International Journal of Algebra and Computation

ALMOST FACTORIZABLE LOCALLY INVERSE SEMIGROUPS

MÁRIA B. SZENDREI
Bolyai Institute, University of Szeged, Aradi vértanúk tere 1
Szeged, H-6720, Hungary
m.szendrei@math.u-szeged.hu

Received (Day Month Year)
Revised (Day Month Year)

Communicated by [editor]

We introduce a notion of almost factorizability within the class of all locally inverse semigroups by requiring a property of order ideals, and we prove that the almost factorizable locally inverse semigroups are just the homomorphic images of Pastijn products of normal bands by completely simple semigroups.

Keywords: Locally inverse semigroup; Pastijn product; almost factorizability.

1. Introduction

A factorizable inverse monoid can be identified, up to isomorphism, with an inverse submonoid M of a symmetric inverse monoid $I(X)$ where each element of M is a restriction of a permutation of X belonging to M. So factorizable inverse monoids are natural objects, and appear in a number of branches of mathematics, cf. [12], [4].

The notion of an almost factorizable inverse semigroup was introduced by Lawson [11] (see also [12]) as the semigroup analogue of a factorizable inverse monoid. Among others, he established (see also McAlister [13] where the main ideas and some of the results were implicit) that the almost factorizable inverse semigroups are just the homomorphic images [or, equivalently, the idempotent separating homomorphic images] of semidirect products of semilattices by groups. Recall that the E-unitary inverse semigroups are just the inverse subsemigroups of semidirect products of semilattices by groups. Thus in the structure theory of inverse semigroups, almost factorizable inverse semigroups have a role dual to that of E-unitary inverse semigroups.

The notion of almost factorizability and the basic results mentioned for the inverse case have been generalized in several directions: for straight locally inverse semigroups by Dombi [2], for orthodox semigroups by Hartmann [7] and for right adequate and for weakly ample semigroups by El Qallali [3], and by Gomes and the author [5], respectively. In all of these classes, the definition of almost factorizability followed, in some sense, the way having been known in the inverse case: given a member S of the class considered, the property required for S to be almost factorizable is formulated within the semigroup of all permissible sets of S, or within the translational hull of S. Furthermore, in most of these classes, the almost factorizable members turned out to have a role dual to the E-unitary-like members introduced and studied much earlier.

In the class of all locally inverse semigroups, the role of semidirect products of semilattices by groups is taken over by Pastijn products of normal bands by completely simple semigroups, and the role of E-unitary inverse semigroups is taken over by weakly E-unitary locally inverse semigroups, see Billhardt and the author [1]. The aim of this paper is to introduce a notion of an almost factorizable locally inverse semigroup so that these semigroups turn out to form a natural class containing all Pastijn products of normal bands by completely simple semigroups and being closed under forming homomorphic images. Recently, Pastijn and Oliveira [15] have provided an analogue of the monoid of permissible subsets of an inverse semigroup for any locally inverse semigroup. However, this does not lead to a notion of almost factorizability we are seeking for. So we find another way to define the appropriate notion. Therefore we consider a much wider oversemigroup, the semigroup of all order ideals, and we formulate the defining property of almost factorizability within this oversemigroup.

The main result of the paper is to establish that this notion fits well into the structure theory of locally inverse semigroups in the sense that a locally inverse semigroup is almost factorizable if and only if it is a homomorphic image of a Pastijn product of a normal band by a completely simple semigroup.

2. Preliminaries

The *natural partial order* $\leq$ is introduced on any regular semigroup S as follows: for any $a, b \in S$, we say that $b \leq a$ if $b \in R_a$ and there exists an idempotent e in R_b such that $b = ea$. An important equivalent description is that $b \leq a$ if and only if there exist idempotents $e, f \in S$ such that $b = ea = af$. It is clear that homomorphisms preserve the natural partial order, that is, if $\phi\colon S \to T$ is a homomorphism between regular semigroups and $b \leq a$ in S then $b\phi \leq a\phi$ holds in T.

A regular semigroup S is called *locally inverse* if eSe is an inverse subsemigroup in S for every idempotent e in S. A fundamental result is that the natural partial order on a regular semigroup S is compatible with the multiplication if and only if S is locally inverse.

An orthodox semigroup S is a regular semigroup where E_S, the set of the idempotents of S, forms a subsemigroup (and so a subband) in S. The least inverse semigroup congruence on an orthodox semigroup S is

$$\gamma = \{(a,b) : V(a) \cap V(b) \neq \emptyset\} = \{(a,b) : V(a) = V(b)\}.$$

Furthermore, for any $a \in S$, we have $a\gamma = D_{aa'}aD_{a'a}$ where $a' \in V(a)$ and D_e ($e \in E_S$) denotes the $\mathcal{D}$-class of the band E_S containing e.

A *normal band* is defined to be a band satisfying the identity $axya = ayxa$. An orthodox semigroup S is locally inverse if and only if E_S is a normal band.

Further important properties of the natural partial order and of locally inverse semigroups are found in [8] and (under the name *pseudoinverse semigroups*) in [6]. For the undefined notions and notation the reader is referred also to these monographs.

Let S be a locally inverse semigroup. We say that a subset $H \subseteq S$ is an *order ideal* in S if the following holds: for every $a \in H$ and $b \in S$, if $b \leq a$ then $b \in H$. Denote by $\mathcal{O}(S)$ the set of all order ideals of S. Moreover, for any $a \in S$ we define

$$(a] = \{b \in S : b \leq a\}.$$

Clearly, $(a]$ is an order ideal, therefore we call it *the principal order ideal generated by* a. Define the mapping

$$\tau\colon S \to \mathcal{O}(S), \quad a \mapsto (a].$$

The following results are proved in [15].

Result 1. *For any regular semigroup S, the set $\mathcal{O}(S)$ forms a semigroup with respect to the usual set product.*

We call $\mathcal{O}(S)$ *the semigroup of order ideals of* S.

Result 2. *Let S be a locally inverse semigroup. Then*

(1) $(a]b = a(b] = (a](b] = (ab]$ for every $a, b \in S$,
(2) $aH = (a]H$ and $Ha = H(a]$ for every $a \in S$ and $H \in \mathcal{O}(S)$,
(3) the mapping τ is an injective homomorphism from S into $\mathcal{O}(S)$.

Now we deduce several easy properties of the images of order ideals under homomorphisms. For any subset A in S, and for any mapping $\phi\colon S \to S'$, put $A\phi = \{a\phi : a \in A\}$.

Lemma 3. *Let $S, \overline{S}$ be locally inverse semigroups, and let $\phi\colon S \to \overline{S}$ be a surjective homomorphism. Then, for every $a \in S$ and $\overline{b} \in \overline{S}$ with $\overline{b} \leq a\phi$, there exists $b \in S$ such that $b\phi = \overline{b}$ and $b \leq a$.*

Proof. Let $b_0 \in S$ such that $b_0\phi = \overline{b}$, and let $b_1 = aa'b_0a'a$ for some $a' \in V(a)$. Then

$$b_1 \leq_{\mathcal{R}} a \quad \text{and} \quad b_1 \leq_{\mathcal{L}} a. \tag{1}$$

