the last statement of Result 2.1, Y has no identity, then we find it convenient to interpret the adjoint identity element in the form $\epsilon = (1, e^1) = (1, e)$ where e is an identity adjoint to Y, and to extend the action of T on Y to an action of T on Y^e by defining $e^t = e$ for every $t \in T$. For, in this case, the equalities (2.9) remain valid for $W^\epsilon(T, Y) = W(T, Y) \cup \{\epsilon\}$. However, let us call the attention to the fact that the extended action need not have property (2.8), so $W^\epsilon(T, Y)$ cannot be considered as a $(2, 1, 1)$-subsemigroup in a W-product of Y^e by T.

Now we recall several definitions and facts of general algebraic nature, but we formulate them only for restriction semigroups; more precisely, for algebras of type $(2, 1, 1)$ with a binary operation $\cdot$ and two unary operations $^+$ and *.

Given a set X of variables, by a *term in X* we mean a formal expression built up from the elements of X by means of the operational symbols — the binary operational symbol $\cdot$ and the unary operational symbols $^+$ and * — in finitely many steps. For example, the left and right hand sides of the equalities in (2.1)–(2.3) are terms in variables x, y. If we work with an associative binary operation then we delete the unnecessary parentheses from the terms. If S is a restriction semigroup then we introduce a nullary operational symbol for every element s in S, and, for simplicity, denote it also by s. By a *polynomial of S* we mean an expression obtained in a way similar to terms, but from variables and these nullary operational symbols. A polynomial can also be interpreted in the way that such nullary operational symbols are substituted for certain variables in a term. For simplicity, later on we just say that elements of S are substituted for the variables. As it is usual for semigroups, we allow to substitute also $1 \in S^1$ for several, but not all, variables to indicate that the variables in question be deleted from the term. For example, if 1 is substituted for the variable y in the terms yxz and $zy^*(x^*y)^+$ then the terms obtained are xz and $z(x^*)^+$, respectively. A *unary polynomial of* S is a polynomial with at most one variable. Their set is denoted by $\mathcal{P}_1(S)$.

If $\mathbf{t} = \mathbf{t}(x_1, x_2, \ldots, x_n)$ is a term or $p = p(x_1, x_2, \ldots, x_n)$ is a polynomial in the variables $x_1, x_2, \ldots, x_n$, and we substitute elements $s_1, s_2, \ldots, s_n$ of S^1 with $\{s_1, s_2, \ldots, s_n\} \cap S \neq \emptyset$ for the variables, then we can evaluate the expression so obtained in S^1. The result is an element of S which is denoted by $\mathbf{t}^S(s_1, s_2, \ldots, s_n)$ and $p^S(s_1, s_2, \ldots, s_n)$, respectively. Notice that the evaluation is compatible with the interpretation of the substitution of $1 \in S^1$ for variables. The *polynomial function* of S corresponding to the polynomial p is the mapping $p^S\colon S^n \to S$, $(s_1, s_2, \ldots, s_n) \mapsto p^S(s_1, s_2, \ldots, s_n)$, which we denote also by $p^S(x_1, x_2, \ldots, x_n)$.

An *identity* is a formal equality $\mathbf{t} = \mathbf{u}$ of two terms, considered with a common set of variables. A restriction semigroup *satisfies the identity* $\mathbf{t} = \mathbf{u}$ if $\mathbf{t}^S(s_1, s_2, \ldots, s_n) = \mathbf{u}^S(s_1, s_2, \ldots, s_n)$ for any $s_1, s_2, \ldots, s_n \in S$. A class $\mathcal{V}$ of algebras of a given type is called a *variety* if it is defined by identites. By definition, we immediately see that the class of restriction semigroups forms a variety of algebras of type $(2, 1, 1)$. Note that weakly ample semigroups and ample semigroups form sub-quasivarieties in the variety of restriction semigroups.

Given a class $\mathcal{V}$ of algebras of type $(2, 1, 1)$, a non-empty set X, a member $F \in \mathcal{V}$ and a mapping $f\colon X \to F$, we say that (F, f), or, briefly, F is a free object in $\mathcal{V}$ on X if it possesses the following universal property: for any $V \in \mathcal{V}$ and any mapping $v\colon X \to V$, there is a unique $(2, 1, 1)$-morphism $\phi\colon F \to V$ such that $f\phi = v$. One can prove that F is, up to $(2, 1, 1)$-isomorphism, unique, provided it exists. It is well known that if $\mathcal{V}$ is a variety then there exists a free object in $\mathcal{V}$ on any non-empty set, and each member of $\mathcal{V}$ is a $(2, 1, 1)$-morphic image of a free object. In particular, this ensures that there exists a free restriction semigroup on any non-empty set, and, up to $(2, 1, 1)$-isomorphism, it is uniquely determined. Moreover, each restriction semigroup is a $(2, 1, 1)$-morphic image of a free one.

A transparent model of the free restriction semigroup on X is given in [3] as a full subsemigroup in the free inverse semigroup on X (cf. [10]). Here we need another model obtained from this in [17] which is given as a subsemigroup in a W-product.

Let X be a set, and consider the free monoid X^* and the free group $F\mathcal{G}(X)$ on X. The elements of X^* are said to be *words in* X. The multiplication in X^* is juxtaposition. The identity element of X^* is the *empty word* which we denote by 1. The subset $X^+ = X^* \setminus \{1\}$ forms a subsemigroup in X^*, and X^+ is the free semigroup on X.

The elements of $F\mathcal{G}(X)$ are supposed to be the *reduced words* in $X \cup X^{-1}$. For any word $w \in (X \cup X^{-1})^*$, the reduced form of w is denoted by $\mathrm{red}(w)$. Thus the product of any elements u, v in $F\mathcal{G}(X)$ is $\mathrm{red}(uv)$. Obviously,

X^* is a subsemigroup in $F\mathcal{G}(X)$ but $F\mathcal{G}(X)$ is not a sub(semi)group in $(X \cup X^{-1})^*$.

The *prefix order* $\leq_p$ is a partial order defined on $F\mathcal{G}(X)$ by $u \leq_p v$ if u is a prefix of v, that is, $v = uw(= \mathrm{red}(uw))$ for some $w \in F\mathcal{G}(X)$. If S is a non-empty subset in $F\mathcal{G}(X)$ then $[S]^{\downarrow}$ denotes the order ideal of $(F\mathcal{G}(X); \leq_p)$ generated by S. In particular, $[u]^{\downarrow}$ is the set of all prefixes of the word $u(\in F\mathcal{G}(X))$ including 1 and u. Therefore each order ideal* but $[1]^{\downarrow} = \{1\}$ has at least two elements.

Denote by $\mathcal{Y}$ the set of all finite order ideals of $(F\mathcal{G}(X); \leq_p)$ with at least two elements. For any $v \in F\mathcal{G}(X)$ and any subset $S \subseteq F\mathcal{G}(X)$, define

$$ {}^{v}S = \{\mathrm{red}(vs) : s \in S\}, \tag{2.10} $$

and let $\mathcal{X} = {}^{F\mathcal{G}(X)}\mathcal{Y}$.

It is well known that the Cayley graph $\Gamma F\mathcal{G}(X)$ of $F\mathcal{G}(X)$, as an X-generated group, is a tree, that is, it is a connected graph without cycles. Moreover, $F\mathcal{G}(X)$ acts on $\Gamma F\mathcal{G}(X)$ on the left by the rule ${}^{v}(u, x) = (vu, x)$ for any $v \in F\mathcal{G}(X)$ and any edge (u, x). Recall that there is a natural bijection β from $\mathcal{X}$ onto the set of all finite connected subgraphs of $\Gamma F\mathcal{G}(X)$ containing at least one edge, where $S \in \mathcal{X}$ is the set of vertices of $S\beta$, and this bijection respects the left action of $F\mathcal{G}(X)$, that is, we have $({}^{v}S)\,\beta = {}^{v}(S\beta)$ for every $v \in F\mathcal{G}(X)$ and $S \in \mathcal{X}$. For example, hence it easily follows that, for any $S \in \mathcal{X}$, we have $S \in \mathcal{Y}$ if and only if $1 \in S$. Moreover, this relation makes it transparent that any two elements $R, S \in \mathcal{X}$ have a least upper bound $R \vee S$ in the partially ordered set $\mathcal{X} = (\mathcal{X}; \supseteq)$, and it can be obtained as follows:

$$ R \vee S = {}^{r}\big({}^{r^{-1}}R \cup \big[r^{-1}s\big]^{\downarrow} \cup {}^{r^{-1}}S\big) $$

for every $r \in R$ and $s \in S$. In particular, $R \vee S = R \cup S$ if $R \cap S \neq \emptyset$. Consequently, $\mathcal{X} = (\mathcal{X}; \vee)$ is a semilattice, and it is also easy to see that $F\mathcal{G}(X)$ acts on this semilattice on the left by automorphisms.

The left action of the group $F\mathcal{G}(X)$ on the semilattice $\mathcal{X}$ naturally defines a right action of $F\mathcal{G}(X)$ on $\mathcal{X}$ by the rule

$$ S^{u} = {}^{u^{-1}}S \quad (u \in F\mathcal{G}(X),\ S \in \mathcal{X}). $$

Its restriction to X^* is clearly a right action of the monoid X^* on the semilattice $\mathcal{X}$ by automorphisms.

Consider the subset

$$ \mathcal{Q} = \mathcal{Y}^{X^*} = \{Q \in \mathcal{X} : Q \cap (X^*)^{-1} \neq \emptyset\} $$

in $\mathcal{X}$, where T^{-1} is used to denote the subset $\{t^{-1} : t \in T\}$ in $F\mathcal{G}(X)$ for any $T \subseteq X^*$. Then $\mathcal{Q}$ forms a subsemilattice in $\mathcal{X} = (\mathcal{X}; \vee)$, and the monoid X^* acts on it on the right by injective endomorphisms. Moreover, for each $t \in X^*$, we have

$$ \mathcal{Q}^{t} = \{Q \in \mathcal{X} : Q \cap (X^* t)^{-1} \neq \emptyset\}, $$

and so it is a dual order ideal in $(\mathcal{Q}; \supseteq)$. These properties of the right action of X^* on $\mathcal{Q}$ allow us to define the W-product $W(X^*, \mathcal{Q})$.

*The empty set is not considered an order ideal.

Result 2.3. *The subset*

$$\begin{aligned} F_W\mathcal{RS}(X) &= \{(t, A^t) \in W(X^*, \mathcal{Q}) : A \in \mathcal{Y} \text{ and } t \in A\} \\ &= \{(t, A^t) \in X^* \times \mathcal{Y} : A \in \mathcal{Y}\} \end{aligned}$$

forms a $(2,1,1)$*-subsemigroup in* $W(X^*, \mathcal{Q})$*. Furthermore,* $F_W\mathcal{RS}(X)$ *together with the injective mapping*

$$X \to F_W\mathcal{RS}(X),\ x \mapsto (x, \{1, x^{-1}\})$$

is a free restriction semigroup on X*.*

Following the general observation above, the adjoint identity to $W(X^*, \mathcal{Q})$ is interpreted as $\epsilon = (1, \emptyset)$ where $\emptyset$ is the identity adjoint to $\mathcal{Q}$, and $\emptyset^t = \emptyset$ for every $t \in X^*$.

