Idealized 2 links

Ferenc A. Bartha
fab2@rice.edu

Generalized Coordinates

We use the following coordinates and notation:

- The middle mass point is given in polar coordinates w.r. to the origin and the fixed x-, y-axes.
  - the connection to the origin (fixed) is a weightless rod,
  - the weight of the mass point is m_1,
  - radius $= d_1$, constant,
  - angle from x-axis to the direction $\overrightarrow{e_{r_1}} = \theta_1$.
- The upper mass point is given in polar coordinates w.r. to middle point
  - the connection to the middle point is a weightless rod,
  - the weight of the mass point is m_2,
  - radius $= d_2$, constant,
  - angle from the direction $\overrightarrow{e_{r_1}}$ to the direction $\overrightarrow{e_{r_2}} = \theta_2$.
- Consequently, we describe our system by θ_1 and θ_2 and their derivatives.

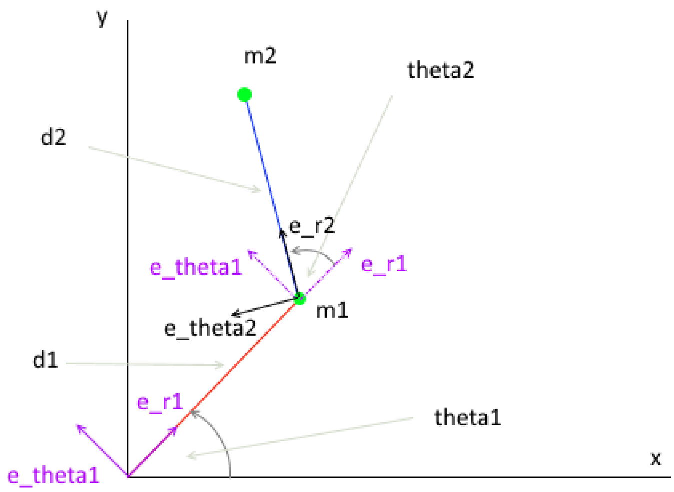

Figure 1: The coordinates

Note that none of these coordinate systems (describing the middle or the upper point) is an inertial coordinate system. Both are rotating (thus accelerating) compared to the one fixed in the origin with axis x and y.

The Lagrangian

We assume that no external forces act on the system, so we have conservation of energy, that is in our case conservation of kinetic energy. The Lagrangian is

$$\mathcal{L}(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2} m_1 |\overrightarrow{v_1}|^2 + \frac{1}{2} m_2 |\overrightarrow{v_2}|^2.$$

We have

- conservation of energy: $\mathcal{L} =$ constant,
- the position vector of the middle mass from the origin is $d_1 \overrightarrow{e_{r_1}}$,
- the velocity vector is the derivative $\overrightarrow{v_1} = \dot{d}_1 \overrightarrow{e_{r_1}} + d_1 \dot{\overrightarrow{e_{r_1}}} = d_1 \dot{\theta}_1 \overrightarrow{e_{\theta_1}}$ as $\dot{d}_1 = 0$ and $\dot{\overrightarrow{e_{r_1}}} = \dot{\theta}_1 \overrightarrow{e_{\theta_1}}$ (see **Appendix**),
- thus, the speed^2 is $|\overrightarrow{v_1}|^2 = (d_1 \dot{\theta}_1)^2$,
- the position vector of the upper mass from the origin is $d_1 \overrightarrow{e_{r_1}} + d_2 \overrightarrow{e_{r_2}}$,
- the velocity vector is $\overrightarrow{v_2} = \overrightarrow{v_1} + \dot{d}_2 \overrightarrow{e_{r_2}} + d_2 \dot{\overrightarrow{e_{r_2}}} = d_1 \dot{\theta}_1 \overrightarrow{e_{\theta_1}} + d_2 (\dot{\theta}_1 + \dot{\theta}_2) \overrightarrow{e_{\theta_2}}$ as $\dot{d}_2 = 0$ and $\dot{\overrightarrow{e_{r_2}}} = (\dot{\theta}_1 + \dot{\theta}_2) \overrightarrow{e_{\theta_2}}$ (see **Appendix**),
- thus, the speed^2 is $|\overrightarrow{v_2}|^2 = (d_1 \dot{\theta}_1)^2 + (d_2 (\dot{\theta}_1 + \dot{\theta}_2))^2 + 2 d_1 d_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\text{angle}(\overrightarrow{e_{\theta_1}}, \overrightarrow{e_{\theta_2}}))$ that is $|\overrightarrow{v_2}|^2 = (d_1 \dot{\theta}_1)^2 + (d_2 (\dot{\theta}_1 + \dot{\theta}_2))^2 + 2 d_1 d_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2$.