Since $\overline{b} \leq a\phi$, we have $b_1\phi = (aa')\phi \cdot b_0\phi \cdot (a'a)\phi = (aa')\phi \cdot \overline{b} \cdot (a'a)\phi = \overline{b}$, and there exists $\overline{e} \in E_{\overline{S}} \cap L_{\overline{b}}$ such that $\overline{b} = a\phi \cdot \overline{e}$. Let $u \in S$ with $u\phi = \overline{e}$, and let $b_1' \in V(b_1)$. Then $(ub_1'b_1)\phi = u\phi \cdot (b_1'b_1)\phi = \overline{e} \cdot (b_1'b_1)\phi$ where $(b_1'b_1)\phi \in E_{\overline{S}}$ and $(b_1'b_1)\phi \,\mathcal{L}\, b_1\phi = \overline{b} \,\mathcal{L}\, \overline{e}$. Therefore we obtain that $(ub_1'b_1)\phi = \overline{e}$. Lallement's lemma implies that there exists $e \in E_S$ such that $e\phi = \overline{e}$ and $e \leq_{\mathcal{L}} ub_1'b_1$. Hence $e \leq_{\mathcal{L}} b_1 \leq_{\mathcal{L}} a$ follows by (1), and so $e \in Sa$. This implies $ae \leq a$. Since we have $(ae)\phi = a\phi \cdot e\phi = a\phi \cdot \overline{e} = \overline{b}$, we see that the element $b = ae$ has the required properties. □

Lemma 4. *Let $S, \overline{S}$ be locally inverse semigroups, and let $\phi\colon S \to \overline{S}$ be a surjective homomorphism. Then we have*

(1) $(a]\phi = (a\phi]$ for every $a \in S$,
(2) $H\phi \in \mathcal{O}(\overline{S})$ for every $H \in \mathcal{O}(S)$.

Moreover, the mapping $\Phi\colon \mathcal{O}(S) \to \mathcal{O}(\overline{S}),\ H \mapsto H\phi$ is a homomorphism.

Proof. If $a \in S$ then the inclusion $(a]\phi \subseteq (a\phi]$ follows since ϕ preserves $\leq$. The reverse inclusion is implied by Lemma 3. The second statement follows easily from the first one, and the last statement is obvoius by the former statement and by Result 1. □

Now we recall the notion of a weakly E-unitary locally inverse semigroup, the definition of a Pastijn product, and we formulate several results needed later.

The notion of a weakly E-unitary locally inverse semigroup was introduced by Veeramony [16] and by Kaďourek [9] as follows. A locally inverse semigroup S is called *weakly E-unitary* if $e \leq a$ implies $a \in E$ for every $e \in E$ and $a \in S$. It is clear from the definition that a regular subsemigroup in a weakly E-unitary locally inverse semigroup is weakly E-unitary. It is proved in [9] that a locally inverse semigroup is weakly E-unitary if and only if its least completely simple semigroup congruence is idempotent pure.

Let K be any semigroup and let T be a completely simple semigroup. Let us choose a Rees matrix representation for T, say $T = \mathcal{M}[G; I, \Lambda; P]$. Suppose that G acts on K by automorphisms on the left — briefly, G acts on K —, that is, for each $g \in G$, an automorphism $a \mapsto {}^{g}a$ of K is given such that ${}^{h}({}^{g}a) = {}^{hg}a$ for every $a \in K$ and $g, h \in G$. Define a multiplication on $K \times T$ by

$$(a, (i, g, \lambda))\,(b, (j, h, \mu)) = (a \cdot {}^{gp_{\lambda j}}b, (i, gp_{\lambda j}h, \mu))\,.$$

It is straightforward to check that in this way a semigroup is obtained. We call it a *Pastijn product of K by T* and denote it by $K \odot T$. This construction was introduced by Pastijn [14] and was applied also by Kadourek [9], [10]. It was noticed in [9] that $K \odot T$ does not essentially depend on the Rees matrix representation of T chosen.

Result 5. *Given a normal band B and a completely simple semigroup T, the Pastijn product $B \odot T$ is a weakly E-unitary locally inverse semigroup and*

$$E_{B\odot T} = \{(a, e) \in B \odot T : a \in B,\ e \in E_T\}.$$

Moreover, the second projection $\pi\colon B \odot T \to T,\ (a, t) \mapsto t$ is an idempotent pure and surjective homomorphism.

The analogue of O'Carroll's embedding thoerem for locally inverse semigroups was proved in [1]:

Result 6. *Each weakly E-unitary locally inverse semigroup is embeddable into a Pastijn product of a normal band by a completely simple semigroup.*

Finally, we recall several further notions needed later. Let S be a locally inverse semigroup and let ρ be a congruence on S. The congruence ρ is called *perfect* if, for any $a, b \in S$, the set product of the ρ-classes $a\rho$ and $b\rho$ is equal to the ρ-class $(ab)\rho$. Recall that, by Lallement's lemma, a ρ-class is idempotent (in S/ρ) if and only if it contains an idempotent element of S. We say that ρ is *idempotent pure* if each idempotent ρ-class consists of idempotent elements only. Given a class $\mathcal{C}$ of semigroups, ρ is termed a *congruence over* $\mathcal{C}$ if every idempotent ρ-class, as a subsemigroup of S, belongs to $\mathcal{C}$. The classes of semigroups we need later in this respect are those of completely simple semigroups and rectangular bands, denoted by $\mathcal{CS}$ and $\mathcal{RB}$, respectively. Note that the concept of a congruence of a locally inverse semigroup over $\mathcal{CS}$ is a natural generalization of that of an idempotent separating congruence of an inverse semigroup. For, a congruence on an inverse semigroup is over $\mathcal{CS}$ if and only if it is idempotent separating. Given locally inverse semigroups S, T and a homomorphism $\phi\colon S \to T$, we say that ϕ is *perfect*, *idempotent pure*, or a *homomorphism over* $\mathcal{C}$ if the congruence induced by ϕ on S has the respective property.

Let S be a locally inverse semigroup and ρ be a congruence [perfect congruence] on S such that S/ρ is a rectangular band. If $\iota\colon S/\rho \to I \times \Lambda$ is an isomorphism, where I is a left zero and Λ is a right zero semigroup, then put

$$S_{i\lambda} = \{a \in S : (a\rho)\iota = (i, \lambda)\} \quad ((i, \lambda) \in I \times \Lambda).$$

Clearly, each $S_{i\lambda}$ is a subsemigroup in S. In this case, we say that S is a *rectangular band* [*perfect rectangular band*] *of the subsemigroups* $S_{i\lambda}$ $((i, \lambda) \in I \times \Lambda)$.

3. A notion of almost factorizability

In this section we introduce a notion of almost factorizability for locally inverse semigroups, and we compare this notion to those introduced earier for inverse semigroups, for straight locally inverse semigroups and for orthodox locally inverse semigroups, that is, for generalized inverse semigroups.

Let S be a locally inverse semigroup. Consider the semigroup $\mathcal{O}(S)$ of all order ideals of S. We say that S is *almost factorizable* if there exists a completely simple subsemigroup $\mathcal{U}$ in $\mathcal{O}(S)$ such that the following conditions are satisfied:

(AF1) $\bigcup E_{\mathcal{U}} = E_S$,
(AF2) $\bigcup \mathcal{U} \supseteq S$.

Notice that the reverse inclusion also holds in condition (AF2) by the definition of $\mathcal{U}$.

Let S be an almost factorizable locally inverse semigroup, and suppose that $\mathcal{U}$ is a completely simple subsemigroup of $\mathcal{O}(S)$ satisfying conditions (AF1) and (AF2). We formulate several basic properties of $\mathcal{U}$.

Lemma 7. *For any $H \in \mathcal{U}$, the following are equivalent:*

(1) $H \in E_{\mathcal{U}}$,
(2) H is a subband in E_S,
(3) H is a normal subband in E_S.