3. Almost left factorizable restriction semigroups

In this section we introduce the notion of a permissible set in a restriction semigroup, and notice that the set of all permissible sets constitutes a restriction monoid in which the original semigroup is naturally embedded. Similarly to the inverse case, we define the notion of an almost left factorizable restriction semigroup by means of permissible sets. Moreover, we establish the analogues of most results known on the general structure of factorizable inverse monoids and of almost factorizable inverse semigroups. Finally, for completeness, we deal with the left-right symmetric versions of factorizability and almost factorizability, although these results are not needed in the rest of the paper.

The notions and results of this section are slight generalizations of those in [5]. All the proofs there can be easily adapted to get proofs of the results here. Therefore all the proofs are left to the reader.

In what follows, let S be a restriction semigroup. A non-empty subset $A \subseteq S$ is said to be *permissible* if A is an order ideal of S with respect to the partial order $\leq$, and the equalities $a^+b = b^+a$ and $ab^* = ba^*$ are valid for every $a, b \in A$. Denote by $C(S)$ the set of all permissible subsets of S, consider the usual set multiplication on $C(S)$, and define, for any $A \in C(S)$,

$$A^+ = \{a^+ : a \in A\} \quad \text{and} \quad A^* = \{a^* : a \in A\}.$$

Theorem 3.1. *The algebra* $C(S) = (C(S); \cdot, ^+, ^*)$ *is a restriction monoid with identity element* $P(S)$*, where the set of projections is*

$$P(C(S)) = \{A \subseteq P(S) : A \text{ is an order ideal in } P(S)\},$$

and the natural partial order is the set inclusion. The mapping $\tau_S \colon S \to C(S)$*,* $a \mapsto (a]$*, where* $(a]$ *stands for the principal order ideal of* S *generated by* a*, is a* $(2,1,1)$*-embedding of* S *into* $C(S)$*. If* $S = S^1$ *then* τ_S *is also a monoid embedding. Moreover,* $C(S)$ *is (left) ample if and only if* S *is (left) ample.*

Notice that if S is a restriction semigroup obtained from an inverse semigroup S_{inv}, and A is a non-empty subset in S then A is a permissible subset of S if and only if it is a permissible subset of S_{inv}. Thus the restriction monoid $C(S)$ is just that corresponding to the inverse monoid $C(S_{\text{inv}})$.

If S is proper then $C(S)$ has important additional properties.

Theorem 3.2. *The restriction monoid $C(S)$ is proper if and only if S is proper. If this is the case then each $\sigma_{C(S)}$-class has a maximum element, and the monoids S/σ_S and $C(S)/\sigma_{C(S)}$ are isomorphic.*

Corollary 3.3. *Every proper restriction semigroup S can be $(2,1,1)$-embedded in a proper restriction semigroup T where T/σ_T is isomorphic to S/σ_S and each σ_T-class has a maximum element.*

It is easy to verify that if M is a restriction monoid then $\widetilde{R}_1(M)$ is a submonoid in M. In particular, if M is left ample then $\widetilde{R}_1(M)$ is right cancellative. We say that a restriction monoid M is *left factorizable* if $M = P(M)\widetilde{R}_1(M)$. Note that, by the dual of the last identity in (2.1), we have $P(M)\,\widetilde{R}_1(M) = \widetilde{R}_1(M)\,P(M)$. A *right factorizable* restriction monoid is defined dually. Notice that left factorizability and right factorizability are independent properties even for ample monoids, see [5, Example 4.1].

We say that a restriction semigroup S is *almost left factorizable* if, for any $a \in S$, there exists $A \in \widetilde{R}_{P(S)}(C(S))$ such that $a \in A$. We define an *almost right factorizable* restriction semigroup dually. Recall that $\widetilde{R}_{P(S)}(C(S)) = \{A \in C(S) : A^+ = P(S)\}$, and it is not difficult to see that S is almost left factorizable if and only if the equality $S\tau_S = P(S)\tau_S \cdot \widetilde{R}_{P(S)}(C(S))$ holds in $C(S)$. This observation and the following statement justify this definition.

Proposition 3.4. *A restriction monoid is almost left factorizable if and only if it is left factorizable.*

As in the inverse case, the following holds.

Proposition 3.5. *Let M be a left factorizable restriction monoid. Then $M \setminus \widetilde{R}_1(M)$ is an almost left factorizable restriction semigroup.*

However, in the contrary to the inverse case, the reverse statement fails even for proper restriction semigroups which are left ample, see [5, Example 4.6].

The main result on almost left factorizable restriction semigroups is the following.

Theorem 3.6. *For every restriction semigroup S, the following conditions are equivalent:*

(1) *S is almost left factorizable;*
(2) *S is a projection separating $(2,1,1)$-morphic image of a W-product of a semilattice by a monoid;*
(3) *S is a $(2,1,1)$-morphic image of a W-product of a semilattice by a monoid.*

The proper and almost left factorizable restriction semigroups are characterized as follows.

Theorem 3.7. *A restriction semigroup is proper and almost left factorizable if and only if it is $(2,1,1)$-isomorphic to a W-product of a semilattice by a monoid.*

Now we turn to considering the left-right symmetric versions of factorizability and almost factorizability. The analogy with the inverse case is, of course, closer in this case.

In a restriction monoid M, the $\widetilde{\mathcal{H}}$-class $\widetilde{H}_1(M)$ forms a monoid. In particular, if M is ample then it is cancellative. A restriction monoid M is said to be *factorizable* if $M = P(M)\,\widetilde{H}_1(M)$. Clearly, a restriction monoid M obtained from an inverse monoid M_{inv} is factorizable if and only M_{inv} is factorizable in the usual sense.

Proposition 3.8. *A restriction monoid is factorizable if and only if it is both left and right factorizable.*

A restriction semigroup S is said to be *almost factorizable* if, for any $a \in S$, there exists $A \in \widetilde{H}_{P(S)}(C(S))$ such that $a \in A$. Notice that a restriction semigroup S obtained from an inverse semigroup S_{inv} is almost factorizable if and only S_{inv} is almost factorizable in the usual sense.

It is clear that, for any restriction semigroup, almost factorizability implies both almost left and almost right factorizability. The converse is true for proper restriction semigroups.

Proposition 3.9. *A proper restriction semigroup is almost factorizable if and only if it is both almost left and almost right factorizable.*

The connection between the structures of almost factorizable restriction semigroups and factorizable restriction monoids is more intimate than in the one-sided case, and it is analogous to the inverse case.

Proposition 3.10. *A restriction monoid is almost factorizable if and only if it is factorizable.*

Theorem 3.11. *If M is a factorizable restriction monoid then $M \setminus \widetilde{H}_1(M)$ is an almost factorizable restriction semigroup. Conversely, each almost factorizable restriction semigroup is $(2,1,1)$-isomorphic to a restriction semigroup of the form $M \setminus \widetilde{H}_1(M)$ where M is a factorizable restriction monoid.*

The left-right symmetric analogues of Theorems 3.6 and 3.7 are the following, cf. Remark 2.2.

Theorem 3.12. *For every restriction semigroup S, the following conditions are equivalent:*

(1) *S is almost factorizable;*
(2) *S is a projection separating $(2,1,1)$-morphic image of a reverse semidirect product $T \ltimes Y$ where Y is a semilattice and T is a monoid acting on Y on the right by automorphisms;*
(3) *S is a $(2,1,1)$-morphic image of a reverse semidirect product $T \ltimes Y$ where Y is a semilattice and T is a monoid acting on Y on the right by automorphisms.*

Theorem 3.13. *A restriction semigroup is proper and almost factorizable if and only if it is $(2,1,1)$-isomorphic to a reverse semidirect product $T \ltimes Y$ where Y is a semilattice and T is a monoid acting on Y on the right by automorphisms.*

4. Embedding in almost left factorizable restriction semigroups

This section is devoted to proving the main result of the paper:

Theorem 4.1. *Each restriction semigroup is* $(2,1,1)$*-embeddable into an almost left factorizable restriction semigroup.*

Let S be any restriction semigroup. Then, by Result 2.3, there exists a non-empty set X such that S is isomorphic to a $(2,1,1)$-factor semigroup of $F_W\mathcal{RS}(X)$ modulo a $(2,1,1)$-congruence τ. Since $F_W\mathcal{RS}(X)$ is a $(2,1,1)$-subsemigroup in $W(X^*,\mathcal{Q})$, the relation τ generates a $(2,1,1)$-congruence $\tau^\#$ on $W(X^*,\mathcal{Q})$, and the mapping

$$\phi\colon F_W\mathcal{RS}(X)/\tau \to W(X^*,\mathcal{Q})/\tau^\#,\ (t,A^t)\tau \mapsto (t,A^t)\tau^\#$$

is obviously a $(2,1,1)$-morphism. In particular, if the restriction of $\tau^\#$ to $F_W\mathcal{RS}(X)$ coincides with τ then ϕ is a $(2,1,1)$-embedding. On the other hand, $W(X^*,\mathcal{Q})/\tau^\#$ is an almost left factorizable restriction semigroup by Theorem 3.6. Therefore, in order to prove Theorem 4.1, it suffices to show the following statement.

Proposition 4.2. *Let X be a non-empty set, and let τ be a* $(2,1,1)$*-congruence on* $F_W\mathcal{RS}(X)$*. Consider the* $(2,1,1)$*-congruence* $\tau^\#$ *on* $W(X^*,\mathcal{Q})$ *generated by* τ*. Then the restriction of* $\tau^\#$ *to* $F_W\mathcal{RS}(X)$ *is equal to* τ*.*

Now we prepare ourselves for the proof of this proposition by describing the $(2,1,1)$-congruence of a restriction semigroup generated by any given relation τ. For notational convenience, we suppose that τ is symmetric. This is enough for our purposes, and this assumption does not essentially reduce generality. For, if τ is an arbitrary relation then $\tau\cup\tau^{-1}$, where τ^{-1} is the converse of τ, is a symmetric relation, and $\tau^\# = (\tau\cup\tau^{-1})^\#$.

A well-known universal algebraic fact implies the following description.

