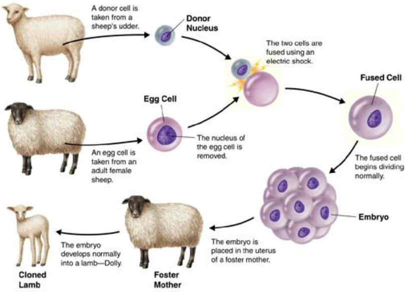
Critical relations of the $2k$-crown poset

Ádám Kunos, Miklós Maróti and László Zádori

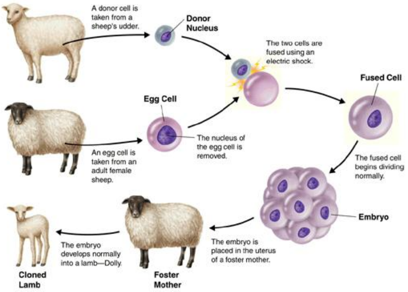
University of Szeged

4th CSM
Szeged, June 29, 2016

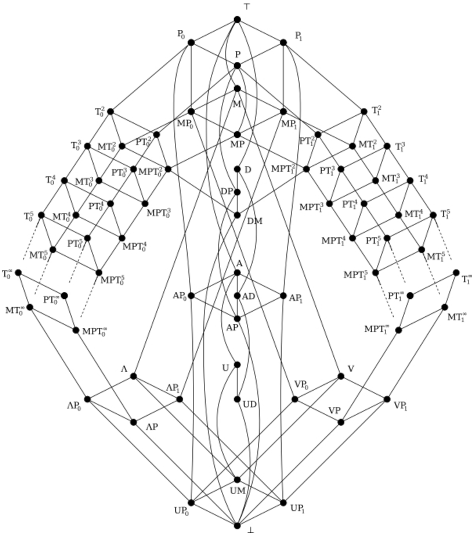
Clones


A donor cell is taken from a sheep's udder.
Donor Nucleus
The two cells are fused using an electric shock.
Fused Cell
Egg Cell
The nucleus of the egg cell is removed.
An egg cell is taken from an adult female sheep.
The fused cell begins dividing normally.
Embryo
The embryo develops normally into a lamb—Dolly.
The embryo is placed in the uterus of a foster mother.
Cloned Lamb
Foster Mother

Clones


A donor cell is taken from a sheep's udder.
Donor Nucleus
The two cells are fused using an electric shock.
Fused Cell
Egg Cell
The nucleus of the egg cell is removed.
An egg cell is taken from an adult female sheep.
The fused cell begins dividing normally.
Embryo
The embryo is placed in the uterus of a foster mother.
The embryo develops normally into a lamb—Dolly.
Cloned Lamb
Foster Mother

Clones



$\top$
P_0
P_1
P
M
T_0^2
MP_0
MP_1
T_1^2
T_0^3
MT_0^2
PT_0^2
MP
PT_1^2
MT_1^2
T_1^3
T_0^4
MT_0^3
PT_0^3
MPT_0^2
D
MPT_1^2
PT_1^3
MT_1^3
T_1^4
DP
PT_0^4
MPT_0^3
MPT_1^3
PT_1^4
T_0^5
MT_0^4
DM
MT_1^4
T_1^5
PT_0^5
MPT_0^4
MPT_1^4
PT_1^5
MT_0^5
MT_1^5
A
T_0^∞
MPT_0^5
MPT_1^5
T_1^∞
AP_0
AD
AP_1
PT_0^∞
PT_1^∞
MT_0^∞
AP
MT_1^∞
MPT_0^∞
MPT_1^∞
Λ
U
V
ΛP_1
VP_0
ΛP_0
UD
VP
VP_1
ΛP
UM
UP_0
UP_1
$\bot$

Clones

Operation: $A^n \to A$ function (n-ary operation).