Proof. Suppose first that $H \in E_{\mathcal{U}}$. Then $H^2 = H$ and, by (AF1), $H \subseteq E_S$. Therefore H is a subsemigroup, and so a subband in E_S. Since S is locally inverse, each subband of S is also locally inverse, that is, a normal subband. Conversely, if H is a (normal) subband in E_S then we clearly have $H^2 = H$ whence $H \in E_{\mathcal{U}}$ follows. □

Lemma 8. *For any $H, H' \in \mathcal{U}$ with $H' \in V(H)$ and for any $a \in H$, there exists $a' \in V(a) \cap H'$.*

Proof. We use the main idea of the proof of [15, Lemma 3.11]. Let $H, H' \in \mathcal{U}$ with $H' \in V(H)$ and let $a \in H$. Since $HH'H = H$, there exist elements $a_1, a_2 \in H$ and $a^\dagger \in H'$ such that $a = a_1 a^\dagger a_2$, and, applying the same idea for a_1, there exist elements $b_1, b_2 \in H$ and $b^\dagger \in H'$ such that $a_1 = b_1 b^\dagger b_2$. Then we have

$$aa^\dagger a = a_1 a^\dagger a_2 a^\dagger a_1 a^\dagger a_2 = b_1 b^\dagger b_2 a^\dagger a_2 a^\dagger a_1 a^\dagger a_2$$

where $b^\dagger b_2,\ a^\dagger a_2,\ a^\dagger a_1 \in H'H$. Since $H'H \in E_{\mathcal{U}}$, Lemma 7 implies that $H'H$ is a normal band. Hence we see that

$$b_1 b^\dagger b_2 a^\dagger a_2 a^\dagger a_1 a^\dagger a_2 = b_1 b^\dagger b_2 a^\dagger a_1 a^\dagger a_2 a^\dagger a_2 = b_1 b^\dagger b_2 a^\dagger a_1 a^\dagger a_2 = a_1 a^\dagger a_1 a^\dagger a_2.$$

Here $a_1 a^\dagger \in HH'$ which also belongs to $E_{\mathcal{U}}$, and so $HH' \subseteq E_S$ by property (AF1). Therefore $a_1 a^\dagger a_1 a^\dagger = a_1 a^\dagger$, and we deduce that

$$aa^\dagger a = a_1 a^\dagger a_1 a^\dagger a_2 = a_1 a^\dagger a_2 = a.$$

Hence it follows that $a' = a^\dagger a a^\dagger \in V(a)$ and $a' \in H'HH' = H'$. □

Now we establish that, for inverse semigroups, almost factorizability in our sense coincides with the usual notion of almost factorizability (see [11], [12]).

Proposition 9. *An inverse semigroup is almost factorizable in the usual sense if and only if it is almost factorizable as a locally inverse semigroup.*

Proof. If S is an inverse semigroup then, by definition, $\Sigma(S)$ is a subgroup in $\mathcal{O}(S)$ with identity element E_S. Moreover, if S is almost factorizable in the usual sense then $\bigcup \Sigma(S) \supseteq S$. Thus $\mathcal{U} = \Sigma(S)$ satisfies conditions (AF1) and (AF2).

Conversely, assume that S is an inverse semigroup and $\mathcal{U}$ is a completely simple subsemigroup in $\mathcal{O}(S)$ within properties (AF1) and (AF2). Let $H_1, H_2 \in E_{\mathcal{U}}$. By Lemma 7, H_1 and H_2 are subsemilattices in the semilattice E_S, whence $H_1H_2 = H_2H_1$ follows. Thus the idempotents commute in $\mathcal{U}$, and so $\mathcal{U}$ is necessarily a group. Now property (AF1) ensures that the identity of $\mathcal{U}$ is E_S. This implies that, for every $H \in \mathcal{U}$ and for its group inverse H^{-1} in $\mathcal{U}$, we have $HH^{-1} = H^{-1}H = E_S$. Thus $\mathcal{U}$ is a subgroup in $\Sigma(S)$, and we deduce by property (AF2) that S is almost factorizable in the usual sense. □

A notion of almost factorizability was defined for straight locally inverse semigroups in [2]. For the precise definition that involves the notion of permissible sets of straight locally inverse semigroups, the reader is referred to [2]. By definition, one can easily see that, for straight locally inverse semigroups, our notion relates to that in [2] as follows.

Proposition 10. *A straight locally inverse semigroup which is almost factorizable in the sense of [2], is almost factorizable as a locally inverse semigroup.*

Finally, let us consider orthodox locally inverse semigroups, that is, generalized inverse semigroups. A notion of almost factorizability for orthodox semigroup was introduced in [7]. Almost factorizability in the latter sense and that as a locally inverse semigroup are different. It is shown in [7] that if S is a generalized inverse semigroup then it is almost factorizable as an orthodox semigroup if and only if it is an idempotent separating homomorphic image of a semidirect product of a normal band by a group, and its greatest inverse semigroup homomorphic image is almost factorizable if and only if it is a homomorphic image of a semidirect product of a normal band by a group. Thus we will be able to deduce by Theorem 22 the following characterization of generalized inverse semigroups which are almost factorizable in our sense.

Proposition 11. *A generalized inverse semigroup is almost factorizable as a locally inverse semigroup if and only if its greatest inverse semigroup homomorphic image is almost factorizable.*

4. Characterization of almost factorizable locally inverse semigroups

In this section we prove that the almost factorizable locally inverse semigroups are just the homomorphic images of Pastijn products of normal bands by completely simple semigroups.

First we show that Pastijn products of normal bands by completely simple semigroups belong to the class of almost factorizable locally inverse semigroups.

Proposition 12. *Each Pastijn product of a normal band by a completely simple semigroup is an almost factorizable locally inverse semigroup.*

Proof. Let B be a normal band, $T = \mathcal{M}[G; I, \Lambda; P]$ a completely simple semigroup, and suppose that G acts on B. We show that the Pastijn product $B \odot T$ is an almost factorizable locally inverse semigroup. By Result 5, $B \odot T$ is a locally inverse semigroup. If $(a, t), (b, u) \in B \odot T$ with $(a, t) \leq (b, u)$ then $t = (a, t)\pi \leq (b, u)\pi = u$ in T whence $t = u$ follows. This implies that the subsets

$$H_t = \{(a, t) \in B \odot T : a \in B\} \quad (t \in T)$$

form order ideals in $B \odot T$. Furthermore, we obviously have $H_t H_u \subseteq H_{tu}$ for every $t, u \in T$. In order to verify the reverse inclusion, suppose that $t = (i, g, \lambda)$, $u = (j, h, \mu)$ and $(a, (i, gp_{\lambda j}h, \mu))$ is an arbitrary element in H_{tu}. It is straightforward to check that

$$(a, (i, gp_{\lambda j}h, \mu)) = (a, (i, g, \lambda)) \left({}^{p_{\lambda j}^{-1} g^{-1}}a, (j, h, \mu)\right)$$

where $(a, (i, g, \lambda)) \in H_t$ and $\left({}^{p_{\lambda j}^{-1} g^{-1}}a, (j, h, \mu)\right) \in H_u$, and so the equality $H_t H_u = H_{tu}$ holds. This implies that

$$\mathcal{U} = \{H_t : t \in T\}$$

is a subsemigroup in $\mathcal{O}(B \odot T)$ which is isomorphic to T, and so $\mathcal{U}$ is completely simple. Result 5 implies that $\mathcal{U}$ possesses property (AF1). Since $B \odot T = \bigcup_{t \in T} H_t$ by definition, condition (AF2) is also satisfied. □

Now we verify that the class of all almost factorizable locally inverse semigroups is closed under taking homomorphic images.