Lemma 4.3. *Let S be a restriction semigroup and τ a symmetric relation on S. Then, for any $s,t\in S$, we have $s\,\tau^\#\,t$ if and only if $s=t$, or there exists $k\in\mathbb{N}$, there exist elements $c_1,d_1,c_2,d_2,\dots,c_k,d_k\in S$ and unary polynomials $p_1,p_2,\dots,p_k\in\mathcal{P}_1(S)$ such that*

$$c_i\,\tau\,d_i\ (i=1,2,\dots,k), \tag{4.1}$$

and

$$\begin{aligned} s=p_1^S(c_1),\ p_1^S(d_1)=p_2^S(c_2),\ p_2^S(d_2)=p_3^S(c_3),\ \dots,\\ p_{k-2}^S(d_{k-2})=p_{k-1}^S(c_{k-1}),\ p_{k-1}^S(d_{k-1})=p_k^S(c_k),\ p_k^S(d_k)=t. \end{aligned} \tag{4.2}$$

We can simplify this description by choosing the unary polynomials involved in a special way. In the language of restriction semigroups, let us define two sequences of terms in variables $x,y,z,y_0,z_0,\dots,y_n,z_n,\dots$. Let

$$\mathbf{t}_+^{(0)}(x,y_0,z_0)=(y_0xz_0)^+,\quad \mathbf{t}_*^{(0)}(x,y_0,z_0)=(y_0xz_0)^*,$$

and, for every $i\in\mathbb{N}$, let

$$\begin{aligned} \mathbf{t}_+^{(i)}(x,y_0,z_0,\dots,y_{i-2},z_{i-1},y_i) &= \big(y_i\mathbf{t}_*^{(i-1)}(x,y_0,z_0,\dots,y_{i-2},z_{i-1})\big)^+,\\ \mathbf{t}_*^{(i)}(x,y_0,z_0,\dots,z_{i-2},y_{i-1},z_i) &= \big(\mathbf{t}_+^{(i-1)}(x,y_0,z_0,\dots,z_{i-2},y_{i-1})z_i\big)^*. \end{aligned}$$

For convenience, we note that

$$\begin{aligned}
\mathbf{t}_+^{(1)}(x, y_0, z_0, y_1) &= \big(y_1(y_0xz_0)^*\big)^+,\\
\mathbf{t}_*^{(1)}(x, y_0, z_0, z_1) &= \big((y_0xz_0)^+z_1\big)^*,\\
\mathbf{t}_+^{(2)}(x, y_0, z_0, z_1, y_2) &= \big(y_2\big((y_0xz_0)^+z_1\big)^*\big)^+,\\
\mathbf{t}_*^{(2)}(x, y_0, z_0, y_1, z_2) &= \big(\big(y_1(y_0xz_0)^*\big)^+z_2\big)^*.
\end{aligned}$$

Define

$$\mathbf{T}_+^{(i)} = \{\mathbf{t}_+^{(i)}\}, \qquad \mathbf{T}_*^{(i)} = \{\mathbf{t}_*^{(i)}\},$$
$$\mathbf{T}_+ = \{\mathbf{t}_+^{(i)} : i \in \mathbb{N}^0\}, \quad \mathbf{T}_* = \{\mathbf{t}_*^{(i)} : i \in \mathbb{N}^0\},$$

and, finally, put

$$\mathbf{T} = \{y\mathbf{u}z : \mathbf{u} = x \text{ or } \mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*\}.$$

Let $\mathbf{t} \in \mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}$. We call the sequence of all variables but x occurring in $\mathbf{t}$ *the* $\mathbf{t}$*-sequence of variables*, and we denote it $\operatorname{var}\mathbf{t}$. More precisely, let

$$\operatorname{var}\mathbf{t}_+^{(0)} = (y_0, z_0) = \operatorname{var}\mathbf{t}_*^{(0)},$$

and

$$\operatorname{var}\mathbf{t}_+^{(i)} = (\operatorname{var}\mathbf{t}_*^{(i-1)}, y_i), \quad \operatorname{var}\mathbf{t}_*^{(i)} = (\operatorname{var}\mathbf{t}_+^{(i-1)}, z_i)$$

for every $i \in \mathbb{N}$. In particular, we have

$$\operatorname{var}\mathbf{t}_+^{(1)} = (y_0, z_0, y_1), \qquad \operatorname{var}\mathbf{t}_*^{(1)} = (y_0, z_0, z_1),$$
$$\operatorname{var}\mathbf{t}_+^{(2)} = (y_0, z_0, z_1, y_2), \quad \operatorname{var}\mathbf{t}_*^{(2)} = (y_0, z_0, y_1, z_2).$$

Moreover, if $\mathbf{t} = y\mathbf{u}z \in \mathbf{T}$ then let

$$\operatorname{var}\mathbf{t} = \begin{cases} (y, z) & \text{if } \mathbf{u} = x,\\ (y, z, \operatorname{var}\mathbf{u}) & \text{if } \mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*. \end{cases}$$

Observe that the members of the sequence $\operatorname{var}\mathbf{t}$ are pairwise distinct. We will find it convenient to use the notation $\operatorname{var}\mathbf{t}$ to denote also the set of variables occurring in the sequence $\operatorname{var}\mathbf{t}$.

Let S be a restriction semigroup, and, as above, let $\mathbf{t} \in \mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}$. A sequence of elements of S^1 is called *a* $\mathbf{t}$*-sequence in* S^1 if its length is equal to the length of $\operatorname{var}\mathbf{t}$. If α is a $\mathbf{t}$-sequence in S^1 then, by writing $\mathbf{t}(x, \alpha)$ we mean the unary polynomial obtained from $\mathbf{t}$ by substituting each member of α for the respective variable in $\operatorname{var}\mathbf{t}$.

For any subset $\mathbf{X}$ in the set of terms $\mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}$, we define the subset

$$\mathbf{X}(S) = \{\mathbf{t}(x, \alpha) : \mathbf{t} \in \mathbf{X}, \text{ and } \alpha \text{ is a } \mathbf{t}\text{-sequence in } S^1\}$$

of $\mathcal{P}_1(S)$, and denote by $\mathbf{X}^S(S)$ the set of polynomial functions of S corresponding to $\mathbf{X}(S)$. Notice that

$$\mathbf{T}_+^S(S) \cup \mathbf{T}_*^S(S) \subseteq \mathbf{T}^S(S). \tag{4.3}$$

For, if $\mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*$ then $\mathbf{t} = y\mathbf{u}z \in \mathbf{T}$, and, for any $\mathbf{u}$-sequence α in S^1, we have $\mathbf{u}^S(x, \alpha) = \mathbf{t}^S(x, (1, 1, \alpha))$ where $(1, 1, \alpha)$ is obviously a $\mathbf{t}$-sequence in S^1.

Before establishing that Lemma 4.3 remains valid if we require that the polynomials $p_1, p_2, \ldots, p_k$ belong to $\mathbf{T}(S)$, we formulate a lemma needed in the proof and also later on.

Lemma 4.4. *Let $i \in \mathbb{N}^0$. For every $a, b \in S^1$ and every $\mathbf{t}_+^{(i)}$-sequence α in S^1, the polynomial function of S corresponding to the unary polynomial $\big(a\mathbf{t}_+^{(i)}(x,\alpha)b\big)^+$ belongs to $\big(\mathbf{T}_+^{(i)}\big)^S(S)$.*

Proof. Assume that $a, b \in S^1$ and α is a $\mathbf{t}_+^{(i)}$-sequence in S^1. If $i = 0$ then let $\alpha = (a_0, b_0)$, and if $i \in \mathbb{N}$ then let $\alpha = (a_0, b_0, \dots, b_{i-1}, a_i)$. In the latter case, denote $(a_0, b_0, \dots, b_{i-1})$ by β, an recall that it is a $\mathbf{t}_*^{(i-1)}$-sequence in S^1. For brevity, put $q = \big(a\mathbf{t}_+^{(i)}(x,\alpha)b\big)^+$. If $i = 0$ then

$$\begin{aligned} q^S(c) &= \big(a(a_0cb_0)^+b\big)^+ = \big(a(a_0cb_0)^+b^+\big)^+ = \big(ab^+(a_0cb_0)^+\big)^+ \\ &= (ab^+a_0cb_0)^+ = \big(\mathbf{t}_+^{(0)}\big)^S(c, ab^+a_0, b_0), \end{aligned}$$

and $\mathbf{t}_+^{(0)}(x, ab^+a_0, b_0) \in \mathbf{T}_+^{(0)}(S)$. Thus $q^S \in \big(\mathbf{T}_+^{(0)}\big)^S(S)$. If $i \geq 1$ then

$$\begin{aligned} q^S(c) &= \big(a\big(\mathbf{t}_+^{(i)}\big)^S(c,\alpha)b\big)^+ = \Big(a\big(a_i\big(\mathbf{t}_*^{(i-1)}\big)^S(c,\beta)\big)^+b\Big)^+ \\ &= \Big(a\big(a_i\big(\mathbf{t}_*^{(i-1)}\big)^S(c,\beta)\big)^+b^+\Big)^+ = \Big(ab^+\big(a_i\big(\mathbf{t}_*^{(i-1)}\big)^S(c,\beta)\big)^+\Big)^+ \\ &= \big(ab^+a_i\big(\mathbf{t}_*^{(i-1)}\big)^S(c,\beta)\big)^+ = \big(\mathbf{t}_+^{(i)}\big)^S(c,(\beta, ab^+a_i)), \end{aligned}$$

and $\mathbf{t}_+^{(i)}(x,(\beta,ab^+a_i)) \in \mathbf{T}_+^{(i)}(S)$. Hence $q^S \in \big(\mathbf{T}_+^{(i)}\big)^S(S)$ follows. □

Proposition 4.5. *Let S be a restriction semigroup and τ a symmetric relation on S. Then, for any $s, t \in S$, we have $s\,\tau^\#\,t$ if and only if $s = t$, or there exists $k \in \mathbb{N}$, there exist elements $c_1, d_1, c_2, d_2, \dots, c_k, d_k \in S$ and unary polynomials $p_1, p_2, \dots, p_k \in \mathbf{T}(S)$ such that conditions (4.1) and (4.2) are fulfilled.*

Proof. For any $s, t \in S$, define $s\,\overline{\tau}\,t$ if $s = t$, or (4.1) and (4.2) hold for some $k \in \mathbb{N}$, $c_1, d_1, c_2, d_2, \dots, c_k, d_k \in S$ and $p_1, p_2, \dots, p_k \in \mathbf{T}(S)$. Obviously, $\tau \subseteq \overline{\tau}$. The inclusion $\mathbf{T}(S) \subseteq \mathcal{P}_1(S)$ implies $\overline{\tau} \subseteq \tau^\#$ by Lemma 4.3. Thus all we have to show is that $\overline{\tau}$ is a $(2,1,1)$-congruence on S. By definition, $\overline{\tau}$ is clearly an equivalence relation. Assume that $s, t \in S$ with $s\,\overline{\tau}\,t$ and $s \neq t$. Let us choose and fix $k \in \mathbb{N}$, $c_1, d_1, c_2, d_2, \dots, c_k, d_k \in S$ and $p_1, p_2, \dots, p_k \in \mathbf{T}(S)$ such that (4.1) and (4.2) are satisfied.