Thus,

$$\mathcal{L}(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2) = \frac{1}{2} (m_1 + m_2) d_1^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 d_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 d_1 d_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2.$$

Equations of Motion

The equations of motion are derived by taking
(Lagrange equations of the 2nd kind a.k.a Euler–Lagrange equations)

$$d/dt[\partial \mathcal{L} / \partial \dot{q}_j] = \partial \mathcal{L} / \partial q_j$$

for all generalized coordinates $q_j \in \{\theta_1, \theta_2\}$.

This gives us

$$\frac{d}{dt} \left[(m_1 + m_2) d_1^2 \dot{\theta}_1 + m_2 d_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 d_1 d_2 (2\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2 \right] = 0 \quad \text{for } q_j = \theta_1$$

and

$$\frac{d}{dt} \left[m_2 d_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + m_2 d_1 d_2 \dot{\theta}_1 \cos\theta_2 \right] = -m_2 d_1 d_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_2) \quad \text{for } q_j = \theta_2.$$

Writing in the time derivatives we get

$$(m_1 + m_2) d_1^2 \ddot{\theta}_1 + m_2 d_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 d_1 d_2 [(2\ddot{\theta}_1 + \ddot{\theta}_2) \cos\theta_2 - (2\dot{\theta}_1 + \dot{\theta}_2) \dot{\theta}_2 \sin\theta_2] = 0$$

that results in

$$\left(m_2d_2^2+m_2d_1d_2\cos\theta_2\right)\ddot{\theta}_2 = -\left((m_1+m_2)d_1^2+m_2d_2^2+2m_2d_1d_2\cos\theta_2\right)\ddot{\theta}_1 + m_2d_1d_2(2\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_2\sin\theta_2 \qquad \text{for } q_j=\theta_1 \tag{1}$$

and

$$m_2d_2^2(\ddot{\theta}_1+\ddot{\theta}_2)+m_2d_1d_2[\ddot{\theta}_1\cos\theta_2-\dot{\theta}_1\dot{\theta}_2\sin\theta_2] = -m_2d_1d_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_2)$$

that is transformed to

$$m_2d_2^2\ddot{\theta}_2 = -m_2d_2^2\ddot{\theta}_1 - m_2d_1d_2\ddot{\theta}_1\cos\theta_2 - m_2d_1d_2\dot{\theta}_1^{\,2}\sin(\theta_2)$$

and simplified to

$$\ddot{\theta}_2 = -\left(\left[1+\frac{d_1}{d_2}\cos\theta_2\right]\ddot{\theta}_1+\frac{d_1}{d_2}\dot{\theta}_1^{\,2}\sin(\theta_2)\right) \qquad \text{for } q_j=\theta_2 \tag{2}$$

Further Transformation of the Equations of Motion

We want to solve the equations (1) and (2) for $\ddot{\theta}_1$ and $\ddot{\theta}_1$ (some tools can model them as they readily are, but we need now a more standard form).

Substituting (2) into (1) yields

$$-\left(m_2d_2^2+m_2d_1d_2\cos\theta_2\right)\left(\left[1+\frac{d_1}{d_2}\cos\theta_2\right]\ddot{\theta}_1+\frac{d_1}{d_2}\dot{\theta}_1^{\,2}\sin(\theta_2)\right) = -\left((m_1+m_2)d_1^2+m_2d_2^2+2m_2d_1d_2\cos\theta_2\right)\ddot{\theta}_1 + m_2d_1d_2(2\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_2\sin\theta_2$$

that is transformed to

$$\left(m_1d_1^2+m_2d_1^2-m_2d_1^2\cos^2\theta_2\right)\ddot{\theta}_1 = m_2d_1d_2(2\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_2\sin\theta_2+\left(m_2d_2^2+m_2d_1d_2\cos\theta_2\right)\frac{d_1}{d_2}\dot{\theta}_1^{\,2}\sin(\theta_2)$$

and simplified to

$$\left(\tfrac{m_1}{m_2}+\sin^2\theta_2\right)\ddot{\theta}_1 = \tfrac{d_2}{d_1}(2\dot{\theta}_1+\dot{\theta}_2)\dot{\theta}_2\sin\theta_2+\left(\tfrac{d_2}{d_1}+\cos\theta_2\right)\dot{\theta}_1^{\,2}\sin(\theta_2) \tag{3}$$

Substitute $\ddot{\theta}_1$ from (3) into (2) and we reach the desired form.