Proposition 13. *Each homomorphic image of an almost factorizable locally inverse semigroup is almost factorizable.*

Proof. Let $S, \overline{S}$ be locally inverse semigroups, and let $\phi\colon S \to \overline{S}$ be a surjective homomorphism. Suppose that S is almost factorizable. We verify that $\overline{S}$ is also almost factorizable. Let $\mathcal{U}$ be a completely simple subsemigroup of $\mathcal{O}(S)$ satisfying conditions (AF1) and (AF2). Consider the homomorphism $\Phi\colon \mathcal{O}(S) \to \mathcal{O}(\overline{S}),\ H \mapsto H\phi$ from Lemma 4. Put $\overline{\mathcal{U}} = \mathcal{U}\Phi$. Obviously, $\overline{\mathcal{U}}$ is a completely simple subsemigroup in $\mathcal{O}(\overline{S})$. Let $\overline{H} \in E_{\overline{\mathcal{U}}}$. By Lallement's lemma, there exists $H \in E_{\mathcal{U}}$ such that $\overline{H} = H\Phi = H\phi$. Since (AF1) holds for S and $\mathcal{U}$, we have $H \subseteq E_S$, and this implies $H\phi \subseteq E_S\phi \subseteq E_{\overline{S}}$. Thus $\bigcup E_{\overline{\mathcal{U}}} \subseteq E_{\overline{S}}$. Furthermore, if $\overline{e} \in E_{\overline{S}}$ then, again applying Lallement's lemma, we obtain $e \in E_S$ such that $\overline{e} = e\phi$. Property (AF1) of S and $\mathcal{U}$ ensures that $e \in H$ for some $H \in E_{\mathcal{U}}$, and so $\overline{e} = e\phi \in H\phi = H\Phi$ where $H\Phi \in E_{\overline{\mathcal{U}}}$. This shows that $\overline{S}$ and $\overline{\mathcal{U}}$ fulfil condition (AF1). Property (AF2) of S and $\mathcal{U}$ and surjectivity of ϕ imply that $\overline{S} = S\phi \subseteq \left(\bigcup \mathcal{U}\right)\phi \subseteq \bigcup \mathcal{U}\phi = \bigcup \overline{\mathcal{U}}$. Therefore property (AF2) is valid also for $\overline{S}$ and $\overline{\mathcal{U}}$. □

The main part of this section is devoted to proving that any almost factorizable locally inverse semigroup can be obtained as a homomorphic image over $\mathcal{CS}$ of a Pastijn product of a normal band by a completely simple semigroup.

First we apply a construction well known in the inverse case. Let S be an almost factorizable locally inverse semigroup, and let $\mathcal{U}$ be a completely simple subsemigroup of $\mathcal{O}(S)$ satisfying conditions (AF1) and (AF2). Consider the following subset of the direct product $S \times \mathcal{U}$:

$$S \bowtie \mathcal{U} = \{(a, H) \in S \times \mathcal{U} : a \in H\}.$$

Lemma 14.

(1) The set $S \bowtie \mathcal{U}$ forms a locally inverse subsemigroup in the direct product $S \times \mathcal{U}$, and

$$\begin{aligned} E_{S \bowtie \mathcal{U}} &= \{(e, H) \in S \bowtie \mathcal{U} : H \in E_{\mathcal{U}}\} \\ &= \{(e, H) \in S \times \mathcal{U} : e \in H,\ H \in E_{\mathcal{U}}\}. \end{aligned}$$

(2) The first projection $\pi_1 \colon S \bowtie \mathcal{U} \to S,\ (a, H) \mapsto a$ is a surjective homomorphism over $\mathcal{CS}$.

(3) The second projection $\pi_2 \colon S \bowtie \mathcal{U} \to \mathcal{U},\ (a, H) \mapsto H$ is a surjective, idempotent pure and perfect homomorphism.

Proof. (1) It is straightforward that $S \bowtie \mathcal{U}$ is a subsemigroup in $S \times \mathcal{U}$. Lemma 8 implies that it is regular, and so it is locally inverse. Since $(e, H) \in E_{S \times \mathcal{U}}$ if and only if $e \in E_S$ and $H \in E_{\mathcal{U}}$, the equality $E_{S \bowtie \mathcal{U}} = \{(e, H) \in S \times \mathcal{U} : e \in H,\ H \in E_{\mathcal{U}}\}$ follows by property (AF1).

(2) It is clear by (AF2) that π_1 is a surjective homomorphism. The idempotent classes of the congruence induced by π_1 are $\mathbf{C}_e = \{(e, H) \in S \times \mathcal{U} : H \in \mathcal{U} \text{ and } e \in H\}$ $(e \in E_S)$, and, for any $e \in E_S$, the semigroup $\mathbf{C}_e$ is isomorphic to the subsemigroup $\mathcal{U}_e = \{H \in \mathcal{U} : e \in H\}$ of the completely simple semigroup $\mathcal{U}$. We show that $\mathcal{U}_e$ is regular for any $e \in E_S$. Let $e \in E_S$ and $H \in \mathcal{U}_e$. Consider the group inverse H^{-1} of H in $\mathcal{U}$. Then we have $HH^{-1} = H^{-1}H \in E_{\mathcal{U}}$ whence it follows by Lemma 7 that HH^{-1} is a subband in E_S. By Lemma 8, there exists $e' \in V(e) \cap H^{-1}$, and so $ee', e'e \in HH^{-1}$, and $e' = e'ee' = e'e \cdot ee' \in HH^{-1}$. Thus $e' \in E_S$, and, similarly, we see that $e = ee'e = ee' \cdot e'e \in HH^{-1}$. This implies that $ee' = ee' \cdot e' \in HH^{-1} \cdot H^{-1} = H^{-1}HH^{-1} = H^{-1}$, and so $e = ee' \cdot e \in H^{-1} \cdot HH^{-1} = H^{-1}$. Therefore $H^{-1} \in \mathcal{U}_e$, and we have established that $\mathcal{U}_e$ is regular. Hence it follows that $\mathcal{U}_e$ is completely simple whence the same holds for $\mathbf{C}_e$ $(e \in E_S)$.

(3) Obviously, π_2 is a surjective homomorphism, and the classes of the congruence κ induced by π_2 are $\mathbf{K}_H = \{(a, H) \in S \times \mathcal{U} : a \in H\}$ $(H \in \mathcal{U})$. In particular, the idempotent classes are just those where $H \in E_{\mathcal{U}}$. Lemma 7 and statement (1) imply that in this case, we have $\mathbf{K}_H \subseteq E_{S \bowtie \mathcal{U}}$, and so κ is idempotent pure. In order to check that κ is perfect, we have to show that $\mathbf{K}_{HJ} \subseteq \mathbf{K}_H \mathbf{K}_J$ for every $H, J \in \mathcal{U}$. However, this is obvious, since, for any $(c, HJ) \in \mathbf{K}_{HJ}$, we have $c \in HJ$ whence there exist $a \in H$ and $b \in J$ with $c = ab$, and so $(a, H) \in \mathbf{K}_H$, $(b, J) \in \mathbf{K}_J$ with $(c, HJ) = (a, H)(b, J)$. □

Lemma 15. *Let T be a locally inverse semigroup which possesses an idempotent pure and perfect congruence τ such that T/τ is completely simple. Then T is weakly E-unitary and almost factorizable.*

Proof. Since T has an idempotent pure completely simple semigroup congruence, it is weakly E-unitary. Notice that the τ-classes form order ideals in T. For, if $b \leq a$ in T then $b\tau \leq a\tau$ in T/τ. However, T/τ is completely simple where the natural partial order is trivial. Thus $b\tau = a\tau$ follows, that is, we deduce that $b \in a\tau$. Since τ is perfect, the τ-classes form a subsemigroup with respect to the set product, and so we obtain that T/τ is a subsemigroup in $\mathcal{O}(T)$. Clearly, condition (AF2) is fulfilled. Finally, property (AF1) follows since τ is idempotent pure. □

We proceed by showing that each T described in the previous lemma can be obtained as a homomorphic image over $\mathcal{CS}$ of a Pastijn product of a normal band by a completely simple semigroup.