In order to check that $\overline{\tau}$ is compatible with the multiplication on the left, notice that, for any $u \in S$, we have

$$\begin{aligned} us = up_1^S(c_1),\ up_1^S(d_1) = up_2^S(c_2),\ up_2^S(d_2) = up_3^S(c_3),\ \dots,& \\ up_{k-1}^S(d_{k-1}) = up_k^S(c_k),\ up_k^S(d_k) = ut.& \end{aligned}$$

For each index j, we have $p_j \in \mathbf{T}(S)$, and so $p_j = \mathbf{t}(x,\alpha)$ for some $\mathbf{t} \in \mathbf{T}$ and a $\mathbf{t}$-sequence $\alpha = (a, b, a_0, b_0, \dots)$ in S^1. By definition, $\mathbf{t} = y\mathbf{u}z$ where $\mathbf{u} \in \{x\} \cup \mathbf{T}_+ \cup \mathbf{T}_*$, and $y \notin \operatorname{var}\mathbf{u}$. Hence $up_j^S(c) = \mathbf{t}^S(c,\alpha')$ for every $c \in S$, that is, $up_j^S(x) = \mathbf{t}^S(x,\alpha')$, where $\alpha' = (ua, b, a_0, b_0, \dots)$. Since $\mathbf{t}(x,\alpha') \in \mathbf{T}(S)$, this implies that $us\,\overline{\tau}\,ut$.

In order to show that $\overline{\tau}$ is compatible with the unary operation $^+$, observe that (4.2) implies

$$\begin{aligned} s^+ = \big(p_1^S(c_1)\big)^+,\ \big(p_1^S(d_1)\big)^+ = \big(p_2^S(c_2)\big)^+,\ \big(p_2^S(d_2)\big)^+ = \big(p_3^S(c_3)\big)^+,\ \dots,& \\ \big(p_{k-1}^S(d_{k-1})\big)^+ = \big(p_k^S(c_k)\big)^+,\ \big(p_k^S(d_k)\big)^+ = t^+.& \end{aligned}$$

Similarly to the previous argument, we verify $s^+ \,\overline{\tau}\, t^+$ by establishing that $\big(p_j^+\big)^S \in \mathbf{T}^S(S)$ for every index j. As above, let $p_j = \mathbf{t}(x,\alpha)$ with $\mathbf{t} \in \mathbf{T}$ and a $\mathbf{t}$-sequence $\alpha = (a, b, a_0, b_0, \ldots)$ in S^1, where $\mathbf{t} = y\mathbf{u}z$ for some $\mathbf{u} \in \{x\} \cup \mathbf{T}_+ \cup \mathbf{T}_*$. Notice that if $\mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*$ then $\alpha_0 = (a_0, b_0, \ldots)$ is a $\mathbf{u}$-sequence in S^1, and $\mathbf{u}^S(c, \alpha_0)$ is a projection in S for any $c \in S$.

Case $\mathbf{u} = x$. For every $c \in S$, we have $(p_j^+)^S(c) = (p_j^S(c))^+ = (acb)^+ = \big(\mathbf{t}_+^{(0)}\big)^S(c,\alpha)$, and so $(p_j^+)^S(x) = \big(\mathbf{t}_+^{(0)}\big)^S(x,\alpha)$. Hence $(p_j^+)^S$ belongs to $\mathbf{T}_+^S(S)$, and $\mathbf{T}_+^S(S) \subseteq \mathbf{T}^S(S)$ by (4.3).

Case $\mathbf{u} = \mathbf{t}_+^{(i)}$. We have $p_j^+ = \big(a\mathbf{t}_+^{(i)}(x,\alpha_0)b\big)^+$, and Lemma 4.4 and inclusion (4.3) imply that $(p_j^+)^S \in \mathbf{T}_+^S(S) \subseteq \mathbf{T}^S(S)$.

Case $\mathbf{u} = \mathbf{t}_*^{(i)}$. Now, we have

$$\begin{aligned}(p_j^+)^S(c) &= \big(a\big(\mathbf{t}_*^{(i)}\big)^S(c,\alpha_0)b\big)^+ = \big(a\big(\mathbf{t}_*^{(i)}\big)^S(c,\alpha_0)b^+\big)^+ \\ &= \big(ab^+\big(\mathbf{t}_*^{(i)}\big)^S(c,\alpha_0)\big)^+ = \big(\mathbf{t}_+^{(i+1)}\big)^S(c,(\alpha_0,ab^+)),\end{aligned}$$

where (α_0, ab^+) is a $\mathbf{t}_+^{(i+1)}$-sequence in S^1, and $\big(\mathbf{t}_+^{(i+1)}\big)^S(x,(\alpha_0,ab^+)) \in \mathbf{T}_+^S(S)$ and $\mathbf{T}_+^S(S) \subseteq \mathbf{T}^S(S)$. This ensures $(p_j^+)^S \in \mathbf{T}^S(S)$.

The compatibility of $\overline{\tau}$ with the multiplication on the right and that with the operation * follow dually. □

In particular, if S is $W(X^*, \mathcal{Q})$, then we can impose additional condition on the substitutions in the definition of $\mathbf{T}(S)$. For brevity, denote $W(X^*, \mathcal{Q})$ by W. Furthermore, let us simplify our arguments by agreeing that any element $u \in W^\epsilon$ is supposed to be $u = (\overline{u}, U^{\overline{u}})$ where $\overline{u} \in X^*$ and $U \in \mathcal{Q}^\emptyset$. If $\delta = (d_0, d_1, \ldots, d_s)$ is a sequence in W^ϵ then we write $\overline{\delta}$ for the sequence $(\overline{d_0}, \overline{d_1}, \ldots, \overline{d_s})$,

By a *reduced* $\mathbf{t}_+^{(0)}$*-sequence* [*reduced* $\mathbf{t}_*^{(0)}$*-sequence*] *in* W^ϵ we mean a $\mathbf{t}_+^{(0)}$-sequence [$\mathbf{t}_*^{(0)}$-sequence] (a_0, b_0) in W^ϵ such that $\overline{b_0} = 1$ [$\overline{a_0} = 1$]. If $i \in \mathbb{N}$ then a $\mathbf{t}_+^{(i)}$-sequence $(a_0, b_0, \ldots, a_{i-2}, b_{i-1}, a_i)$ in W^ϵ is called *reduced* if $(a_0, b_0, \ldots, a_{i-2}, b_{i-1})$ is a reduced $\mathbf{t}_*^{(i-1)}$-sequence in W^ϵ, $\overline{a_i} \neq 1$, and the word $\overline{a_i}\,\overline{b_{i-1}}^{-1} \in (X \cup X^{-1})^*$ is reduced. Dually, a $\mathbf{t}_*^{(i)}$-sequence $(a_0, b_0, \ldots, b_{i-2}, a_{i-1}, b_i)$ in W^ϵ is termed *reduced* if $(a_0, b_0, \ldots, b_{i-2}, a_{i-1})$ is a reduced $\mathbf{t}_+^{(i-1)}$-sequence in W^ϵ, $\overline{b_i} \neq 1$, and the word $\overline{a_{i-1}}^{-1}\overline{b_i} \in (X \cup X^{-1})^*$ is reduced.

Now we characterize the reduced $\mathbf{t}_+^{(i)}$-sequences directly. Let $i \in \mathbb{N}$, and consider a $\mathbf{t}_+^{(i)}$-sequence $\alpha = (a_0, b_0, \ldots, a_{i-2}, b_{i-1}, a_i)$ in W^ϵ. Observe that $\alpha = (a_0, b_0, \alpha^-)$ where

$$\alpha^- = \begin{cases} (a_1, b_2, \ldots, a_{i-2}, b_{i-1}, a_i) & \text{if } i \text{ is odd,} \\ (b_1, a_2, \ldots, a_{i-2}, b_{i-1}, a_i) & \text{if } i \text{ is even.}\end{cases}$$

Define a word in X^* corresponding to α by

$$\underline{w}(\alpha) = \begin{cases} \overline{a_0} & \text{if } i \text{ is odd,} \\ \overline{b_0} & \text{if } i \text{ is even,}\end{cases}$$

and another one in $(X \cup X^{-1})^*$ by

$$\mathbf{w}(\alpha) = \begin{cases} \overline{a_i}\,\overline{b_{i-1}}^{-1}\,\overline{a_{i-2}} \dots \overline{b_2}^{-1}\,\overline{a_1}\,\overline{b_0}^{-1} & \text{if } i \text{ is odd,} \\ \overline{a_i}\,\overline{b_{i-1}}^{-1}\,\overline{a_{i-2}} \dots \overline{a_2}\,\overline{b_1}^{-1}\,\overline{a_0} & \text{if } i \text{ is even.} \end{cases}$$

Lemma 4.6. *For every $i \in \mathbb{N}$, a $\mathbf{t}_+^{(i)}$-sequence α in W^ϵ is reduced if and only if $\underline{w}(\alpha) = 1$, the members of $\overline{\alpha^-}$ are in X^+, and $\mathbf{w}(\alpha)$ is a reduced word in $(X \cup X^{-1})^*$.*

A characterization of a reduced $\mathbf{t}_*^{(i)}$-sequence ($i \in \mathbb{N}$) in W^ϵ could be given dually. For example, if $\alpha = (a_0, b_0, \dots, b_{i-2}, a_{i-1}, b_i)$ is a $\mathbf{t}_*^{(i)}$-sequence in W^ϵ then $\mathbf{w}(\alpha)$ would be $\dots \overline{b_{i-2}}\,\overline{a_{i-1}}^{-1}\overline{b_i}$. However, we will find it more convenient to consider the inverse of this word to be $\mathbf{w}(\alpha)$, that is, we define

$$\mathbf{w}(\alpha) = \begin{cases} \overline{b_i}^{-1}\,\overline{a_{i-1}}\,\overline{b_{i-2}}^{-1} \dots \overline{a_2}\,\overline{b_1}^{-1}\,\overline{a_0} & \text{if } i \text{ is odd,} \\ \overline{b_i}^{-1}\,\overline{a_{i-1}}\,\overline{b_{i-2}}^{-1} \dots \overline{b_2}^{-1}\,\overline{a_1}\,\overline{b_0}^{-1} & \text{if } i \text{ is even.} \end{cases}$$

For, in this case, we have $\mathbf{w}((\alpha_1, a_i)) = \overline{a_i}\mathbf{w}(\alpha_1)$ for any $\mathbf{t}_*^{(i-1)}$-sequence α_1 and $\mathbf{t}_+^{(i)}$-sequence (α_1, a_i), and, similarly, $\mathbf{w}((\alpha_1, b_i)) = \overline{b_i}^{-1}\mathbf{w}(\alpha_1)$ for any $\mathbf{t}_+^{(i-1)}$-sequence α_1 and $\mathbf{t}_*^{(i)}$-sequence (α_1, b_i). Since the word $\mathbf{w}(\alpha) \in (X \cup X^{-1})^*$ is reduced if and only if its inverse is, the formulation of the dual of Lemma 4.6 does not change with this modification.

One can easily prove Lemma 4.6 and its dual simultaneously by induction on i, therefore it is left to the reader.

Now let $\mathbf{t} = y\mathbf{u}z \in \mathbf{T}$ with $\mathbf{u} \in \{x\} \cup \mathbf{T}_+ \cup \mathbf{T}_*$. A $\mathbf{t}$-sequence $\alpha = (a, b, \beta)$ in W^ϵ, where β is empty if $\mathbf{u} = x$ and is a $\mathbf{u}$-sequence in W^ϵ otherwise, is said to be a *reduced* $\mathbf{t}$*-sequence in* W^ϵ if β is empty or is a reduced $\mathbf{u}$-sequence in W^ϵ.

