ON THE STRUCTURE OF CANCELLATIVE CONJUGATION SEMIGROUPS

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Abstract. As an abstraction of the conjugation on multiplicative semigroups of quaternions, Garrão, Martins-Ferreira, Raposo, and Sobral [2] introduced the notion of a conjugation semigroup, and studied the category of commutative cancellative semigroups and monoids. In this paper the conjugations of a group G are shown to be in one-to-one correspondence with the endomorphisms of G whose ranges are in the center. Moreover, cancellative conjugation semigroups are proved to be, up to isomorphism, the subsemigroups of conjugation groups that are closed under conjugation.

1. Introduction

The notion of a conjugation semigroup is introduced by Garrão, Martins-Ferreira, Raposo, and Sobral [2] in order to present and investigate a new class of weakly Mal'tsev categories that fail to be Mal'tsev. A *conjugation* on a semigroup $(S;\cdot)$ is a unary operation $^-$ on $(S;\cdot)$ such that the following equalities hold for every $x, y \in S$:

$$\overline{x}x = x\overline{x}, \tag{1.1}$$

$$x\overline{y}y = y\overline{y}x, \tag{1.2}$$

$$\overline{xy} = \overline{y}\,\overline{x}. \tag{1.3}$$

By a *conjugation semigroup* we mean a unary semigroup $S = (S;\cdot,^-)$ where $^-$ is a conjugation on the semigroup $(S;\cdot)$. If $(S;\cdot)$ is a monoid with identity element 1, and $\overline{1} = 1$ then S is called a *conjugation monoid*. Notice that if $(S;\cdot)$ is a cancellative monoid then the equality $\overline{1} = 1$ is satisfied by any conjugation on $(S;\cdot)$. If $(S;\cdot)$ is a cancellative semigroup (monoid) or, in particular, a group then S is termed a *cancellative conjugation semigroup (monoid)* or, in particular, a *conjugation group*. For example, rules $\overline{x} = x$, $\overline{x} = 1$, and $\overline{x} = x^{-1}$ define a conjugation on every commutative semigroup, commutative monoid and group, respectively. We denote these conjugations by id, **1** and inv, respectively.

Notice that conjugation semigroups (monoids, groups) form a variety of unary semigroups (monoids, groups). Therefore a unary subsemigroup (submonoid, subgroup) of a conjugation semigroup (monoid, group) is a conjugation semigroup (monoid, group), and so we call it a *conjugation subsemigroup (submonoid, subgroup)*. Obviously, if conjugation is id (**1**, inv) in a commutative conjugation semigroup (commutative conjugation monoid, conjugation group) S then each subsemigroup (submonoid, subgroup) of $(S;\cdot)$ is a conjugation subsemigroup (submonoid, subgroup) of S.

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It is clear by (1.3) that each conjugation on a semigroup (monoid, group) is necessarily a (monoid, group) anti-endomorphism, that is, a (monoid, group) homomorphism from the semigroup (monoid, group) into its dual. In particular, on a commutative semigroup (commutative monoid, Abelian group), the conjugations are just the (monoid, group) endomorphisms since (1.1) and (1.2) are implied by commutativity. In particular, conjugations id and **1** are of this kind.

The examples in [2] motivating the notion of a conjugation semigroup are the multiplicative semigroups with underlying sets

$$T_{\mathbb{K}} = \{u \in \mathbb{K} : 0 < |u| < 1\} \quad (\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\})$$

together with conjugation id if $\mathbb{K} = \mathbb{R}$ and with the usual conjugation otherwise. The same conjugation on the multiplicative group of the field $\mathbb{K}$ defines a conjugation group $\mathbb{K}^*$, therefore $T_{\mathbb{K}}$ is a conjugation subsemigroup in $\mathbb{K}^*$ and $T_{\mathbb{K}}$ is cancellative. Note also that $T_{\mathbb{K}} = T_{\mathbb{H}} \cap \mathbb{K}^*$, and $\mathbb{R}^*$ is a conjugation subgroup in $\mathbb{C}^*$ and $\mathbb{C}^*$ in $\mathbb{H}^*$. It is an important observation in [2] that a cancellative semigroup possessing a conjugation is necessarily emdeddable in a group. It is a natural question whether every cancellative conjugation semigroup is isomorphic to a conjugation subsemigroup of a conjugation group.