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition):

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

Definition

Clone:

- a set of operations on a set A,

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

Definition

Clone:

- a set of operations on a set A,
- contains all projections,

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

Definition

Clone:

- a set of operations on a set A,
- contains all projections,
- closed under superpositions.

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

Definition

Clone:

- a set of operations on a set A,
- contains all projections,
- closed under superpositions.

Examples: all operations on a set,

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \dots, x_i, \dots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \dots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \dots, g_n)$.

Definition

Clone:

- a set of operations on a set A,
- contains all projections,
- closed under superpositions.

Examples: all operations on a set, all projections,

Clones

Operation: $A^n \to A$ function (n-ary operation).

ith projection: $\pi_i(x_1, \ldots, x_i, \ldots x_n) = x_i$.

Superposition (composition): substitute m-ary operations $g_1, \ldots, g_n$ into an n-ary operation f to get a new m-ary operation $f(g_1, \ldots, g_n)$.

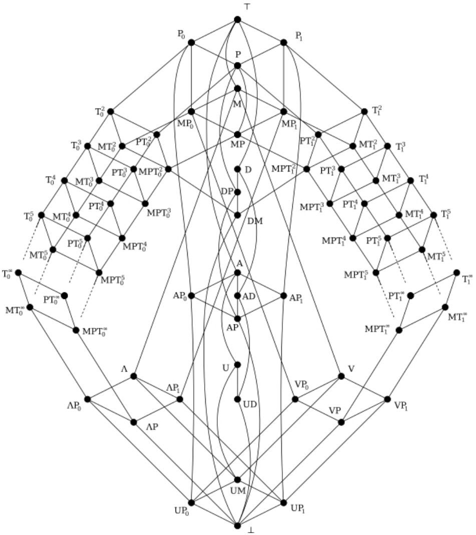
Definition

Clone:

- a set of operations on a set A,
- contains all projections,
- closed under superpositions.

Examples: all operations on a set, all projections, ...

Clones on a two element set



Clones on a three element set

?

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

In 1966, I. G. Rosenberg classified the maximal clones (over finite sets) into six classes.

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

In 1966, I. G. Rosenberg classified the maximal clones (over finite sets) into six classes.

For five of the six classes it has been shown that the clones of these classes are finitely generated. (Generating is adding all the projections and taking all possible superpositions.)

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

In 1966, I. G. Rosenberg classified the maximal clones (over finite sets) into six classes.

For five of the six classes it has been shown that the clones of these classes are finitely generated. (Generating is adding all the projections and taking all possible superpositions.)

The sixth class is the clones of monotone operations of bounded partial orders.

Maximal clones

If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

In 1966, I. G. Rosenberg classified the maximal clones (over finite sets) into six classes.

For five of the six classes it has been shown that the clones of these classes are finitely generated. (Generating is adding all the projections and taking all possible superpositions.)

The sixth class is the clones of monotone operations of bounded partial orders. For small partial orders, the finitely generatedness of the corresponding clone has been shown.

Maximal clones

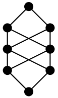
If clones in general are that difficult, we should investigate special ones...

Maximal clone: such a clone that the only larger one is the clone of all operations

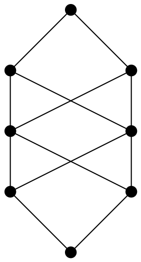
In 1966, I. G. Rosenberg classified the maximal clones (over finite sets) into six classes.

For five of the six classes it has been shown that the clones of these classes are finitely generated. (Generating is adding all the projections and taking all possible superpositions.)

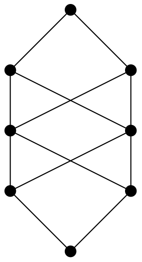
The sixth class is the clones of monotone operations of bounded partial orders. For small partial orders, the finitely generatedness of the corresponding clone has been shown.