For any subset $\mathbf{X}$ in the set of terms $\mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}$, we define the subset

$$\mathbf{X}[W] = \{\mathbf{t}(x, \alpha) : \mathbf{t} \in \mathbf{X}, \text{ and } \alpha \text{ is a reduced } \mathbf{t}\text{-sequence in } W^\epsilon\}$$

of $\mathcal{P}_1(W)$, and denote by $\mathbf{X}^W[W]$ the set of polynomial functions of W corresponding to $\mathbf{X}[W]$. Obviously, we have

$$\mathbf{X}^W[W] \subseteq \mathbf{X}^W(W) \quad \text{for any } \mathbf{X} \subseteq \mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}. \tag{4.4}$$

We intend to show that the reverse inclusion also holds, that is, the equality is valid.

Lemma 4.7. *Let $i \in \mathbb{N}$, and let $\alpha = (a_0, b_0, \dots, a_{i-2}, b_{i-1}, a_i)$ be a $\mathbf{t}_+^{(i)}$-sequence in W^ϵ.*

(1) *If $\overline{a_i} = 1$ then $\left(\mathbf{t}_+^{(i)}\right)^W(x, \alpha) \in \left(\mathbf{T}_*^{(i-1)}\right)^W(W)$.*
(2) *If $\overline{a_i} \neq 1$ then there exist elements $a_i' \in W$, $b_{i-1}' \in W^\epsilon$ such that*

$$\operatorname{red}\left(\overline{a_i}\,\overline{b_{i-1}}^{-1}\right) = \overline{a_i'}\,\overline{b_{i-1}'}^{-1},$$

and we have

$$\left(\mathbf{t}_+^{(i)}\right)^W(x, \alpha) = \left(\mathbf{t}_+^{(i)}\right)^W(x, \alpha') \tag{4.5}$$

for the $\mathbf{t}_+^{(i)}$-sequence $\alpha' = (a_0, b_0, \dots, a_{i-2}, b_{i-1}', a_i')$ in W^ϵ.

Proof. If $i \geq 2$ then denote the $\mathbf{t}_+^{(i-2)}$-sequence $(a_0, b_0, \ldots, a_{i-2})$ by α_2. For brevity, let $p = \mathbf{t}_+^{(i)}(x, \alpha)$, and define

$$r = \begin{cases} a_0 x & \text{if } i = 1, \\ \mathbf{t}_+^{(i-2)}(x, \alpha_2) & \text{otherwise.} \end{cases}$$

Then, by definition, we have

$$p = \big(a_i (r b_{i-1})^*\big)^+. \tag{4.6}$$

(1) If $\overline{a_i} = 1$ then $a_i \in P(W)$. Therefore, for every $c \in W$, we see that $p^W(c) = \big(a_i(r^W(c)b_{i-1})^*\big)^+ = \big((r^W(c)b_{i-1})^* a_i\big)^+ = \big((r^W(c)b_{i-1}a_i)^*\big)^+ = (r^W(c)b_{i-1}a_i)^*$. Hence $p^W(c) = \big(\mathbf{t}_*^{(i-1)}\big)^W(c, (\beta, b_{i-1}a_i))$, where $\beta = a_0$ if $i = 1$ and $\beta = \alpha_2$ otherwise. Clearly, we have $\mathbf{t}_*^{(i-1)}(x, (\beta, b_{i-1}a_i)) \in \mathbf{T}_*^{(i-1)}(W)$ which implies that $p^W \in \big(\mathbf{T}_*^{(i-1)}\big)^W(W)$.

(2) For brevity, write a for a_i and b for b_{i-1}. Suppose that $\overline{a} = ut$ and $\overline{b} = vt$ where $u, v, t \in X^*$ and $\operatorname{red}\big(\overline{a}\overline{b}^{-1}\big) = uv^{-1}$. For an arbitrary $c \in W$, let $r^W(c) = (\overline{r}, R^{\overline{r}})$. Thus we obtain by (4.6) that

$$\begin{aligned} p^W(c) &= \big(a_i(r^W(c)b_{i-1})^*\big)^+ = \big((\overline{a}, A^{\overline{a}})\big((\overline{r}, R^{\overline{r}})(\overline{b}, B^{\overline{b}})\big)^*\big)^+ \\ &= \big((\overline{a}, A^{\overline{a}})\big(1, (R \vee {}^{\overline{r}}B)^{\overline{r}\overline{b}}\big)\big)^+ = \big(1, A \vee {}^{\overline{a}}\big((R \vee {}^{\overline{r}}B)^{\overline{r}\overline{b}}\big)\big) \\ &= \big(1, A \vee {}^{\overline{a}\overline{b}^{-1}\overline{r}^{-1}}(R \vee {}^{\overline{r}}B)\big) = \big(1, A \vee {}^{uv^{-1}\overline{r}^{-1}}(R \vee {}^{\overline{r}}B)\big). \end{aligned}$$

Obviously, $(u, A^u) \in W$ since $A \neq \emptyset$ is implied by $\overline{a} = \overline{a_i} \neq 1$, and $(v, B^v) \in W^\epsilon$. Hence a similar calculation to the previous one but in the reverse order shows that

$$p^W(c) = \big(1, A \vee {}^{uv^{-1}\overline{r}^{-1}}(R \vee {}^{\overline{r}}B)\big) = \big((u, A^u)\big((\overline{r}, R^{\overline{r}})(v, B^v)\big)^*\big)^+.$$

Choosing $a_i' = (u, A^u)$ and $b_{i-1}' = (v, B^v)$, we deduce that

$$p^W(c) = \big(a_i'(r^W(c)b_{i-1}')^*\big)^+ = \big(\mathbf{t}_+^{(i)}\big)^W(c, \alpha').$$

This shows that $\big(\mathbf{t}_+^{(i)}\big)^W(x, \alpha) = \big(\mathbf{t}_+^{(i)}\big)^W(x, \alpha')$ where

$$\operatorname{red}\big(\overline{a_i}\,\overline{b_{i-1}}^{-1}\big) = uv^{-1} = \overline{a_i'}\,\overline{b_{i-1}'}^{-1}.$$

□

Lemma 4.8. *For every $i \in \mathbb{N}$, $\mathbf{t} \in \mathbf{T}_+^{(i)} \cup \mathbf{T}_*^{(i)}$ and every $\mathbf{t}$-sequence α in W^ϵ, we have either*

$$\mathbf{t}^W(x, \alpha) \in \bigcup_{j=0}^{i-1} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W],$$

or

$$\mathbf{t}^W(x, \alpha) = \mathbf{t}^W(x, \alpha')$$

for some reduced $\mathbf{t}$-sequence α' in W^ϵ with $\operatorname{red}(\mathbf{w}(\alpha)) = \mathbf{w}(\alpha')$.

Proof. We prove this statement by induction on i. First let $i = 0$ and $\mathbf{t} = \mathbf{t}_+^{(0)}$. For $p = \mathbf{t}_+^{(0)}(x, \alpha)$ where $\alpha = (a_0, b_0)$, we have $p^W(c) = (a_0cb_0)^+ = (a_0cb_0^+)^+$ for any $c \in W$. Thus $p^W = \big(\mathbf{t}_+^{(0)}\big)^W(x, \alpha') \in \big(\mathbf{T}_+^{(0)}\big)^W[W]$ follows for $\alpha' = (a_0, b_0^+)$. For, $\overline{b_0^+} = 1$, and so α' is, indeed, reduced. This and the dual argument show that, in case $i = 0$, the assertion of Lemma 4.8 holds.

Let $i \in \mathbb{N}$, and suppose that the statement of the lemma holds for any index smaller than i. In particular, it implies that

$$\big(\mathbf{T}_+^{(k)} \cup \mathbf{T}_*^{(k)}\big)^W(W) \subseteq \bigcup_{j=0}^{k} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W] \quad \text{for every } k < i. \tag{4.7}$$

Let $\mathbf{t} = \mathbf{t}_+^{(i)}$, $\alpha = (a_0, b_0, \ldots, a_{i-2}, b_{i-1}, a_i)$, and put $p = \mathbf{t}_+^{(i)}(x, \alpha)$. If $\overline{a_i} = 1$ then Lemma 4.7(1) implies $p^W \in \big(\mathbf{T}_*^{(i-1)}\big)^W(W)$, and (4.7) ensures that $p^W \in \bigcup_{j=0}^{i-1} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W]$.

Assume that $\overline{a_i} \neq 1$. Now Lemma 4.7(2) implies that equality (4.5) holds for some $a_i' \in W$, $b_{i-1}' \in W^\epsilon$ with $\mathrm{red}\big(\overline{a_i}\,\overline{b_{i-1}}^{-1}\big) = \overline{a_i'}\,\overline{b_{i-1}'}^{-1}$, and for the $\mathbf{t}_+^{(i)}$-sequence $\alpha' = (a_0, b_0, \ldots, a_{i-2}, b_{i-1}', a_i')$. If $\overline{a_i'} = 1$ then the argument in the previous paragraph applies for $\tilde{p} = \mathbf{t}_+^{(i)}(x, \alpha')$, and $p^W = \tilde{p}^W \in \bigcup_{j=0}^{i-1} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W]$ follows.