From now on, let T be a locally inverse semigroup, and let τ be an idempotent pure and perfect congruence on T such that T/τ is completely simple. Suppose that $T/\tau = \mathcal{M}[G; I, \Lambda; P]$. Denote by $G_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$ the $\mathcal{H}$-class $\{(i, g, \lambda) : g \in G\}$ of $\mathcal{M}[G; I, \Lambda; P]$. Put

$$T_{i\lambda} = \{a \in T : a\tau \in G_{i\lambda}\} \quad (i \in I,\ \lambda \in \Lambda).$$

Lemma 16.

(1) For any $i \in I,\ \lambda \in \Lambda$, the set $T_{i\lambda}$ forms a generalized inverse subsemigroup in T, and T is a perfect rectangular band of these subsemigroups $T_{i\lambda}$ $((i, \lambda) \in I \times \Lambda)$.

(2) For any $i \in I$, the set $T_i = \bigcup_{\lambda \in \Lambda} T_{i\lambda}$ forms a generalized inverse subsemigroup in T.

Proof. (1) It is obvious that $T_{i\lambda}$ is a subsemigroup for any $i \in I,\ \lambda \in \Lambda$, and T is a rectangular band of these subsemigroups. Perfectness follows from the facts that τ is a perfect congruence on T and $\mathcal{H}$ is a perfect congruence on T/τ. Let $i \in I,\ \lambda \in \Lambda$. Recall from Lemma 15 and its proof that T is almost factorizable and T/τ is a completely simple subsemigroup of $\mathcal{O}(T)$ satisfying (AF1) and (AF2). Hence it follows by Lemma 7 that

$$E_{T_{i\lambda}} = \{a \in T : a\tau = (i, p_{\lambda i}^{-1}, \lambda)\} \tag{2}$$

is a normal subband in $T_{i\lambda}$. Moreover, applying Lemma 8 for any $a \in T_{i\lambda}$, for its τ-class and its group inverse within T/τ, we obtain an inverse a' of a within $T_{i\lambda}$. Thus $T_{i\lambda}$ is regular, and so it is a generalized inverse semigroup.

(2) Since T is a rectangular band of $T_{i\lambda}$ $((i, \lambda) \in I \times \Lambda)$, it is immediate that T_i $(i \in I)$ is a subsemigroup in T. Moreover, it is also regular since each $T_{i\lambda}$ is regular. Finally, (2) implies that E_{T_i} is a subband in E_T whence it follows that T_i is a generalized inverse semigroup. □

Denote by $\gamma_{i\lambda}$ the least inverse semigroup congruence on $T_{i\lambda}$, and consider the equivalence relation γ on T whose classes are just the $\gamma_{i\lambda}$-classes of the pairwise disjoint subsemigroups $T_{i\lambda}$. Recall that two elements $a, b \in T_{i\lambda}$ are $\gamma_{i\lambda}$-related if and only if $V(a) \cap V(b) \neq \emptyset$, or, equivalently, if and only if $V(a) = V(b)$ in $T_{i\lambda}$.

Lemma 17.

(1) For any $a \in T$ and $(i, \lambda) \in I \times \Lambda$, there exists an inverse a' of a in $T_{i\lambda}$, and the set of all inverses of a belonging to $T_{i\lambda}$ is just the $\gamma_{i\lambda}$-class of a' in $T_{i\lambda}$.

(2) For any $a, b \in T$, we have $a \,\gamma\, b$ if and only if a and b have a common inverse in T.

(3) The relation γ is a congruence on T over $\mathcal{RB}$, and it is contained in τ.

Proof. (1) By Lemma 15 and its proof, T is almost factorizable, and conditions (AF1), (AF2) are satisfied by the completely simple subsemigroup $\mathcal{U} = T/\tau$ of $\mathcal{O}(T)$. Applying Lemma 8 for this $\mathcal{U}$, we obtain that, for each $a \in T$ and $(i, \lambda) \in I \times \Lambda$, there exists an inverse a' of a in $T_{i\lambda}$.

In order to verify the second statement, let $a \in T_{j\mu}$ and $a', a'' \in V(a) \cap T_{i\lambda}$. Since $T_{i\lambda}$ is regular by Lemma 16(1), there is an inverse $(a'')'$ of a'' in $T_{i\lambda}$. Thus $a' = a'a \cdot a''(a'')' \cdot a'' \cdot (a'')'a'' \cdot aa'$ where $a'a \in E_{T_{i\mu}} \subseteq E_{T_i}$ and $a''(a'')' \in E_{T_{i\lambda}} \subseteq E_{T_i}$. Furthermore, we have $a'a \,\mathcal{L}\, a''a \,\mathcal{R}\, a''(a'')'$ in E_{T_i}, and so, by Lemma 16(2), we obtain that $a''(a'')'$ and $a'a \cdot a''(a'')'$ are $\mathcal{D}$-related elements in $E_{T_{i\lambda}}$. Similarly, one can see that $(a'')'a''$ and $(a'')'a'' \cdot aa'$ are also $\mathcal{D}$-related elements in $E_{T_{i\lambda}}$. This implies that $a' \,\gamma_{i\lambda}\, a''$ in $T_{i\lambda}$.

Conversely, let $a \in T_{j\mu}$, $a' \in V(a) \cap T_{i\lambda}$, and consider an element $b \in T_{i\lambda}$ which is $\gamma_{i\lambda}$-related to a'. Then $b = ea'f$ for some $e, f \in E_{T_{i\lambda}}$ where $e \,\mathcal{D}\, a'(a')'$ and $f \,\mathcal{D}\, (a')'a'$ in $E_{T_{i\lambda}}$ for any $(a')' \in V(a') \cap T_{i\lambda}$. Then we have

$$aba = aea'fa = a \cdot a'aea'a \cdot a' \cdot aa'faa' \cdot a$$

where $a'a \,\mathcal{R}\, a'(a')'\mathcal{D}e$ in E_{T_i} and $aa' \,\mathcal{R}\, (a')'a'\mathcal{D}f$ in E_{T_i}. Hence $a'aea'a = a'a$ and $aa'faa' = aa'$ follows, and so we deduce that $aba = a \cdot a'a \cdot a' \cdot aa' \cdot a = a$. Similarly, we also see that $bab = ea'faea'f = ea' \cdot aa'faa' \cdot a \cdot a'aea'a \cdot a'f = ea' \cdot aa' \cdot a \cdot a'a \cdot a'f = ea'f = b$. Thus we have verified that b is an inverse of a.

(2) This statement is a direct consequence of (1) and of the description of the least inverse semigroup congruence on a generalized inverse semigroup, mentioned above.

(3) We verify left compatibility of γ, the proof of right compatibility being dual. Let $a, b \in T_{i\lambda}$ with $a \,\gamma_{i\lambda}\, b$, and let $c \in T_{j\mu}$. We have to show that $ca \,\gamma\, cb$. Suppose that $x \in T_{i\lambda} \cap V(a) \cap V(b)$. Moreover, by (1), let $c' \in T_{i\lambda} \cap V(c)$. Notice that $c'c \in E_{T_{i\mu}} \subseteq E_{T_i}$ and $ax \,\mathcal{D}\, bx$ in $E_{T_{i\lambda}} \subseteq E_{T_i}$. Since E_{T_i} is a (normal) band by Lemma 16(2) where $\mathcal{D}$ is a congruence, and $c'cax, c'cbx \in E_{T_{i\lambda}}$, we obtain that $c'cax \,\mathcal{D}\, c'cbx$ in E_{T_i}, and so also in $E_{T_{i\lambda}}$. Hence it follows that $x(c'cax)c' \in V(cb)$. For, we have $cb \cdot x(c'cax)c' \cdot cb = c(c'cbx)(c'cax)(c'cbx)b = c(c'cbx)b = cb$ and $x(c'cax)c' \cdot cb \cdot x(c'cax)c' = x(c'cax)(c'cbx)(c'cax)c' = x(c'cax)c'$. A similar but

easier calculation shows that $x(c'cax)c' \in V(ca)$ also holds. This shows by (2) that $ca\,\gamma\,cb$.

