Hopf bifurcations in Nicholson's blowfly equation are always supercritical

István Balázs*, Gergely Röst†

Abstract

We prove that all Hopf bifurcations in the Nicholson's blowfly equation are supercritical as we increase the delay. Earlier results treated only the first bifurcation point, and to determine the criticality of the bifurcation, one needed to substitute the parameters into a lengthy formula of the first Lyapunov coefficient. With our result, there is no need for such calculations at any bifurcation point.

Keywords: delay differential equation; Hopf bifurcation; supercritical; normal form

1 Introduction

Nicholson's blowfly equation

$$N'(t) = -\gamma N(t) + pN(t-\tau)e^{-aN(t-\tau)} \tag{1}$$

is one of the most studied nonlinear delay differential equations, yet its dynamics is not fully understood. The equation can be interpreted as a population dynamical model with maturation delay and intraspecific competition, with $N(t)$ denoting the population size, and all parameters being positive. There exists a positive equilibrium $N^* = (1/a)\ln(p/\gamma)$ of (1) if and only if $a > 0$ and $p > \gamma$. These relations are assumed throughout this paper, since we are interested in the bifurcations of periodic orbits from the positive equilibrium.

We can easily see that Nicholson's equation has Hopf bifurcations at critical delays τ_k with critical eigenvalues ω_k, $k \in \mathbb{Z}$. There is a well known method for determining the direction of Hopf bifurcations for this type of equations [3], however, the calculations are rather tedious and rarely followed through completely. In [4], the next theorem was proven.

*MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged
Aradi vértanúk tere 1, Szeged, H-6720, Hungary
Department of Mathematics, University of Klagenfurt
Universitätsstraße 65-67, Klagenfurt am Wörthersee, 9020, Austria
†Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, Szeged, H-6720, Hungary

Theorem 1. *Let*

$$\begin{aligned}\operatorname{Re} C_1(0) =&\frac{\gamma}{2\left(aN^*-1\right)}\left\{\frac{\tau_0\left(aN^*-2\right)^2}{3\Delta^2\omega_0\gamma^3\left(aN^*-1\right)^3}\left[2\gamma\left(\gamma^2-\omega_0^2\right)\left(\left(1+\tau_0\gamma\right)^2-\omega_0^2\tau_0^2\right)\right.\right.\\ &\left.+\tau_0\left(1+\tau_0\gamma\right)\left(\gamma^4+\omega_0^4-6\omega_0^2\gamma_0^2\right)\right]-\frac{2\tau_0\left(aN^*-2\right)^2}{\Delta aN^*\gamma}\left(\gamma+\tau_0\gamma^2+\omega_0^2\tau_0\right)\\ &+\frac{\tau_0\left(aN^*-3\right)}{\Delta\gamma}\left(\gamma+\tau_0\gamma^2+\omega_0^2\tau_0\right)+\frac{\tau_0\left(aN^*-2\right)^2}{\Delta}\\ &\left.\times\frac{\left(\omega^2\tau_0-\gamma-\tau_0\gamma^2\right)\frac{\gamma^2aN^*-\omega_0^2}{\gamma(aN^*-1)}+\left(\omega_0+2\omega_0\tau_0\gamma\right)2\omega_0\frac{aN^*}{aN^*-1}}{\frac{\left(\gamma^2aN^*-\omega_0^2\right)^2}{\left(\gamma(aN^*-1)\right)^2}+4\omega^2\frac{a^2N^{*2}}{(aN^*-1)^2}}\right\},\end{aligned}$$

where $\Delta=(1+\tau_0\gamma)^2+\omega_0^2\tau_0^2$*. Then,*

(a) the Hopf bifurcation occurs as τ *crosses* τ_0 *to the right if* $\operatorname{Re} C_1(0)<0$*, and to the left if* $\operatorname{Re} C_1(0)>0$*; and*

(b) the bifurcating periodic solution is stable if $\operatorname{Re} C_1(0)<0$ *and unstable if* $\operatorname{Re} C_1(0)>0$*.*

In Section 2, we prove that this long formula for $\operatorname{Re} C_1(0)$ is always negative, and this holds not only for the first bifurcation point τ_0, but for all critical parameter values τ_k, $k\in\mathbb{N}_0$. Here we use the method of [2] (see also [1]) to obtain our main result, which is stated in the following theorem.