From now on, assume that $\overline{a_i'} \neq 1$, and, introducing the notation $\alpha_1 = (a_0, b_0, \ldots, a_{i-2}, b_{i-1}')$ and $q = \mathbf{t}_*^{(i-1)}(x, \alpha_1)$, we obtain that $\tilde{p} = (a_i'q)^+$. If $\overline{b_{i-1}'} = 1$ then the dual of the argument in the previous paragraph verifies that $q^W \in \bigcup_{j=0}^{i-2} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W]$. If $q^W \in \big(\mathbf{T}_*^{(j)}\big)^W[W]$ for some $j \leq i-2$ then $q^W = \big(\mathbf{t}_*^{(j)}\big)^W(x, \gamma)$ for some $\mathbf{t}_*^{(j)}$-sequence γ in W^ϵ, and so

$$\tilde{p}^W = \big(a_i'q^W\big)^+ = \big(\mathbf{t}_+^{(j+1)}\big)^W(x, (\gamma, a_i')) \in \big(\mathbf{T}_+^{(j+1)}\big)^W(W)$$

follows. If $q^W \in \big(\mathbf{T}_+^{(j)}\big)^W[W]$ for some $j \leq i-2$ then $q^W = \big(\mathbf{t}_+^{(j)}\big)^W(x, \gamma)$ for some $\mathbf{t}_+^{(j)}$-sequence γ in W^ϵ, and so Lemma 4.4 implies that

$$\tilde{p}^W = \big(a_i'q^W\big)^+ = \big(a_i'\big(\mathbf{t}_+^{(j)}\big)^W(x, \gamma)\big)^+ \in \big(\mathbf{T}_+^{(j)}\big)^W(W).$$

Thus we deduce that if $q^W \in \bigcup_{j=0}^{i-2} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W]$ then $p^W = \tilde{p}^W \in \bigcup_{j=0}^{i-1} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W(W)$, which implies by (4.7) that

$$\big(\mathbf{t}_+^{(i)}\big)^W(x, \alpha) = p^W \in \bigcup_{j=0}^{i-1} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W].$$

In the opposite case, that is, if

$$\big(\mathbf{t}_*^{(i-1)}\big)^W(x, \alpha_1) = q^W \notin \bigcup_{j=0}^{i-2} \big(\mathbf{T}_+^{(j)} \cup \mathbf{T}_*^{(j)}\big)^W[W],$$

and, consequently, $\overline{b_{i-1}'} \neq 1$, then the induction hypothesis implies that

$$\big(\mathbf{t}_*^{(i-1)}\big)^W(x, \alpha_1) = \big(\mathbf{t}_*^{(i-1)}\big)^W(x, \alpha_1'')$$

for some reduced $\mathbf{t}_*^{(i-1)}$-sequence $\alpha_1'' = (a_0'', b_0'', \ldots, a_{i-2}'', b_{i-1}'')$, where

$$\text{red}\,\big(\mathbf{w}(\alpha_1)\big) = \mathbf{w}(\alpha_1''). \tag{4.8}$$

Hence $p^W = \tilde{p}^W = (a_i' q^W)^+ = \big(\mathbf{t}_+^{(i)}\big)^W(x, \alpha'')$ where $\alpha'' = (\alpha_1'', a_i')$ is a $\mathbf{t}_+^{(i)}$-sequence in W^ϵ such that $\overline{a_i'} \neq 1$. We intend to verify that α'' is reduced. Taking into account the properties of α_1'', all we have to check is that the word $\overline{a_i'}\,\overline{b_{i-1}''}^{-1} \in (X \cup X^{-1})^*$ is reduced.

By the dual of Lemma 4.6, the members of $\overline{(\alpha_1'')^-}$ belong to X^+. Since the lengths of α_1 and α_1'' are the same, equality (4.8) implies that no member of $\overline{\alpha_1^-}$ is the empty word, and, when reducing the word $\mathbf{w}(\alpha_1)$, no member of $\overline{\alpha_1^-}$ is fully deleted. Therefore $\overline{b_{i-1}''}^{-1}$ is a prefix of $\overline{b_{i-1}'}^{-1}$. Since, by the choice of a_i' and b_{i-1}', we know that $\overline{a_i'}\,\overline{b_{i-1}'}^{-1}$ is reduced, this implies that $\overline{a_i'}\,\overline{b_{i-1}''}^{-1}$ is also reduced. □

An immediate consequence of this lemma is that the inclusion

$$(\mathbf{T}_+ \cup \mathbf{T}_*)^W(W) \subseteq (\mathbf{T}_+ \cup \mathbf{T}_*)^W[W]$$

is valid. Hence $\mathbf{T}^W(W) \subseteq \mathbf{T}^W[W]$ follows, and so we obtain the following statement by (4.4).

Lemma 4.9. *The equality $\mathbf{T}^W(W) = \mathbf{T}^W[W]$ holds.*

This lemma allows us to replace $\mathbf{T}(W)$ by $\mathbf{T}[W]$ in Proposition 4.5 if $S = W$. Denote by $(4.2)_{S=W}$ the condition obtained from (4.2) by replacing S by W.

Proposition 4.10. *If τ is a symmetric relation on $W = W(X^*, \mathcal{Q})$ then, for any $s, t \in W$, we have $s\,\tau^\#\,t$ if and only if $s = t$, or there exists $k \in \mathbb{N}$, there exist elements $c_1, d_1, c_2, d_2, \ldots, c_k, d_k \in W$ and unary polynomials $p_1, p_2, \ldots, p_k \in \mathbf{T}[W]$ such that conditions (4.1) and $(4.2)_{S=W}$ are satisfied.*

The main idea of the proof of Proposition 4.2 is that if τ is a $(2,1,1)$-congruence on $F_W = F_W\mathcal{RS}(X)$, s, t are distinct $\tau^\#$-related elements in F_W, and $k \in \mathbb{N}$, $c_1, d_1, c_2, d_2, \ldots, c_k, d_k \in F_W$, $p_1, p_2, \ldots, p_k \in \mathbf{T}[W]$ fulfil conditions (4.1) and $(4.2)_{S=W}$, then we find $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k \in \mathbf{T}(F_W)$ such that

$$\tilde{p}_j^{F_W}(c_j) \le p_j^W(c_j) \quad \text{and} \quad \tilde{p}_j^{F_W}(d_j) \le p_j^W(d_j) \quad \text{for} \quad j = 1, 2, \ldots, k,$$

and

$$\tilde{p}_j^{F_W}(d_j) = \tilde{p}_{j+1}^{F_W}(c_{j+1}) \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

In order to help finding $\tilde{p}_{j+1}$ to $\tilde{p}_j$ with the above properties, we show several lemmas. Recall that, for any $u = (\overline{u}, U^{\overline{u}}) \in W$, we have $u \in F_W$ if and only if $1, u \in U$.

Lemma 4.11. *Let $a, b \in W^\epsilon$, $q \in W$ and $s \in F_W$ such that $s \le aqb$.*

1. *If $q \in F_W$ then there exist $\tilde{a}, \tilde{b} \in F_W$ such that $\tilde{a} \le a$, $\tilde{b} \le b$ and $s = \tilde{a}q\tilde{b}$.*
2. *If $q \in P(W)$ then there exist $\tilde{a}, \tilde{b}, \tilde{q} \in F_W$ such that $\tilde{a} \le a$, $\tilde{b} \le b$, $\tilde{q} \le q$ and $s = \tilde{a}\tilde{q}\tilde{b}$.*

Proof. Recall that $u \le v$ in W^ϵ if and only if $\overline{u} = \overline{v}$ in X^* and $U \supseteq V$ in $\mathcal{Q}$. Let $a, b \in W^\epsilon$, $q \in W$ and $s \in F_W$ with $s \le aqb$. The latter relation is equivalent to requiring that

$$\overline{s} = \overline{aq}\overline{b}, \quad \text{and} \quad S \supseteq A \vee {}^{\overline{a}}Q \vee {}^{\overline{aq}}B.$$

This implies

$$S \supseteq A, \quad S^{\overline{a}} \supseteq Q \quad \text{and} \quad S^{\overline{aq}} \supseteq B. \tag{4.9}$$

Since $s \in F_W$, we have $1, \overline{s} \in S$. Hence $\overline{a}, \overline{aq} \in S$ follows because $\overline{s} = \overline{aq}\overline{b}$ and S is prefix closed. Therefore we also see that $1 \in S^{\overline{a}}$ and $1, \overline{b} \in S^{\overline{aq}}$. This implies that the elements $\tilde{a} = (\overline{a}, S^{\overline{a}})$, $\tilde{q} = (1, S^{\overline{a}})$ and $\tilde{b} = \big(\overline{b}, (S^{\overline{aq}})^{\overline{b}}\big)$ belong to F_W. It is straightforward by (4.9) that $\tilde{a} \le a$ and $\tilde{b} \le b$. Moreover, if $q \in P(W)$, that is, $\overline{q} = 1$, then we also have $\tilde{q} \le q$. Finally, we check that s is of the form required.

(1) We have

$$\begin{aligned} \tilde{a}q\tilde{b} &= (\overline{a}, S^{\overline{a}})(\overline{q}, Q^{\overline{q}})\big(\overline{b}, (S^{\overline{aq}})^{\overline{b}}\big) = \big(\overline{aq}\overline{b}, S^{\overline{aq}\overline{b}} \vee Q^{\overline{q}\overline{b}} \vee S^{\overline{aq}\overline{b}}\big) \\ &= (\overline{s}, S^{\overline{s}} \vee Q^{\overline{q}\overline{b}}) = (\overline{s}, S^{\overline{s}}) = s, \end{aligned}$$

where we apply in the last step that (4.9) implies $Q^{\overline{q}\overline{b}} \subseteq (S^{\overline{a}})^{\overline{q}\overline{b}} = S^{\overline{s}}$.

(2) Now we have $\overline{q} = 1$, and so

$$\tilde{a}\tilde{q}\tilde{b} = (\overline{a}, S^{\overline{a}})(1, S^{\overline{a}})\big(\overline{b}, (S^{\overline{a}})^{\overline{b}}\big) = \big(\overline{a}\overline{b}, S^{\overline{a}\overline{b}}\big) = (\overline{s}, S^{\overline{s}}) = s.$$

□

Lemma 4.12. *Let $a \in W^\epsilon$, $q \in W$ and $s \in F_W$ such that $s \le (aq)^+$.*

1. *If $q = rb$ for some $r \in F_W$ and $b \in P(W^\epsilon)$, then there exist $\tilde{a}, \tilde{b} \in F_W$ such that $\tilde{a} \le a$, $\tilde{b} \le b$ and $s = (\tilde{a}r\tilde{b})^+$.*
2. *If $q = (erf)^*$ for some $r \in F_W$ and $e, f \in P(W^\epsilon)$, then there exist $\tilde{a}, \tilde{q} \in F_W$ such that $\tilde{a} \le a$, $\tilde{q} \le q$ and $s = (\tilde{a}\tilde{q})^+$.*
3. *If $\overline{a} \ne 1$, and $q = (pb)^*$ for some $p \in P(W)$ and $b \in W$ where $\overline{b} \ne 1$ and $\overline{a}\overline{b}^{-1}$ is reduced, then there exist $\tilde{a}, \tilde{q} \in F_W$ such that $\tilde{a} \le a$, $\tilde{q} \le q$ and $s = (\tilde{a}\tilde{q})^+$.*

Proof. (1) Let $a \in W^\epsilon$, $q \in W$, $r, s \in F_W$ and $b \in P(W^\epsilon)$ such that $q = rb$ and $s \le (aq)^+$. Hence $s \le (arb)^+$, or, equivalently,

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq A \vee {}^{\overline{a}}R \vee {}^{\overline{ar}}B.$$

This implies (cf. (4.9))

$$S \supseteq A, \quad S^{\overline{a}} \supseteq R \quad \text{and} \quad S^{\overline{ar}} \supseteq B. \tag{4.10}$$

Notice that $b \in P(W^\epsilon)$ ensures $\overline{b} = 1$. Since $r, s \in F_W$, we have $1, \overline{r} \in R$ and $1 \in S$. The former relation implies $\overline{a}, \overline{ar} \in {}^{\overline{a}}R \subseteq S$, whence we obtain that $1 \in S^{\overline{ar}}$. Thus we see that the elements $\tilde{a} = (\overline{a}, S^{\overline{a}})$ and $\tilde{b} = (1, S^{\overline{ar}})$ belong to F_W. It is straightforward by (4.10) that $\tilde{a} \le a$ and $\tilde{b} \le b$. Moreover, we have

$$(\tilde{a}r\tilde{b})^+ = \big((\overline{a}, S^{\overline{a}})(\overline{r}, R^{\overline{r}})(1, S^{\overline{ar}})\big)^+ = (1, S \vee {}^{\overline{a}}R \vee S) = (1, S) = s.$$