The first problematic poset has turned out to be: 

Nonfinitely generated maximal clones found

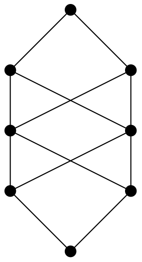


Nonfinitely generated maximal clones found



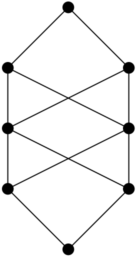
In 1986, G. Tardos published an ingenious paper showing that the clone of the poset above is NOT finitely generated. This was the first maximal clone to be shown nonfinitely generated.

Nonfinitely generated maximal clones found



In 1986, G. Tardos published an ingenious paper showing that the clone of the poset above is NOT finitely generated. This was the first maximal clone to be shown nonfinitely generated.
In 1993, L. Zádori generalised Tardos's result for series-paralell posets.

Nonfinitely generated maximal clones found



In 1986, G. Tardos published an ingenious paper showing that the clone of the poset above is NOT finitely generated. This was the first maximal clone to be shown nonfinitely generated.

In 1993, L. Zádori generalised Tardos's result for series-paralell posets.

No one has found nonfinitely generated maximal clones since, though one may conjecture there are a lot of them.

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset.

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

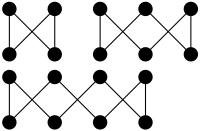
Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

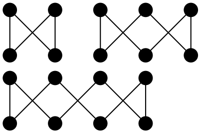
The crown posets—C_4, C_6, and C_8:



Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

The crown posets—C_4, C_6, and C_8:

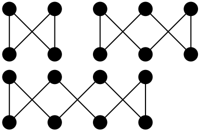


For crowns, the obstacles can get very difficult, there is no known description of them so far.

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

The crown posets—C_4, C_6, and C_8:

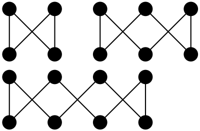


For crowns, the obstacles can get very difficult, there is no known description of them so far. $\rightarrow$ We need something instead of obstacles that we can understand.

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

The crown posets—C_4, C_6, and C_8:

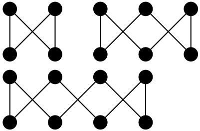


For crowns, the obstacles can get very difficult, there is no known description of them so far. → We need something instead of obstacles that we can understand.
→ critical relations

Main objective, difficulties and our approach

Goal: find more nonfinitely generated maximal clones!
Can we modify Tardos's proof to be working on other posets?
Tardos's proof heavily relies on a lemma in which he describes the so-called obstacles of his poset. PROBLEM: obstacles are very difficult to describe in general.

The crown posets—C_4, C_6, and C_8:



For crowns, the obstacles can get very difficult, there is no known description of them so far. → We need something instead of obstacles that we can understand. → critical relations → **We described the critical relations of the crowns.**

Invariant relations of posets

Let P be a finite poset.

Invariant relations of posets

Let P be a finite poset.
$\alpha \subseteq P^n$ is called *invariant* if for any monotone m-ary operation f and $x_1, \ldots, x_m \in \alpha$:

Invariant relations of posets

Let P be a finite poset.
$\alpha \subseteq P^n$ is called *invariant* if for any monotone m-ary operation f and $x_1, \ldots, x_m \in \alpha$: $f(x_1, \ldots, x_m) \in \alpha$ holds, where $f(x_1, \ldots, x_m)$ is taken componentwise.

Invariant relations of posets

Let P be a finite poset.
$\alpha \subseteq P^n$ is called *invariant* if for any monotone m-ary operation f and $x_1, \ldots, x_m \in \alpha$: $f(x_1, \ldots, x_m) \in \alpha$ holds, where $f(x_1, \ldots, x_m)$ is taken componentwise.

Theorem

Let P be an arbitrary finite poset and $\alpha \subseteq P^n$. Then α is invariant if and only if there exists a finite poset Q and $x_1, \ldots, x_n \in Q$ for which

$$\alpha = \{(f(x_1), f(x_2), \ldots, f(x_n)) \mid \ f : Q \to P \ \text{ monotone}\}.$$

Invariant relations of posets

Let P be a finite poset.
$\alpha \subseteq P^n$ is called *invariant* if for any monotone m-ary operation f and $x_1, \ldots, x_m \in \alpha$: $f(x_1, \ldots, x_m) \in \alpha$ holds, where $f(x_1, \ldots, x_m)$ is taken componentwise.