Theorem 2. *If* $p>e^2\gamma$*, equation* (1) *undergoes a supercritical Hopf bifurcation at* N^* *when* $\tau=\tau_k$*, for all* $k\in\mathbb{N}_0$*.*

2 Proof of the main result

2.1 Preliminary calculations

Let $N(t)$ be an arbitrary solution of equation (1). Setting $N(t)=N^*+(1/a)x(t)$, $x(t)$ satisfies

$$x'(t)=-\gamma x(t)-a\gamma N^*\left[1-e^{-x(t-\tau)}\right]+\gamma x(t-\tau)e^{-x(t-\tau)}. \tag{2}$$

For normalizing the delay, we use the transformation $y(s)=x(\tau s)$, and obtain

$$y'(s)=-\tau\gamma\left(y(s)+aN^*\left[1-e^{-y(s-1)}\right]-y(s-1)e^{-y(s-1)}\right). \tag{3}$$

Using the Taylor expansion of the exponential function, the linearization of (3) is

$$z'(s)=-\tau\gamma\big(z(s)+aN^*z(s-1)-z(s-1)\big).$$

By introducing the new parameter $b=aN^*-1$, it can be written as

$$z'(s)=-\tau\gamma\big(z(s)+bz(s-1)\big). \tag{4}$$

Substituting the exponential Ansatz, we find the characteristic equation

$$\lambda=-\tau\gamma\left(1+be^{-\lambda}\right). \tag{5}$$

For $\omega > 0$, $\pm i\omega$ is a pair of complex conjugate roots of (5) if and only if

$$i\omega = -\tau\gamma\big(1 + b(\cos\omega - i\sin\omega)\big).$$

Separating the real and imaginary parts, we obtain

$$\tau\gamma b\cos\omega = -\tau\gamma, \tag{6}$$
$$\tau\gamma b\sin\omega = \omega. \tag{7}$$

Equation (6) can be simplified to

$$\cos\omega = -\frac{1}{b}.$$

As $\omega > 0$, (7) implies $\sin\omega > 0$. Then we have the critical imaginary parts

$$\omega_k = \arccos\left(-\frac{1}{b}\right) + 2k\pi, \qquad k \in \mathbb{N}_0.$$

The inequality $b > 1$ implies the estimates $-1 < -\frac{1}{b} < 0$ and

$$\frac{\pi}{2} < \arccos\left(-\frac{1}{b}\right) < \pi. \tag{8}$$

From $\sin\omega_k > 0$, we get

$$\sin\omega_k = \sin\left(\arccos\left(-\frac{1}{b}\right) + 2k\pi\right) = \sin\left(\arccos\left(-\frac{1}{b}\right)\right) = \sqrt{1 - \frac{1}{b^2}}.$$

Thus, the critical parameter values are

$$\tau_k = \frac{\omega_k}{\gamma b\sin\omega_k} = \frac{\arccos\left(-\frac{1}{b}\right) + 2k\pi}{\gamma b\sqrt{1 - \frac{1}{b^2}}} = \frac{\arccos\left(-\frac{1}{b}\right) + 2k\pi}{\gamma\sqrt{b^2 - 1}}, \qquad k \in \mathbb{N}_0.$$

For checking the transversality condition, we differentiate the characteristic equation (5) with respect to the parameter τ:

$$\lambda' = -\gamma\left(1 + be^{-\lambda}\right) + \tau\gamma be^{-\lambda}\lambda',$$

and express the derivative

$$\lambda' = -\frac{\gamma\left(1 + be^{-\lambda}\right)}{1 - \tau\gamma be^{-\lambda}}.$$

Substituting $-\tau\gamma be^{-\lambda} = \lambda + \tau\gamma$ and $-\gamma(1 + be^{-\lambda}) = \lambda/\tau$ from (5), we can see that

$$\lambda' = \frac{\lambda}{\tau(1 + \lambda + \tau\gamma)}.$$

Considering λ in the form $\mu + i\omega$, where $\mu, \omega \in \mathbb{R}$, and taking the real part, we get

$$\mu' = \operatorname{Re}\frac{i\omega}{\tau(1 + \mu + i\omega + \tau\gamma)} = \operatorname{Re}\frac{i\omega(1 + \mu + \tau\gamma - i\omega)}{\tau\left((1 + \mu + \tau\gamma)^2 + \omega^2\right)} = \frac{\omega^2}{\tau\left((1 + \mu + \tau\gamma)^2 + \omega^2\right)} > 0.$$

As the real parts of the eigenvalues are strictly increasing in the parameter τ, the transversality condition holds, and we have Hopf bifurcations at critical values τ_k, $k \in \mathbb{Z}$, if $b > 1$. The calculation in this section is equivalent with Section 2 of [4], however, our notations will be more convenient for us in the sequel.