(2) Let $a \in W^\epsilon$, $q \in W$, $r, s \in F_W$ and $e, f \in P(W^\epsilon)$ such that $q = (erf)^*$ and $s \leq (aq)^+$. This is equivalent to requiring that

$$\overline{q} = 1, \quad \text{and} \quad Q = (E \vee R)^{\overline{r}} \vee F$$

and

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq A \vee {}^{\overline{a}}Q = A \vee {}^{\overline{ar}^{-1}}(E \vee R) \vee {}^{\overline{a}}F. \tag{4.11}$$

Since $r, s \in F_W$, we have $\overline{r} \in R$ and $1 \in S$. By (4.11) we obtain that $\overline{a} = \overline{ar}^{-1}\overline{r} \in S$, whence $1 \in S^{\overline{a}}$ follows. Thus the elements $\tilde{a} = (\overline{a}, S^{\overline{a}})$ and $\tilde{q} = (1, S^{\overline{a}})$ belong to F_W, and the inequalities $\tilde{a} \leq a$, $\tilde{q} \leq q$ hold by (4.11). Moreover, we have

$$(\tilde{a}\tilde{q})^+ = \big((\overline{a}, S^{\overline{a}})(1, S^{\overline{a}})\big)^+ = (\overline{a}, S^{\overline{a}})^+ = (1, S) = s.$$

(3) Let $a, b, q \in W$, $p \in P(W)$ and $s \in F_W$ such that $\overline{a}, \overline{b} \neq 1$, $\overline{a}\overline{b}^{-1}$ is reduced, and $q = (pb)^*$, $s \leq (aq)^+$. The last two relations are equivalent to

$$\overline{q} = 1, \quad \text{and} \quad Q = (P \vee B)^{\overline{b}}$$

and

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq A \vee {}^{\overline{a}}Q = A \vee {}^{\overline{a}\overline{b}^{-1}}(P \vee B), \tag{4.12}$$

respectively, since $\overline{p} = 1$. Here $A, P \vee B \in \mathcal{Q}$, and so there exist $u, v \in X^*$ such that $u^{-1} \in A$ and $v^{-1} \in P \vee B$. This implies by the definition of the operation $\vee$ that $u^{-1}\left[\mathrm{red}\left(u\overline{a}\overline{b}^{-1}v^{-1}\right)\right]^{\downarrow} \in A \vee {}^{\overline{a}\overline{b}^{-1}}(P \vee B) \subseteq S$. Since $\overline{a}, \overline{b} \neq 1$ and $\overline{a}\overline{b}^{-1}$ is reduced, so is $u\overline{a}\overline{b}^{-1}v^{-1}$, whence we obtain that $u\overline{a} \in \left[\mathrm{red}\left(u\overline{a}\overline{b}^{-1}v^{-1}\right)\right]^{\downarrow}$. Thus we deduce that $\overline{a} \in S$ and $1 \in S^{\overline{a}}$. Therefore we can again consider the elements $\tilde{a} = (\overline{a}, S^{\overline{a}})$ and $\tilde{q} = (1, S^{\overline{a}})$, as in the proof of the previous statement, and they fulfil all the conditions required. □

Since a W-product is not a left-right symmetric construction, we need to verify the dual of Lemma 4.12 separately.

Lemma 4.13. *Let $b \in W^\epsilon$, $q \in W$ and $s \in F_W$ such that $s \leq (qb)^*$.*

1. *If $q = ar$ for some $r \in F_W$ and $a \in P(W^\epsilon)$ then there exist $\tilde{a}, \tilde{b} \in F_W$ such that $\tilde{a} \leq a$, $\tilde{b} \leq b$ and $s = (\tilde{a}r\tilde{b})^*$.*
2. *If $q = (erf)^+$ for some $r \in F_W$ and $e, f \in P(W^\epsilon)$ then there exist $\tilde{b}, \tilde{q} \in F_W$ such that $\tilde{b} \leq b$, $\tilde{q} \leq q$ and $s = (\tilde{q}\tilde{b})^*$.*
3. *If $\overline{b} \neq 1$, and $q = (ap)^+$ for some $p \in P(W)$ and $a \in W$ where $\overline{a} \neq 1$ and $\overline{b}^{-1}\overline{a}$ is reduced, then there exist $\tilde{b}, \tilde{q} \in F_W$ such that $\tilde{b} \leq b$, $\tilde{q} \leq q$ and $s = (\tilde{q}\tilde{b})^*$.*

Proof. (1) Let $a \in P(W^\epsilon)$, $b \in W^\epsilon$ and $r, s \in F_W$ such that $s \leq (arb)^*$. Then

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq (A \vee R)^{\overline{r}\overline{b}} \vee B^{\overline{b}}.$$

This implies ${}^{\overline{r}\overline{b}}S \supseteq A, R$ and ${}^{\overline{b}}S \supseteq B$. Since $r, s \in F_W$, and so $1, \overline{r} \in R$ and $1 \in S$, we obtain that $1 \in {}^{\overline{r}\overline{b}}S$ and $1, \overline{b} \in {}^{\overline{b}}S$. Thus the elements $\tilde{a} = (1, {}^{\overline{r}\overline{b}}S)$

and $\tilde{b} = \big(\overline{b}, ({}^{\overline{b}}S)^{\overline{b}}\big)$ belong to F_W, and $\tilde{a} \leq a$ and $\tilde{b} \leq b$ by the above inclusions. Furthermore, we have

$$(\tilde{a}r\tilde{b})^* = \Big((1, {}^{\overline{r}\overline{b}}S)(\overline{r}, R^{\overline{r}})\big(\overline{b}, ({}^{\overline{b}}S)^{\overline{b}}\big)\Big)^* = (1, S \vee R^{\overline{r}\overline{b}} \vee S) = (1, S) = s.$$

(2) Let $b \in W^\epsilon$, $q \in W$, $r, s \in F_W$ and $e, f \in P(W^\epsilon)$ such that $q = (erf)^+$ and $s \leq (qb)^*$. These relations are equivalent to

$$\overline{q} = 1, \quad \text{and} \quad Q = E \vee R \vee {}^{\overline{r}}F$$

and

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq (Q \vee B)^{\overline{b}} = (E \vee R \vee {}^{\overline{r}}F \vee B)^{\overline{b}}, \tag{4.13}$$

respectively. Since $r, s \in F_W$, we have $1 \in R, S$. Therefore $\overline{b} \in {}^{\overline{b}}S$ follows, and (4.13) implies that $\overline{b}^{-1} \in R^{\overline{b}} \subseteq S$. From the latter observation we see that $1 \in {}^{\overline{b}}S$. Thus we deduce that the elements $\tilde{b} = \big(\overline{b}, ({}^{\overline{b}}S)^{\overline{b}}\big), \tilde{q} = (1, {}^{\overline{b}}S)$ belong to F_W, and the inequalities $\tilde{b} \leq b$ and $\tilde{q} \leq q$ hold by (4.13). We also have

$$(\tilde{q}\tilde{b})^* = \Big((1, {}^{\overline{b}}S)\big(\overline{b}, ({}^{\overline{b}}S)^{\overline{b}}\big)\Big)^* = \big(1, ({}^{\overline{b}}S)^{\overline{b}}\big) = (1, S) = s.$$

(3) Let $a, b, q \in W$, $p \in P(W)$ and $s \in F_W$ such that $\overline{a}, \overline{b} \neq 1$, $\overline{b}^{-1}\overline{a}$ is reduced, and $q = (ap)^+$, $s \leq (qb)^*$. Since $\overline{p} = 1$, the last two relations are equivalent to

$$\overline{q} = 1, \quad \text{and} \quad Q = A \vee {}^{\overline{a}}P$$

and

$$\overline{s} = 1, \quad \text{and} \quad S \supseteq (Q \vee B)^{\overline{b}} = A^{\overline{b}} \vee ({}^{\overline{a}}P)^{\overline{b}} \vee B^{\overline{b}}, \tag{4.14}$$

respectively. Notice that $1 \in S$ implies $\overline{b} \in {}^{\overline{b}}S$. The inclusion in (4.14) implies $S \supseteq ({}^{\overline{a}}P)^{\overline{b}} = {}^{\overline{b}^{-1}\overline{a}}P$. Since $P \in \mathcal{Q}$, there exists $u \in X^*$ with $u^{-1} \in P$, whence $\operatorname{red}(\overline{b}^{-1}\overline{a}u^{-1}) \in S$ follows. However, $\overline{b}, \overline{a} \neq 1$ and $\overline{b}^{-1}\overline{a}$ is reduced, therefore $\operatorname{red}(\overline{b}^{-1}\overline{a}u^{-1}) = \overline{b}^{-1}\operatorname{red}(\overline{a}u^{-1})$. Since the set S is prefix closed, this implies that $\overline{b}^{-1} \in S$, and so $1 \in {}^{\overline{b}}S$. Thus we can consider the elements $\tilde{b} = \big(\overline{b}, ({}^{\overline{b}}S)^{\overline{b}}\big)$, $\tilde{q} = (1, {}^{\overline{b}}S)$, as above, and they satisfy all the conditions required. □

Now we are ready to prove the crucial statement that allows us to find the unary polynomials $\tilde{p}_j$ mentioned above. If $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ are sequences of the same length consisting of elements in W^ϵ then we write $\alpha \leq \beta$ to denote that $a_k \leq b_k$ for $k = 1, 2, \ldots, n$.

Lemma 4.14. *If $\mathbf{t} \in \mathbf{T}_+ \cup \mathbf{T}_* \cup \mathbf{T}$, α is a reduced $\mathbf{t}$-sequence in W^ϵ and $r, s \in F_W$ such that $s \leq \mathbf{t}^W(r, \alpha)$, then there exists a $\mathbf{t}$-sequence $\widetilde{\alpha}$ in F_W such that $\widetilde{\alpha} \leq \alpha$ and $s = \mathbf{t}^{F_W}(r, \widetilde{\alpha})$.*

Proof. First we verify the assertion for $\mathbf{T}_+ \cup \mathbf{T}_* = \bigcup_{i \in \mathbb{N}^0}\big(\mathbf{T}_+^{(i)} \cup \mathbf{T}_*^{(i)}\big)$ by induction on i. If $\alpha = (a_0, b_0)$ is a reduced $\mathbf{t}_+^{(0)}$-sequence in W^ϵ, that is, $\overline{b_0} = 1$, and $r, s \in F_W$ such that $s \leq \big(\mathbf{t}_+^{(0)}\big)^W(r, \alpha) = (a_0 r b_0)^+$, then Lemma 4.12(1) shows the existence of elements $\tilde{a}_0, \tilde{b}_0 \in F_W$ such that the statement is valid for the $\mathbf{t}_+^{(0)}$-sequence $\widetilde{\alpha} = (\tilde{a}_0, \tilde{b}_0)$ in F_W. This observation and its dual (where we apply Lemma 4.13(1) instead of Lemma 4.12(1)) verify the assertion for $\mathbf{t} \in \mathbf{T}_+^{(0)} \cup \mathbf{T}_*^{(0)}$.