Theorem

Let P be an arbitrary finite poset and $\alpha \subseteq P^n$. Then α is invariant if and only if there exists a finite poset Q and $x_1, \ldots, x_n \in Q$ for which

$$\alpha = \{(f(x_1), f(x_2), \ldots, f(x_n)) \mid f : Q \to P \text{ monotone}\}.$$

Example. It is easy to see that the relation $\{(x, y) : \exists z \text{ such that } z \leq x, y\}$ is invariant for every poset.

Invariant relations of posets

Let P be a finite poset.
$\alpha \subseteq P^n$ is called *invariant* if for any monotone m-ary operation f and $x_1, \ldots, x_m \in \alpha$: $f(x_1, \ldots, x_m) \in \alpha$ holds, where $f(x_1, \ldots, x_m)$ is taken componentwise.

Theorem

Let P be an arbitrary finite poset and $\alpha \subseteq P^n$. Then α is invariant if and only if there exists a finite poset Q and $x_1, \ldots, x_n \in Q$ for which

$$\alpha = \{(f(x_1), f(x_2), \ldots, f(x_n)) \mid f : Q \to P \ \text{monotone}\}.$$

Example. It is easy to see that the relation $\{(x, y) : \exists z \text{ such that } z \leq x, y\}$ is invariant for every poset.

Two examples for how the theorem works.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

For every critical relation α there exists a minimal Q and $x_1, \dots, x_n$ defining α.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

For every critical relation α there exists a minimal Q and $x_1, \ldots, x_n$ defining α. From minimality, we can get a tuple $y = (y_1, \ldots, y_n) \in P^n$ for which the partial function $Q \to P : x_i \mapsto y_i$ is not extendible monotonically but for every Q' properly contained in Q, it is.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

For every critical relation α there exists a minimal Q and $x_1, \ldots, x_n$ defining α. From minimality, we can get a tuple $y = (y_1, \ldots, y_n) \in P^n$ for which the partial function $Q \to P : x_i \mapsto y_i$ is not extendible monotonically but for every Q' properly contained in Q, it is.

An *obstacle* for P is a pair (Q, f), where Q is a poset and f is a partial function that is not extendible but for all Q' properly contained in Q, it is.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

For every critical relation α there exists a minimal Q and $x_1, \ldots, x_n$ defining α. From minimality, we can get a tuple $y = (y_1, \ldots, y_n) \in P^n$ for which the partial function $Q \to P : x_i \mapsto y_i$ is not extendible monotonically but for every Q' properly contained in Q, it is.

An *obstacle* for P is a pair (Q, f), where Q is a poset and f is a partial function that is not extendible but for all Q' properly contained in Q, it is.

We see that there is a connection between critical relations and obstacles: to every critical relation we can assign an obstacle.

Critical relations and obstacles

Definition

Critical relations are invariant relations that are not (non-trivial) direct products and not (non-trivial) intersections of invariant relations.

For every critical relation α there exists a minimal Q and $x_1, \ldots, x_n$ defining α. From minimality, we can get a tuple $y = (y_1, \ldots, y_n) \in P^n$ for which the partial function $Q \to P : x_i \mapsto y_i$ is not extendible monotonically but for every Q' properly contained in Q, it is.

An *obstacle* for P is a pair (Q, f), where Q is a poset and f is a partial function that is not extendible but for all Q' properly contained in Q, it is.

We see that there is a connection between critical relations and obstacles: to every critical relation we can assign an obstacle.

Tardos used the obstacles to decide if $Q \to T$ partial functions are extendible monotonically or not.