Finally, observe that, for any $(i,\lambda) \in I \times \Lambda$, the congruence $\gamma_{i\lambda}$ is over $\mathcal{RB}$, and so the same holds for γ. Moreover, the restriction $\tau_{i\lambda} = \tau \cap (T_{i\lambda} \times T_{i\lambda})$ of τ to $T_{i\lambda}$ is a group congruence. Since $\gamma_{i\lambda}$ is the least inverse semigroup congruence on $T_{i\lambda}$, we clearly have $\gamma_{i\lambda} \subseteq \tau_{i\lambda}$ which implies $\gamma \subseteq \tau$. □

Remark 18. Let T be, as above, a locally inverse semigroup which possesses an idempotent pure and perfect congruence τ such that T/τ is completely simple.

(1) The previous lemma implies that T/γ is a rectangular band $I \times \Lambda$ of the inverse semigroups $T_{i\lambda}/\gamma_{i\lambda}$ $((i,\lambda) \in I\times\Lambda)$. Notice that γ is the smallest congruence ρ on T such that T/ρ is a rectangular band of the inverse semigroups $T_{i\lambda}/\rho_{i\lambda}$ $((i,\lambda) \in I \times \Lambda)$, where $\rho_{i\lambda} = \rho \cap (T_{i\lambda} \times T_{i\lambda})$.
(2) In particular, if $T = B \odot \mathcal{M}[G; I, \Lambda; P]$ where B is a normal band then we have $(a,(i,g,\lambda))\ \gamma\ (b,(j,h,\mu))$ if and only if $a\,\mathcal{D}\,b$ and $(i,g,\lambda) = (j,h,\mu)$. Since $\mathcal{D}$ is the least semilattice congruence on B, the action of G on B induces an action of G on the semilattice $B/\mathcal{D}$ by the rule ${}^{g}D_a = D_{{}^{g}a}$ $(g \in G,\ a \in B)$. The Pastijn product $(B/\mathcal{D}) \odot \mathcal{M}[G; I, \Lambda; P]$ defined by this action is isomorphic to T/γ.

We have seen that T/γ is a rectangular band $I \times \Lambda$ of the inverse semigroups $T_{i\lambda}/\gamma_{i\lambda}$ $((i,\lambda) \in I\times\Lambda)$. Furthermore, τ/γ is an idempotent pure and perfect congruence on T/γ such that $(T/\gamma)/(\tau/\gamma)$ is a completely simple semigroup isomorphic to T/τ. By Lemma 15, T/γ is weakly E-unitary, and so each inverse semigroup $T_{i\lambda}/\gamma_{i\lambda}$ $((i,\lambda) \in I \times \Lambda)$ is E-unitary. The same lemma and its proof implies that the completely simple semigroup $\mathcal{U} = (T/\gamma)/(\tau/\gamma)$ satisfies conditions (AF1) and (AF2). Hence it follows that, for any $(i,\lambda) \in I \times \Lambda$, the $\tau_{i\lambda}/\gamma_{i\lambda}$-classes form a subgroup in $\mathcal{U}$ with identity element $E_{T_{i\lambda}/\gamma_{i\lambda}}$. It is easy to deduce (see the proof of Proposition 9) that the $\tau_{i\lambda}/\gamma_{i\lambda}$-classes form permissible sets, whence we deduce that the inverse semigroup $T_{i\lambda}/\gamma_{i\lambda}$ is almost factorizable. These two properties imply that the inverse semigroup $T_{i\lambda}/\gamma_{i\lambda}$ is isomorphic to a semidirect product of a semilattice by a group for any $(i,\lambda) \in I\times\Lambda$. Thus [14, Theorem 4.6] implies that T/γ is isomorphic to a Pastijn product $Y \odot \mathcal{M}[G; I, \Lambda; P]$ where $\mathcal{M}[G; I, \Lambda; P] = T/\tau$ and, moreover, Y is isomorphic to $E_{T_{i\lambda}/\gamma_{i\lambda}}$ and G to $(T_{i\lambda}/\gamma_{i\lambda})/(\sigma_{i\lambda}/\gamma_{i\lambda})$, and so to $T_{i\lambda}/\sigma_{i\lambda}$ for any $(i,\lambda) \in I \times \Lambda$. Notice that the Pastijn product $Y \odot \mathcal{M}[G; I, \Lambda; P]$ involves an action of G on Y.

Let us fix an isomorphism $\iota\colon Y \odot \mathcal{M}[G; I, \Lambda; P] \to T/\gamma$, and put

$$E^{\alpha}_{i\lambda} = \left\{e \in T : (e\gamma)\iota^{-1} = (\alpha,(i,p^{-1}_{\lambda i},\lambda))\right\} \quad (\alpha \in Y,\ (i,\lambda) \in I \times \Lambda).$$

Notice that

$$E^{\alpha}_{i\lambda}E^{\alpha}_{i\mu} \subseteq E^{\alpha}_{i\mu} \quad \text{and} \quad E^{p^{-1}_{\lambda i}\alpha}_{i\lambda}E^{p^{-1}_{\lambda j}\alpha}_{j\lambda} \subseteq E^{p^{-1}_{\lambda i}\alpha}_{i\lambda}, \tag{3}$$

since

$$(\alpha,(i,p_{\lambda i}^{-1},\lambda))(\alpha,(i,p_{\mu i}^{-1},\mu)) = (\alpha,(i,p_{\mu i}^{-1},\mu)),$$
$$({}^{p_{\lambda i}^{-1}}\alpha,(i,p_{\lambda i}^{-1},\lambda))({}^{p_{\lambda j}^{-1}}\alpha,(j,p_{\lambda j}^{-1},\lambda)) = ({}^{p_{\lambda i}^{-1}}\alpha,(i,p_{\lambda i}^{-1},\lambda)).$$

Moreover, $E_{T_{i\lambda}}$, or briefly $E_{i\lambda}$, is a semilattice Y of the rectangular bands $E_{i\lambda}^{\alpha}$ ($\alpha \in Y$). For our later convenience, denote by $\nu_{i\lambda}$ the surjective homomorphism $E_{i\lambda} \to Y$ which assigns α to every element in $E_{i\lambda}^{\alpha}$, and put $E_i = \bigcup_{\lambda\in\Lambda} E_{i\lambda},\ E_\lambda = \bigcup_{i\in I} E_{i\lambda}$. Here $E_i = E_{T_i}$ is a normal band by Lemma 16(2), and it can be seen by (3) to be a semilattice Y of the rectangular bands $E_i^{\alpha} = \bigcup_{\lambda\in\Lambda} E_{i\lambda}^{\alpha}$ ($\alpha \in Y$). Dually, $E_\lambda = E_{T_\lambda}$ is a normal band, and a semilattice Y of the rectangular bands $E_\lambda^{p_\lambda^{-1}\alpha} = \bigcup_{i\in I} E_{i\lambda}^{p_{\lambda i}^{-1}\alpha}$ ($\alpha \in Y$).

We shall need the following property.