The conjugation group $\mathbb{K}^*$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) has a nice and simple structure. It is straightforward that

$$E_{\mathbb{K}} = \{u \in \mathbb{K} : |u| = 1\} \quad \text{and} \quad \mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$$

form conjugation subgroups in $\mathbb{K}^*$, and the conjugations on them are inv and id, respectively. Moreover, it is also easy to check that the conjugation group $\mathbb{K}^*$ is (isomorphic, as a unary group, to) their direct product. Obviously, further conjugations can be obtained on the group $(\mathbb{K}^*; \cdot)$ by considering any endomorphism of $(\mathbb{R}^+; \cdot)$ as conjugation instead of id. The idea naturally arises whether conjugations on groups can be obtained in general by combining these two kinds of conjugations.

The aim of this note is to answer these questions by proving the following theorems.

Theorem 1.1. *For every group $G = (G; \cdot)$, there is a one-to-one correspondence between the conjugations on G and the endomorphisms of G whose ranges are in the center of G. For any such endomorphism ζ of G, the respective conjugation is defined by the rule $\overline{g} = g\zeta \cdot g^{-1}$ ($g \in G$).*

Theorem 1.2. *The cancellative conjugation semigroups (monoids) are, up to isomorphism, just the conjugation subsemigroups (submonoids) of conjugation groups.*

2. Conjugation groups

In this section we prove Theorem 1.1.

Our first observation (Lemma 2.1(1)) associates an endomorphism to any conjugation on a semigroup. Let $S = (S; \cdot)$ be an arbitrary semigroup (monoid, group), and consider its center $Z(S)$. By definition, it consists of all elements of S which commute with every element of S. If $Z(S)$ is non-empty then it forms a subsemigroup (submonoid, subgroup) in S. Note that $1 \in Z(S)$ if S is a monoid or, in particular, a group. In the rest of the section, we deal with conjugations on groups, and $G = (G; \cdot)$ will denote an arbitrary group.

Lemma 2.1. (1) *Let S be a semigroup, and let $^{-}$ be a conjugation on S. Then the mapping*

$$\zeta\colon S \to S,\ a\zeta = a\overline{a} \quad (a \in S)$$

is an endomorphism of the semigroup $(S; \cdot)$ such that $S\zeta \subseteq Z(S)$.

(2) *In particular, if $S = G$ is a group then*

$$\overline{g} = g^{-1} \cdot g\zeta = g\zeta \cdot g^{-1} \quad \textit{for every } g \in G.$$

Proof. Properties (1.1) and (1.2) imply that $S\zeta \subseteq Z(S)$. By applying (1.3), (1.2) and (1.1), we obtain for evey $a, b \in S$ that

$$(ab)\zeta = ab\overline{ab} = ab\overline{b}\overline{a} = a\overline{a}\overline{b}b = a\overline{a}b\overline{b} = a\zeta \cdot b\zeta.$$

Thus ζ is, indeed, an endomorphism of the semigroup S.

If $S = G$ is a group then the equalities in statement (2) are obvious from the definition of ζ and equality (1.1). □

Conversely, now we associate a conjugation to each endomorphism of a group whose range is contained in the center.

Lemma 2.2. *Suppose that $\zeta\colon G \to G$ is an endomorphism of G such that $G\zeta \subseteq Z(G)$, and define a unary operation $^{-}$ on G by the rule*

$$\overline{g} = g\zeta \cdot g^{-1} (= g^{-1} \cdot g\zeta) \quad (g \in G),$$

where the second equality follows by the assumption $G\zeta \subseteq Z(G)$. Then the operation $^{-}$ is a conjugation on G.