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \ldots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \ldots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \ldots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \ldots p_n = y\}.$$

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \ldots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \ldots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \ldots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \ldots p_n = y\}.$$

It is easy to see that $|d^U(x,y) - d^D(x,y)| \leq 1$.

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \dots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \dots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \dots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \dots p_n = y\}.$$

It is easy to see that $|d^U(x,y) - d^D(x,y)| \leq 1$.
Let $d(x,y) = (d^U(x,y), d^D(x,y))$

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \dots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \dots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\ p_0, \dots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \dots p_n = y\}.$$

It is easy to see that $|d^U(x,y) - d^D(x,y)| \leq 1$.
Let $d(x,y) = (d^U(x,y), d^D(x,y))$ and $R_{m,n} = \{(x,y) \in C_{2k}^2 : d(x,y) \leq (m,n)\}$.

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \ldots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \ldots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \ldots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \ldots p_n = y\}.$$

It is easy to see that $|d^U(x,y) - d^D(x,y)| \leq 1$.
Let $d(x,y) = (d^U(x,y), d^D(x,y))$ and $R_{m,n} = \{(x,y) \in C_{2k}^2 : d(x,y) \leq (m,n)\}$.

Lemma

C_{2k}'s all nonempty binary invariant relations are $R_{m,n}$, where m and n are nonnegative integeres with $|m-n| \leq 1$.

Binary critical relations of crowns

Let P be an arbitrary finite poset and let $x, y \in P$ be in the same connected component. Let

$$d^U(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \dots p_n \in P : x \leq p_0 \geq p_1 \leq p_2 \geq \dots p_n = y\},$$

$$d^D(x,y) = \min\{n \in \mathbb{N}_0 \ : \ \exists\, p_0, \dots p_n \in P : x \geq p_0 \leq p_1 \geq p_2 \leq \dots p_n = y\}.$$

It is easy to see that $|d^U(x,y) - d^D(x,y)| \leq 1$.
Let $d(x,y) = (d^U(x,y), d^D(x,y))$ and $R_{m,n} = \{(x,y) \in C_{2k}^2 : d(x,y) \leq (m,n)\}$.

Lemma

C_{2k}'s all nonempty binary invariant relations are $R_{m,n}$, where m and n are nonnegative integeres with $|m-n| \leq 1$.

Corollary

C_{2k}'s binary critical relations are those $R_{n,n+1}$ and $R_{n+1,n}$ which are not full relations, where n is a nonnegative integer.

Critical relations of crowns

A tuple $\overline{a} = (a_1, \ldots, a_n)$ is called *large* if there is no $(k+1)$-element subfence of C_{2k} that contains $\{a_1, \ldots, a_n\}$.

Critical relations of crowns

A tuple $\overline{a} = (a_1, \ldots, a_n)$ is called *large* if there is no $(k+1)$-element subfence of C_{2k} that contains $\{a_1, \ldots, a_n\}$. Let

$$R_{(a_1,\ldots,a_n)} = C_{2k}^n \setminus \{(\sigma(a_1), \ldots, \sigma(a_n)) : \sigma \in \mathrm{Aut}\, C_{2k}\}$$

Critical relations of crowns

A tuple $\overline{a} = (a_1, \ldots, a_n)$ is called *large* if there is no $(k+1)$-element subfence of C_{2k} that contains $\{a_1, \ldots, a_n\}$. Let

$$R_{(a_1,\ldots,a_n)} = C_{2k}^n \setminus \{(\sigma(a_1), \ldots, \sigma(a_n)) : \sigma \in \text{Aut } C_{2k}\}$$

Theorem

The critical relations of C_{2k} are:

- *the unary $\emptyset$ relation,*
- *the binary critical relations: those $R_{n,n+1}$ and $R_{n+1,n}$ which are not full relations, where n is a nonnegative integer, and*
- *for all large range tuples $\overline{a} \in C_{2k}^n$, the relations $R_{\overline{a}}$.*

Thank you!