Deriving the Initial Conditions

Our experiment is the following. The initial alignment is $\theta_1(0)=\frac{\pi}{2}$ and $\theta_2(0)=0$. A momentary impulse gives velocity to the middle point and does not affect the upper point.

- $\overrightarrow{v_1}_{,init} = \overrightarrow{v_1}(0) = d_1\dot{\theta}_1(0)\overrightarrow{e_{\theta_1}}(0)$
  $\Rightarrow$ we obtain the initial value $\dot{\theta}_1(0)$.
- The upper point is not affected that is $\overrightarrow{0} = \overrightarrow{v_2}(0) = d_1\dot{\theta}_1(0)\overrightarrow{e_{\theta_1}}(0)+d_2(\dot{\theta}_1(0)+\dot{\theta}_2(0))\overrightarrow{e_{\theta_2}}(0)$
  $\Rightarrow$ we obtain the initial value $\dot{\theta}_2(0) = -\frac{d_1}{d_2}\dot{\theta}_1(0)-\dot{\theta}_1(0)$
  as $\overrightarrow{e_{\theta_1}}(0)=\overrightarrow{e_{\theta_2}}(0)$ due to the initial alignment.

Recall that $\dot{\theta}_2$ is a relative coordinate with respect to the middle point that is in motion at time 0. Thus, as the upper point is not in motion viewing from the fixed coordinate system centered at the origin, it is moving when viewed from the middle point, therefore, it has nonzero **relative** velocity $d_2(\dot{\theta}_1(0)+\dot{\theta}_2(0))\overrightarrow{e_{\theta_2}}(0) = -d_1\dot{\theta}_1(0)\overrightarrow{e_{\theta_2}}(0) = -d_1\dot{\theta}_1(0)\overrightarrow{e_{\theta_1}}(0) = -\overrightarrow{v_1}(0)$.

Appendix

The derivation for $\dot{\overrightarrow{e_{r_1}}}(t) = \dot{\theta}_1(t)\overrightarrow{e_{\theta_1}}(t)$ is pretty straightforward. Centered at the origin, $\overrightarrow{e_{r_1}}(t)$ and $\overrightarrow{e_{\theta_1}}(t)$ form a coordinate system that rotates with speed $\dot{\theta}_1(t)$ around the origin (thus, rotates with respect to the inertial coordinate system). This rotation is actually around the z-axis (lifting it into 3D).

Given a position vector $\overrightarrow{p}_{relative}(t)$ of a moving point in this rotating coordinate system, its velocity (within this rotating system) is $\dot{\overrightarrow{p}}_{relative}(t)$. Looking from the origin in the inertial coordinate system, the effect of the rotation needs to be taken into account:

$$\dot{\overrightarrow{p}}_{global}(t) = \dot{\overrightarrow{p}}_{relative}(t) + \dot{\theta}_1(t)\overrightarrow{k} \times \overrightarrow{p}_{global}(t).$$

Here $\overrightarrow{k}$ is the axis of rotation, the direction of the z-axis.

Applying this for $\overrightarrow{p}_{global}(t) = \overrightarrow{e_{r_1}}(t)$, we get

$$\dot{\overrightarrow{e_{r_1}}}(t) = \dot{(1,0)} + \dot{\theta}_1(t)\overrightarrow{k} \times \overrightarrow{e_{r_1}}(t) = \dot{\theta}_1(t)\overrightarrow{k} \times \overrightarrow{e_{r_1}}(t).$$

Now as $\overrightarrow{e_{\theta_1}}(t) = \overrightarrow{k} \times \overrightarrow{e_{r_1}}(t)$ by definition, we have shown the sought relation.

Similar argument works for $\dot{\overrightarrow{e_{r_2}}}(t) = (\dot{\theta}_1(t) + \dot{\theta}_2(t))\overrightarrow{e_{\theta_2}}(t)$ as well. Translate $\dot{\overrightarrow{e_{r_2}}}(t)$ to the origin and observe that it actually rotates with speed $(\dot{\theta}_1(t) + \dot{\theta}_2(t))$ around the z-axis.

Further reading

- Wikipedia - Lagrangian Mechanics
- MIT OpenCourseware - Dynamics - Relative Motion using Rotating Axis
- Wikibooks - 2D Coordinate Systems
- Wikipedia - Double Pendulum