Lemma 19. *For every $i \in I$, $\lambda,\mu \in \Lambda$ and for every $a \in E_{i\lambda}$, there exists $b \in E_{i\mu}$ such that $a\,\mathcal{R}\,b$.*

Proof. Since τ is a perfect congruence on T, we have $E_{i\mu}E_{i\lambda} = E_{i\lambda}$, and so $a = xy$ for some $x \in E_{i\mu}$ and $y \in E_{i\lambda}$. The band E_i is normal, therefore $a = axaya$ follows where $a\,\mathcal{D}\,ax\,\mathcal{D}\,aya$ in E_i, $ax \in E_{i\mu}$ and $aya \in E_{i\lambda}$. This implies $a = ba$ where $b = ax \in E_{i\mu}$ and $a = aya \in E_{i\lambda}$. Hence we obtain $a\,\mathcal{R}\,b$. □

Given T as above, we find a normal band B, which is a semilattice Y of rectangular bands B^{α} ($\alpha \in Y$), and we define a homomorphism $\psi\colon B \odot \mathcal{M}[G;I,\Lambda;P] \to T$ with certain properties. If ψ is such a homomorphism, then, by Lemma 17 and Remark 18, there is a unique homomorphism $\psi'\colon Y \odot \mathcal{M}[G;I,\Lambda;P] \to T/\gamma$ such that, for any $b \in B^{\alpha}$, we have $((b,(i,g,\lambda))\,\psi)\gamma = (\alpha,(i,g,\lambda))\,\psi'$. In this case, we say that ψ' is *induced by* ψ.

Lemma 20. *Let T be a locally inverse semigroup which possesses an idempotent pure and perfect congruence τ such that T/τ is completely simple. Put an isomorphism $\iota\colon Y \odot \mathcal{M}[G;I,\Lambda;P] \to T/\gamma$ as above. Then there exists a normal band B acted upon by G and a surjective homomorphism $\psi\colon B \odot \mathcal{M}[G;I,\Lambda;P] \to T$ with the following properties:*

(a) B is a semilattice Y of rectangular bands B^{α} ($\alpha \in Y$),
(b) the homomorphism $Y \odot \mathcal{M}[G;I,\Lambda;P] \to T/\gamma$ induced by ψ is just ι.

Consequently, ψ is a surjective homomorphism over $\mathcal{RB}$.

Proof. The main idea of the construction is similar to that in the proof of [7, Theorem 11].

Let U be a set such that $|U| > |E_i^{\alpha}/\mathcal{R}|, \left|E_\lambda^{p_\lambda^{-1}\alpha}/\mathcal{L}\right|$ for every $\alpha \in Y$, $i \in I$ and $\lambda \in \Lambda$. Consider a set $X = \{x_u^{\alpha} : \alpha \in Y,\ u \in U\}$ where $Y \times U \to X,\ (\alpha,u) \mapsto x_u^{\alpha}$

is a bijection. Based on the action of G on Y, let us define an action of G on X by putting ${}^g(x_u^\alpha) = x_u^{{}^g\alpha}$, and extend it in the unique possible way to an action of G on the free normal band B_X. Furthermore, consider the mapping $\pi\colon X \to Y,\ x_u^\alpha \mapsto \alpha$. For any $\alpha \in Y$, $i \in I$ and $\lambda \in \Lambda$, define surjective mappings $r_i^\alpha\colon U \to E_i^\alpha/\mathcal{R}$ and $\ell_\lambda^{p_\lambda^{-1}\alpha}\colon U \to E_\lambda^{p_\lambda^{-1}\alpha}/\mathcal{L}$. By Lemma 19 and its dual, the mapping $d_{i\lambda}^\alpha\colon E_{i\lambda}^\alpha \to E_i^\alpha/\mathcal{R} \times E_\lambda^{p_{\lambda i}p_\lambda^{-1}\alpha}/\mathcal{L},\ e \mapsto (R_e, L_e)$ is a bijection. Define $\varepsilon_{i\lambda}\colon X \to E_{i\lambda}$ by $x_u^\alpha\varepsilon_{i\lambda} = \left(ur_i^\alpha, u\ell_\lambda^{p_{\lambda i}p_\lambda^{-1}\alpha}\right)(d_{i\lambda}^\alpha)^{-1}$. Hence it is clear that

$$x\varepsilon_{i\lambda}\,\mathcal{R}\,x\varepsilon_{i\mu} \quad \text{for every} \quad x \in X \quad \text{and for every} \quad i \in I,\ \lambda,\mu \in \Lambda, \tag{4}$$

and

$$x\varepsilon_{i\lambda}\,\mathcal{L}\,x\varepsilon_{j\lambda} \quad \text{for every} \quad x \in X \quad \text{and for every} \quad i,j \in I,\ \lambda \in \Lambda. \tag{5}$$

Furthermore, $\{x_u^\alpha\varepsilon_{i\lambda} : u \in U\}$ generates $E_{i\lambda}^\alpha$, and so $X\varepsilon_{i\lambda}$ generates $E_{i\lambda}$ for any $(i,\lambda) \in I \times \Lambda$. Therefore, since Y and $E_{i\lambda}$ are normal bands, the mappings π and $\varepsilon_{i\lambda}$ can be uniquely extended to surjective homomorphisms $\pi\colon B_X \to Y$ and $\varepsilon_{i\lambda}\colon B_X \to E_{i\lambda}$. One can easily see that properties (4) and (5) remain valid now for any $z \in B_X$. For, the $\mathcal{R}[\mathcal{L}]$-class of $z\varepsilon_{i\lambda}$ is uniquely determined by $z\pi$ and by the $\mathcal{R}[\mathcal{L}]$-class of $x\varepsilon_{i\lambda}$ where x is the first [last] letter in the word z. Moreover, we have $\varepsilon_{i\lambda}\nu_{i\lambda} = \pi$ since $x_u^\alpha\varepsilon_{i\lambda} \in E_{i\lambda}^\alpha$ by definition. This implies that, for any $i \in I$ and $\lambda \in \Lambda$, the congruence $\rho_{i\lambda}$ on B_X induced by $\varepsilon_{i\lambda}$ is contained in the congruence induced by π, and if $a, b \in B_X$ are such that $a\pi = b\pi$ then $a\rho_{i\lambda}$ and $b\rho_{i\lambda}$ belong to the same $\mathcal{D}$-class in $B_X/\rho_{i\lambda}$. Consider the smallest congruence ρ on B_X such that $a\rho\,\mathcal{D}\,b\rho$ in B_X/ρ for any $a, b \in B_X$ with $a\pi = b\pi$, and denote it by δ. [It is routine to see that δ is the congruence on B_X generated by $\{(aba, a) : a, b \in B_X \text{ with } a\pi = b\pi\}$.]

Put $B = B_X/\delta$. By definition, δ is contained in the congruence induced by π, and also $\delta \subseteq \rho_{i\lambda}$ for any $i \in I,\ \lambda \in \Lambda$. Therefore there is a unique homomorphism $\overline{\pi}\colon B \to Y$ with $\pi = \delta^\natural\overline{\pi}$. Similarly, for any $i \in I$ and $\lambda \in \Lambda$, there is a unique homomorphism $\overline{\varepsilon_{i\lambda}}\colon B \to E_{i\lambda}$ with $\varepsilon_{i\lambda} = \delta^\natural\overline{\varepsilon_{i\lambda}}$. These homomorphisms are surjective, the congruence on B induced by $\overline{\pi}$ is $\mathcal{D}$, and we have $\overline{\pi} = \overline{\varepsilon_{i\lambda}}\nu_{i\lambda}$. It is routine to check that the analogues of properties (4) and (5) hold in B: for any $a \in B$ and for any $i, j \in I,\ \lambda, \mu \in \Lambda$, we have

$$a\overline{\varepsilon}_{i\lambda}\,\mathcal{R}\,a\overline{\varepsilon}_{i\mu} \quad \text{and} \quad a\overline{\varepsilon}_{i\lambda}\,\mathcal{L}\,a\overline{\varepsilon}_{j\lambda}. \tag{6}$$