Proof. Equality (1.1) clearly holds by definition, and (1.2) is implied by (1.1) and the inclusion $G \subseteq Z(G)$. Finally, property (1.3) can be checked as follows:

$$\begin{aligned}\overline{gh} &= (gh)\zeta \cdot (gh)^{-1} = g\zeta \cdot h\zeta \cdot h^{-1}g^{-1} \\ &= h\zeta \cdot h^{-1} \cdot g\zeta \cdot g^{-1} \quad (\text{since } g\zeta \in Z(G)) \\ &= \overline{h}\overline{g}.\end{aligned}$$

□

In order to conclude the proof of Theorem 1.1, notice that the assignments

$$^{-} \mapsto \zeta \qquad \text{and} \qquad \zeta \mapsto {}^{-}$$

defined in Lemmas 2.1(1) and 2.2, respectively, provide mappings from the set of all conjugations on G to the set of all endomorphisms of G whose ranges are contained in $Z(G)$, and in the reverse direction, respectively. It is straightforward to verify that they are mutual inverses.

3. Cancellative conjugation semigroups

This section is devoted to proving Theorem 1.2.

Throughout the section, let $S = (S; \cdot, ^{-})$ be a cancellative conjugation semigroup. It was observed in [2] that, due to identities (1.1) and (1.2), conjugation semigroups satisfy the condition that $aS \cap bS \neq \emptyset$ for every $a, b \in S$, and this implies by Ore's theorem [1, Theorem 1.23] that the semigroup $(S; \cdot)$ is embeddable in a group. Since the identities mentioned are left-right symmetric, the dual condition $Sa \cap Sb \neq \emptyset$ $(a, b \in S)$ also holds in S. Thus we obtain by [1, Theorem 1.24 and Exercise 1.10.3] that there exists a group $G = (G; \cdot)$ containing $(S; \cdot)$ as a subsemigroup such that

(3.1) every $g \in G$ is of the form $g = ab^{-1} = c^{-1}d$ for some $a, b, c, d \in S$.

Such a group G is uniquely determined up to isomorphism, and called the *group of quotients of* $(S; \cdot)$.

The main step of the proof of Theorem 1.2 is that, given a cancellative conjugation semigroup $S = (S; \cdot, ^{-})$, we extend the conjugation $^{-}$ in S to the group of quotients G of $(S; \cdot)$ such that we obtain a conjugation $^{\sim}$ on G. Before introducing $^{\sim}$, we need two lemmas.

Lemma 3.1. *For every $a, b, c, d \in S$, the following implications hold in G:*

(1) *if $ab^{-1} = c^{-1}d$ then $\overline{b}^{-1}\overline{a} = \overline{d}\,\overline{c}^{-1}$;*
(2) *if $ab^{-1} = cd^{-1}$ then $\overline{b}^{-1}\overline{a} = \overline{d}^{-1}\overline{c}$.*

Proof. If $ab^{-1} = c^{-1}d$ in G then $ca = db$ in S whence we see by (1.3) that $\overline{a}\,\overline{c} = \overline{ca} = \overline{db} = \overline{b}\,\overline{d}$ in S. This implies the equality $\overline{b}^{-1}\overline{a} = \overline{d}\,\overline{c}^{-1}$ in G, and (1) is shown.

If $ab^{-1} = cd^{-1}$ in G then property (3.1) ensures that $ab^{-1} = x^{-1}y = cd^{-1}$ for some $x, y \in S$. Thus applying (1) for both equalities, we obtain the equality to be verified in (2). □

Lemma 3.2. *For every $a \in S$ and $g \in G$, we have $a\overline{a}g = g\overline{a}a$.*

Proof. By (3.1), assume that $g = bc^{-1} = x^{-1}y$ for some $b, c, x, y \in S$. Then $xb = yc$ and $a\overline{a}xb = a\overline{a}yc$ in S. By applying (1.2), we obtain that $x\overline{a}ab = y\overline{a}ac$ in S. This implies the equality $\overline{a}abc^{-1} = x^{-1}y\overline{a}a$ in G, and we deduce by (1.1) that $a\overline{a}g = \overline{a}abc^{-1} = x^{-1}y\overline{a}a = g\overline{a}a$. □

Now we are ready to define the unary operation $^{\sim}$ on G as follows: for any $g \in G$, if $g = ab^{-1}$ for some $a, b \in S$ then let $\widetilde{g} = \overline{b}^{-1}\overline{a}$. Lemma 3.1 shows that $^{\sim}$ is well defined.