Let $i \in \mathbb{N}$, and suppose that, for any $\mathbf{t} \in \big(\mathbf{T}_+^{(i-1)} \cup \mathbf{T}_*^{(i-1)}\big)$, the proposition holds. Consider a reduced $\mathbf{t}_+^{(i)}$-sequence $\alpha = (a_0, b_0, \dots, a_{i-2}, b_{i-1}, a_i)$ in W^ϵ. By definition, $\alpha_1 = (a_0, b_0, \dots, a_{i-2}, b_{i-1})$ is a reduced $\mathbf{t}_*^{(i-1)}$-sequence, $\overline{a_i} \neq 1$ and $\overline{a_i}\,\overline{b_{i-1}}^{-1}$ is reduced. Moreover, let $r, s \in F_W$ such that $s \leq \big(\mathbf{t}_+^{(i)}\big)^W(r, \alpha)$. Putting

$$q = \big(\mathbf{t}_*^{(i-1)}\big)^W(r, \alpha_1), \tag{4.15}$$

we have

$$s \leq \big(\mathbf{t}_+^{(i)}\big)^W(r, \alpha) = (a_i q)^+. \tag{4.16}$$

We distinguish two cases according to whether $\overline{b_{i-1}}$ equals 1 or not. If $\overline{b_{i-1}} = 1$ then, by Lemma 4.6, α_1 can be a reduced $\mathbf{t}_*^{(i-1)}$-sequence only if $i = 1$. Therefore we have $\alpha_1 = (a_0, b_0)$ and $\overline{a_0} = \overline{b_0} = 1$, and so $a_0, b_0 \in P(W^\epsilon)$ such that $q = (a_0 r b_0)^*$. Thus Lemma 4.12(2) implies the existence of elements $\tilde{a}_i$ (where $i = 1$) and $\tilde{q}$ in F_W such that

$$\tilde{a}_i \leq a_i, \quad \tilde{q} \leq q \quad \text{and} \quad s = (\tilde{a}_i \tilde{q})^+. \tag{4.17}$$

Now assume that $\overline{b_{i-1}} \neq 1$. If $i = 1$ then we have $q = (a_0(rb_0))^*$ where $a_0 \in P(W^\epsilon)$ and $rb_0 \in W$ with $\overline{rb_0} \neq 1$. Furthermore, $\overline{a_1}\,\overline{rb_0}^{-1} = \overline{a_1}\,\overline{b_0}^{-1}\overline{r}^{-1}$ is reduced since $\overline{a_1}\,\overline{b_0}^{-1}$ is reduced and $\overline{b_0} \neq 1$. Therefore Lemma 4.12(3) can be applied to obtain elements $\tilde{a}_i$ (where $i = 1$) and $\tilde{q}$ in F_W such that (4.17) holds. If $i \geq 2$ then $\alpha_2 = (a_0, b_0, \dots, a_{i-2})$ is a reduced $\mathbf{t}_+^{(i-2)}$-sequence, and, by (4.15), we have $q = (pb_{i-1})^*$ where $p = \big(\mathbf{t}_+^{(i-2)}\big)^W(r, \alpha_2) \in P(W)$ and $\overline{b_{i-1}} \neq 1$. Again, Lemma 4.12(3) implies the existence of elements $\tilde{a}_i, \tilde{q} \in F_W$ such that (4.17) is valid. Thus, in each case, we have found an element $\tilde{q} \in F_W$ such that, by (4.15) and (4.17), we have $\tilde{q} \leq \big(\mathbf{t}_*^{(i-1)}\big)^W(r, \alpha_1)$ where $r \in F_W$ and α_1 is a reduced $\mathbf{t}_*^{(i-1)}$-sequence in W^ϵ. Therefore the induction hypothesis can be applied to get a $\mathbf{t}_*^{(i-1)}$-sequence $\widetilde{\alpha}_1$ in F_W such that $\widetilde{\alpha}_1 \leq \alpha_1$ and $\tilde{q} = \big(\mathbf{t}_*^{(i-1)}\big)^{F_W}(r, \widetilde{\alpha}_1)$. Hence it follows that $\widetilde{\alpha} = (\widetilde{\alpha}_1, \tilde{a}_i)$ is a $\mathbf{t}_+^{(i)}$-sequence in F_W, and, by (4.17), we have $\widetilde{\alpha} \leq \alpha$ and $s = (\tilde{a}_i \tilde{q})^+ = \Big(\tilde{a}_i \big(\mathbf{t}_*^{(i-1)}\big)^{F_W}(r, \widetilde{\alpha}_1)\Big)^+ = \big(\mathbf{t}_+^{(i)}\big)^{F_W}(r, \widetilde{\alpha})$. This argument together with its dual (where Lemma 4.13 is applied instead of Lemma 4.12) completes the proof for $\mathbf{T}_+ \cup \mathbf{T}_*$.

Now let $\mathbf{t} \in \mathbf{T}$, and assume that α is a reduced $\mathbf{t}$-sequence in W^ϵ and $r, s \in F_W$ such that $s \leq \mathbf{t}^W(r, \alpha)$. If $\mathbf{t} = yxz$ and $\alpha = (a, b)$ with $a, b \in W^\epsilon$, then $\mathbf{t}^W(r, \alpha) = arb$. Thus Lemma 4.11(1) implies that there exist $\tilde{a}, \tilde{b} \in F_W$ with $s = \tilde{a} r \tilde{b}$, and so $\widetilde{\alpha} = (\tilde{a}, \tilde{b})$ is a $\mathbf{t}$-sequence in F_W where $\widetilde{\alpha} \leq \alpha$ and $s = \mathbf{t}^{F_W}(r, \widetilde{\alpha})$. If $\mathbf{t} = y\mathbf{u}z$ with $\mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*$, and $\alpha = (a, b, \beta)$ where $a, b \in W^\epsilon$ and β is a reduced $\mathbf{u}$-sequence in W^ϵ, then $\mathbf{t}^W(r, \alpha) = aqb$ where $q = \mathbf{u}^W(r, \beta) \in P(W)$. Therefore Lemma 4.11(2) can be applied to obtain elements $\tilde{a}, \tilde{b}, \tilde{q} \in F_W$ such that $\tilde{a} \leq a$, $\tilde{b} \leq b$, $\tilde{q} \leq q$ and $s = \tilde{a}\tilde{q}\tilde{b}$. Thus $\tilde{q} \leq \mathbf{u}^W(r, \beta)$ where β is a reduced $\mathbf{u}$-sequence in W^ϵ. Since $\mathbf{u} \in \mathbf{T}_+ \cup \mathbf{T}_*$, it follows by the previous part of the proof that there exists a $\mathbf{u}$-sequence $\widetilde{\beta}$ in F_W such that $\widetilde{\beta} \leq \beta$ and $\tilde{q} = \mathbf{u}^{F_W}(r, \widetilde{\beta})$. Hence $\widetilde{\alpha} = (a, b, \widetilde{\beta})$ is a $\mathbf{t}$-sequence in F_W, $\widetilde{\alpha} \leq \alpha$ and $s = \tilde{a}\tilde{q}\tilde{b} = \tilde{a}\mathbf{u}^{F_W}(r, \widetilde{\beta})\tilde{b} = \mathbf{t}^{F_W}(r, \widetilde{\alpha})$. □

Finally, we turn to proving Proposition 4.2.

Proof of Proposition 4.2. For brevity, denote $F_W\mathcal{RS}(X)$ by F_W as before. Let τ be a $(2,1,1)$-congruence on F_W, and let s, t be distinct $\tau^{\#}$-related elements in F_W. By Proposition 4.10, let us choose and fix $k \in \mathbb{N}$, $c_1, d_1, c_2, d_2, \ldots, c_k, d_k \in F_W$, and $p_1, p_2, \ldots, p_k \in \mathbf{T}[W]$ such that conditions (4.1) and $(4.2)_{S=W}$ are satisfied. By definition, for every j $(j = 1, 2, \ldots, k)$, we have $p_j = \mathbf{t}_j(x, \alpha_j)$ for some $\mathbf{t}_j \in \mathbf{T}$ and for some reduced $\mathbf{t}_j$-sequence α_j in W^ϵ. Let us define $s_0, s_1, \ldots, s_k \in F_W$ and $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_k \in \mathbf{T}(F_W)$ in the following manner. Put $s_0 = s$, and notice that, by assumption, $s_0 \in F_W$ and $s_0 \leq p_1^W(c_1) = \mathbf{t}_1^W(c_1, \alpha_1)$. If s_{j-1} is defined for some j $(1 \leq j \leq k)$ such that $s_{j-1} \in F_W$ and $s_{j-1} \leq p_j^W(c_j) = \mathbf{t}_j^W(c_j, \alpha_j)$, then, by applying Lemma 4.14, let us choose and fix a $\mathbf{t}_j$-sequence $\widetilde{\alpha}_j$ in F_W such that $\widetilde{\alpha}_j \leq \alpha_j$ and $s_{j-1} = \mathbf{t}_j^{F_W}(c_j, \widetilde{\alpha}_j)$. Consider the unary polynomial $\tilde{p}_j = \mathbf{t}_j(x, \widetilde{\alpha}_j)$ in $\mathbf{T}(F_W)$, and define $s_j = \tilde{p}_j^{F_W}(d_j)$. Obviously, $s_j \in F_W$ and $s_j = \mathbf{t}_j^{F_W}(d_j, \widetilde{\alpha}_j) \leq \mathbf{t}_j^W(d_j, \alpha_j) = p_j^W(d_j)$. Furthermore, if $j < k$ then we have $s_j \leq p_j^W(d_j) = p_{j+1}^W(c_{j+1}) = \mathbf{t}_{j+1}^W(c_{j+1}, \alpha_{j+1})$. Thus $s_0, s_1, \ldots, s_k \in F_W$ are defined, $s_0 = s$ and $s_k = \mathbf{t}_k^{F_W}(d_k, \widetilde{\alpha}_k) \leq \mathbf{t}_k^W(d_k, \alpha_k) = p_k^W(d_k) = t$. Moreover, for every j $(1 \leq j \leq k)$, we have $s_{j-1} = \mathbf{t}_j^{F_W}(c_j, \widetilde{\alpha}_j) = \tilde{p}_j^{F_W}(c_j)\,\tau\,\tilde{p}_j^{F_W}(d_j) = s_j$ since $c_j\,\tau\,d_j$ in F_W and τ is a $(2,1,1)$-congruence on F_W. Hence we obtain $s\,\tau\,s_k$, and we deduce that $s\tau = s_k\tau \leq t\tau$ in F_W/τ. By symmetry, we also have $t\tau \leq s\tau$ whence it follows that $s\,\tau\,t$. □

As it was mentioned at the beginning of this section, the main result, Theorem 4.1 is immediately implied by Proposition 4.2.

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