Consider the restriction $\iota_{i\lambda}\colon Y \rtimes G_{i\lambda} \to T_{i\lambda}/\gamma_{i\lambda}$ of ι. Here $Y \rtimes G_{i\lambda}$ is the subsemigroup of $Y \odot \mathcal{M}[G; I, \Lambda; P]$ with underlying set $Y \times G_{i\lambda}$, and it is, indeed, a semidirect product of Y by $G_{i\lambda}$. This allows us to define the mapping $\psi\colon B \odot \mathcal{M}[G; I, \Lambda; P] \to T$ such that $(a, (i, g, \lambda))\psi$ is the unique element $t \in T$ such that $t\gamma = (a\overline{\pi}, (i, g, \lambda))\iota$ [or, equivalently, $t \in T_{i\lambda}$ such that $t\gamma_{i\lambda} = (a\overline{\pi}, (i, g, \lambda))\iota_{i\lambda}$; see Lemma 17] and $a\overline{\varepsilon_{i\lambda}}\,\mathcal{R}\,t\,\mathcal{L}\,\left({}^{p_{\lambda i}^{-1}g^{-1}}a\right)\overline{\varepsilon_{i\lambda}}$. Such a t exists and is unique since $T_{i\lambda}$ is a generalized inverse semigroup, $\iota_{i\lambda}$ is an isomorphism, and $a\overline{\pi}\,\mathcal{R}\,t\gamma_{i\lambda}\,\mathcal{L}\,\left({}^{p_{\lambda i}^{-1}g^{-1}}a\right)\overline{\pi}$. Thus the mapping ψ is well defined. It is straightforward that ψ is surjective. All that remains to be checked is that ψ is a homomorphism over $\mathcal{RB}$.

Let $(a,(i,g,\lambda)),(b,(j,h,\mu)) \in B \odot \mathcal{M}[G;I,\Lambda;P]$, and assume that

$$(a,(i,g,\lambda))\psi = s \quad \text{and} \quad (b,(j,h,\mu))\psi = t.$$

By definition, we have

$$(st)\gamma = s\gamma \cdot t\gamma = ((a \cdot {}^{gp_{\lambda j}}b)\,\overline{\pi},(i,gp_{\lambda j}h,\mu))\,. \tag{7}$$

Furthermore, we have $a \cdot {}^{gp_{\lambda j}}b \leq_{\mathcal{R}} a$ in B, and the relation

$$(a \cdot {}^{gp_{\lambda j}}b)\,\overline{\varepsilon}_{i\mu} \leq_{\mathcal{R}} a\overline{\varepsilon}_{i\mu}\,\mathcal{R}\,a\overline{\varepsilon}_{i\lambda} \tag{8}$$

is implied by (6) in the normal band E_i. Applying Lemma 17(1), consider inverses s' of s and $(st)'$ of st belonging to $T_{i\lambda}$ and $T_{i\mu}$, respectively. Then we have

$$st(st)' \leq_{\mathcal{R}} ss'\,\mathcal{R}\,a\overline{\varepsilon}_{i\lambda}. \tag{9}$$

By (7), we have $st(st)'\,\mathcal{D}\,(a \cdot {}^{gp_{\lambda j}}b)\,\overline{\varepsilon_{i\mu}}$, and so we obtain by (8) and (9) that $st\,\mathcal{R}\,st(st)'\,\mathcal{R}\,(a \cdot {}^{gp_{\lambda j}}b)\,\overline{\varepsilon_{i\mu}}$. This property and its dual imply that $((a,(i,g,\lambda))(b,(j,h,\mu)))\,\psi = ((a \cdot {}^{gp_{\lambda j}}b)\,,(i,gp_{\lambda j}h,\mu))\,\psi = st$, completing the proof that ψ is a homomorphism.

It is clear from the construction that the idempotent classes of the congruence induced by $\psi\gamma^{\natural}$ are rectangular bands, and so the same holds for the idempotent classes of the congruence induced by ψ. This implies the last statement. □

Thus we can prove the following statement.

Proposition 21. *Each almost factorizable locally inverse semigroup is a homomorphic image over $\mathcal{CS}$ of a Pastijn product of a normal band by a completely simple semigroup.*

Proof. Let S be an almost factorizable locally inverse semigroup and $\mathcal{U}$ a completely simple subsemigroup in $\mathcal{O}(S)$ such that conditions (AF1) and (AF2) are fulfilled. Lemma 14 implies that $T = S \bowtie \mathcal{U}$ is a locally inverse semigroup such that the congruence τ induced by the second projection is an idempotent pure and perfect congruence, and T/τ is a completely simple semigroup (isomorphic to $\mathcal{U}$). Moreover, the first projection $\pi_1 \colon T \to S$ is a surjective homomorphism over $\mathcal{CS}$. Applying Lemma 20, we obtain a homomorphism over $\mathcal{RB}$ from a Pastijn product P onto T. Thus $\psi\pi_1 \colon P \to S$ is also a surjective homomorphism over $\mathcal{CS}$. □

Combining Propositions 12, 13 and 21, we can deduce the main result of the paper.

Theorem 22. *For any locally inverse semigroup S, the following statements are equivalent:*

(1) S is almost factorizable,
(2) S is a homomorphic image of a Pastijn product of a normal band by a completely simple semigroup,

(3) S is a homomorphic image over $\mathcal{CS}$ of a Pastijn product of a normal band by a completely simple semigroup.

Acknowledgments

Research partially supported by the Hungarian National Foundation for Scientific Research grant no. K60148 and K77409.

References

[1] B. Billhardt and M. B. Szendrei, Weakly E-unitary locally inverse semigroups, *J. Algebra* **267** (2003) 559–576.
[2] E. Dombi, Almost factorisable straight locally inverse semigroups, *Acta Sci. Math. (Szeged)* **69** (2003) 569–589.
[3] A. El Qallali, Factorisable right adequate semigroups, *Proc. Edinburgh Math. Soc.* **24** (1981) 171–178.
[4] D. G. FitzGerald, Factorizable inverse monoids, *Semigroup Forum* **80** (2010) 484–509; err. 510.
[5] G. M. S. Gomes and M. B. Szendrei, Almost factorizable weakly ample semigroups, *Comm. Algebra* **35** (2007) 3503–3523.
[6] P. A. Grillet, *Semigroups: An introduction to the structure theory*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 193 (Marcel Decker, 1995).
[7] M. Hartmann, Almost factorizable orthodox semigroups, *Semigroup Forum* **74** (2007) 106–124.
[8] J. M. Howie, *Fundamentals of Semigroup Theory* (Clarendon Press, 1995).
[9] J. Kaďourek, On some existence varieties of locally inverse semigroups, *Internat. J. Algebra Comput.* **6** (1996) 761–788.
[10] J. Kaďourek, On some existence varieties of locally orthodox semigroups, *Internat. J. Algebra Comput.* **7** (1997) 93–131.
[11] M. V. Lawson, Almost factorisable inverse semigroups, *Glasgow Math. J.* **36** (1994) 97–111.
[12] M. V. Lawson, *Inverse semigroups: The Theory of Partial Symmetries* (World Scientific, 1998).
[13] D. B. McAlister, Some covering and embedding theorems for inverse semigroups, *J. Austral. Math. Soc.* **22** (1976) 188–211.
[14] F. Pastijn, Rectangular bands of inverse semigroups, *Simon Stevin* **56** (1982) 1–97.
[15] F. J. Pastijn and L. Oliveira, Maximal dense ideal extensions of locally inverse semigroups, *Semigroup Forum* **72** (2006) 441–458.
[16] R. Veeramony, Proper pseudo-inverse semigroups, *Simon Stevin* **58** (1984) 65–86.