2.2 Directions of the Hopf bifurcations

We follow the argument of [2], and apply it to equation (3). We denote the difference between the parameter and the critical value by $\alpha = \tau - \tau_k$, and use the notation $y_s(u) = y(s+u)$, $-1 \le u \le 0$ for solutions segments, as usual. Let L and F be defined by the relation

$$L(\alpha)y_s + F(y_s,\alpha) = (\tau_k + \alpha)\left(-\gamma y(s) - a\gamma N^*\left[1 - e^{-y(s-1)}\right] + \gamma y(s-1)e^{-y(s-1)}\right),$$

where $L(\alpha)$ is a linear operator, $F(0,0) = 0$ and $D_1F(0,0) = 0$. Then we have

$$L(\alpha)\varphi = -(\tau_k + \alpha)\gamma\big(\varphi(0) + b\varphi(-1)\big)$$

and

$$F(\varphi,\alpha) = -(\tau_k+\alpha)\gamma\left((b+1)\left(1 - e^{-\varphi(-1)}\right) - \varphi(-1)e^{-\varphi(-1)} - b\varphi(-1)\right). \tag{9}$$

For $L_0 = L(0)$ we get

$$L_0(\varphi) = -\tau_k\gamma(\varphi(0) + b\varphi(-1)).$$

By substitution,

$$L_0(1) = -\tau_k\gamma(1+b), \quad L_0\left(\theta e^{i\omega_k\theta}\right) = \tau_k\gamma b e^{-i\omega_k}, \quad L_0\left(e^{2i\omega_k\theta}\right) = -\tau_k\gamma\left(1 + be^{-2i\omega_k}\right)$$

follows. Expanding (9) into a Taylor series with respect to the first variable and substituting $\alpha = 0$, we have

$$F(\varphi,0) = \tau_k\gamma\frac{b-1}{2}\varphi^2(-1) - \tau_k\gamma\frac{b-2}{6}\varphi^3(-1) + h.o.t.$$

The coefficients of the right-hand side of

$$\begin{aligned} &F(y_1e^{i\omega_k\theta} + y_2e^{-i\omega_k\theta} + y_3\cdot 1 + y_4e^{2i\omega_k\theta},0) \\ &= B_{(2,0,0,0)}y_1^2 + B_{(1,1,0,0)}y_1y_2 + B_{(1,0,1,0)}y_1y_3 + B_{(0,1,0,1)}y_2y_4 + B_{(2,1,0,0)}y_1^2y_2 + \dots \end{aligned} \tag{10}$$

are

$$B_{2,0,0,0} = \tau_k\gamma\frac{b-1}{2}e^{-2i\omega_k}, \quad B_{1,1,0,0} = \tau_k\gamma(b-1), \quad B_{1,0,1,0} = \tau_k\gamma(b-1)e^{-i\omega_k},$$
$$B_{0,1,0,1} = \tau_k\gamma(b-1)e^{-i\omega_k}, \quad B_{2,1,0,0} = -\tau_k\gamma\frac{b-2}{2}e^{-i\omega_k}.$$

The direction of the bifurcation is determined by the sign of

$$K = \mathrm{Re}\left[\frac{1}{1 - L_0(\theta e^{i\omega_k\theta})}\left(B_{(2,1,0,0)} - \frac{B_{(1,1,0,0)}B_{(1,0,1,0)}}{L_0(1)} + \frac{B_{(2,0,0,0)}B_{(0,1,0,1)}}{2i\omega_k - L_0(e^{2i\omega_k})}\right)\right]. \tag{11}$$

We use the notation $a \sim b$ for real numbers a and b having the same sign. Substituting all terms into K, we have

$$\begin{aligned} K &\sim \left(2b^5 - 12b^4 + 23b^3 - 23b^2 + 4b + 4\right)\sqrt{b^2-1} \\ &\quad + b^2\left(-6b^4 + 19b^3 - 27b^2 + 10b + 2\right)\left(\arccos\left(-\frac{1}{b}\right) + 2k\pi\right) =: K_* \end{aligned} \tag{12}$$