Proposition 3.3. *The unary operation $^{\sim}$ on G is a conjugation, and it extends the conjugation $^{-}$ in S.*

Proof. First we check that $^{\sim}$ extends $^{-}$. Assume that $c = ab^{-1}$ for some $a, b, c \in S$. Then $a = cb$ in S, and we have $\overline{a} = \overline{cb} = \overline{b}\,\overline{c}$ by (1.3) which implies that $\overline{c} = \overline{b}^{-1}\overline{a} = \widetilde{c}$.

Now we prove that the unary group $(G; \cdot, ^{\sim})$ satisfies the identities (1.1)–(1.3). Let $g = ab^{-1} \in G$ and $h = cd^{-1} \in G$ where $a, b, c, d \in S$.

(1.1) By definition and Lemma 3.2, we have

$$\widetilde{g}g = \widetilde{ab^{-1}}ab^{-1} = \overline{b}^{-1}\overline{a}ab^{-1} = a\overline{a}\overline{b}^{-1}b^{-1} = a\overline{a}(b\overline{b})^{-1}.$$

Similarly, we see that

$$g\widetilde{g} = ab^{-1}\widetilde{ab^{-1}} = ab^{-1}\overline{b}^{-1}\overline{a} = a(\overline{b}b)^{-1}\overline{a} = a(\overline{a}^{-1}\overline{b}b)^{-1} = a(b\overline{b}\overline{a}^{-1})^{-1} = a\overline{a}(b\overline{b})^{-1}$$

whence $\widetilde{g}g = g\widetilde{g}$ follows.

(1.2) Now we show that $h\widetilde{g}g = g\widetilde{g}h$. Due to the equalities in the previous paragraph, it suffices to check that $ha\overline{a}(b\overline{b})^{-1} = a\overline{a}(b\overline{b})^{-1}h$. Applying (1.1) and Lemma 3.2, we obtain that

$$ha\overline{a}(b\overline{b})^{-1} = a\overline{a}h(b\overline{b})^{-1} = a\overline{a}(b\overline{b}h^{-1})^{-1} = a\overline{a}(h^{-1}b\overline{b})^{-1} = a\overline{a}(b\overline{b})^{-1}h,$$

completing the proof.

(1.3) In order to check the equality $\widetilde{gh} = \widetilde{h}\widetilde{g}$, notice that $\widetilde{h}\widetilde{g} = \widetilde{cd^{-1}}\widetilde{ab^{-1}} = \overline{d}^{-1}\overline{c}\overline{b}^{-1}\overline{a}$ by definition. On the other hand, we have $b^{-1}c = xy^{-1}$ for some $x, y \in S$ by (3.1). This implies that $cy = bx$ in S, and $\overline{y}\,\overline{c} = \overline{x}\overline{b}$ follows by (1.3). Hence we obtain that $\overline{y}^{-1}\overline{x} = \overline{c}\overline{b}^{-1}$ in G. By applying (1.3) and this equality, we see that

$$\widetilde{gh} = a\widetilde{b^{-1}cd}^{-1} = a\widetilde{xy^{-1}d}^{-1} = a\widetilde{x(dy)}^{-1} = \overline{dy}^{-1}\overline{ax} = (\overline{y}\overline{d})^{-1}\overline{x}\,\overline{a} = \overline{d}^{-1}\overline{y}^{-1}\overline{x}\,\overline{a} = \overline{d}^{-1}\overline{c}\overline{b}^{-1}\overline{a}.$$

Thus the equality $\widetilde{gh} = \widetilde{h}\widetilde{g}$ is verified. □

This proposition shows that the conjugation semigroup $S = (S; \cdot, ^{-})$ is, indeed, a conjugation subsemigroup in the conjugation group $G = (G; \cdot, ^{\sim})$, and so Theorem 1.2 is proved for cancellative conjugation semigroups. The statement for cancellative conjugation monoids is a straightforward consequence since (1.3) implies that each conjugation on a cancellative monoid is a monoid conjugation.

References

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