The polynomial $p_1(b) := -6b^4 + 19b^3 - 27b^2 + 10b + 2$ is the solution of the 4th order IVP (initial value problem)

$$p_1(1) = -2, \quad \frac{dp_1}{db}(1) = -11, \quad \frac{d^2p_1}{db^2}(1) = -12, \quad \frac{d^3p_1}{db^3}(1) = -30, \quad \frac{d^4p_1}{db^4}(b) = -144,$$

so $p_1(b)$ is negative for all $b > 1$. In (12), the coefficient of $2k\pi$ is also negative for all $b > 1$. From (8), for $k \in \mathbb{N}_0$, we conclude

$$K_* < \left(2b^5 - 12b^4 + 23b^3 - 23b^2 + 4b + 4\right)\sqrt{b^2 - 1} + b^2 p_1(b)\frac{\pi}{2}.$$

This expression is negative for all $b > 1$ if

$$\left(2b^5 - 12b^4 + 23b^3 - 23b^2 + 4b + 4\right)\sqrt{b^2 - 1} < -b^2 p_1(b)\frac{\pi}{2}.$$

As the terms $\sqrt{b^2 - 1}$ and $-b^2 p_1(b)$ are positive, the last inequality holds if

$$p_2(b) := \left(2b^5 - 12b^4 + 23b^3 - 23b^2 + 4b + 4\right)^2 \left(b^2 - 1\right) - b^4 p_1(b)^2 \frac{\pi^2}{4} < 0.$$

The polynomial $p_2(b)$ is the unique solution of the 12th order IVP

$$p_2(1) = -\pi^2 < 0,$$
$$\frac{dp_2}{db}(1), = 8 - 15\pi^2 < 0,$$
$$\frac{d^2p_2}{db^2}(1), = 184 - \frac{345\pi^2}{2} < 0,$$
$$\frac{d^3p_2}{db^3}(1), = 2004 - 1518\pi^2 < 0,$$
$$\frac{d^4p_2}{db^4}(1), = 10776 - 10968\pi^2 < 0,$$
$$\frac{d^5p_2}{db^5}(1), = 55200 - 72240\pi^2 < 0,$$
$$\frac{d^6p_2}{db^6}(1), = 249120 - 474840\pi^2 < 0,$$
$$\frac{d^7p_2}{db^7}(1), = 534240 - 3064320\pi^2 < 0,$$
$$\frac{d^8p_2}{db^8}(1), = -120960 - 18103680\pi^2 < 0,$$
$$\frac{d^9p_2}{db^9}(1), = -13063680 - 98340480\pi^2 < 0,$$
$$\frac{d^{10}p_2}{db^{10}}(1), = -116121600 - 501681600\pi^2 < 0,$$
$$\frac{d^{11}p_2}{db^{11}}(1), = -2035756800\pi^2 < 0,$$
$$\frac{d^{12}p_2}{db^{12}}(b), = 1916006400 - 4311014400\pi^2 < 0,$$

so $p_2(b)$ is negative for all $b > 1$. Hence, we conclude that $K < 0$ for all $b > 1$ and $k \in \mathbb{N}_0$, by [2], the theorem holds.

We have proven that all Hopf bifurcations for the Nicholson's blowfly equation are supercritical, hence there is no need to calculate the complicated first Lyapunov coefficients in the future. Our theorem significantly improves Theorem 2 of [4], where only the first bifurcation point was studied, and even for that a lengthy formula needed to be checked for any particular parameter combination to ensure supercriticality.

Acknowledgments

IB was supported by NKFI K129322 and EFOP-3.6.2-16-2017-00015. GR was supported by NKFI FK124016 and TUDFO/47138-1/2019-ITM.

References

[1] I. Balázs and G. Röst. “Hopf bifurcation for Wright-type delay differential equations: The simplest formula, period estimates, and the absence of folds”. In: *Communications in Nonlinear Science and Numerical Simulation* 84 (2020).

[2] T. Faria and L. T. Magalhães. “Normal Forms for Retarded Functional Differential Equations with Parameters and Applications to Hopf Bifurcation”. In: *Journal of Differential Equations* 122.2 (1995), pp. 201–224.

[3] B. D. Hassard, N. D. Kazarinoff, and Y-H. Wan. *Theory and Applications of Hopf Bifurcation*. Cambridge University Press, 1981.

[4] J. Wei and M. Y. Li. “Hopf bifurcation analysis in a delayed Nicholson blowflies equation”. In: *Nonlinear Analysis* 60.7 (2005), pp. 1351–1367.