Definability in the embeddability ordering of finite directed graphs

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First-order definability in posets

For example, consider a poset with elements 1, 2, 3, 5. The covers of 1 are 1, 2, 4, 5, 60. The covers of 3 are 3, 5, 6, 15, 30. The covers of 5 are 5, 60. The covers of 60 are 60.

Proof: An automorphism:

- 1 ↦ 1
- 2 ↦ 2
- 4 ↦ 4
- 5 ↦ 5
- 6 ↦ 6
- 10 ↦ 10
- 15 ↦ 15
- 30 ↦ 30
- 12 ↦ 12
- 20 ↦ 20
- 12 ↦ 12
- 60 ↦ 60.

This shows that the poset has the desired properties.
First-order definability in posets

\{1\}

Proof: an automorphism:

- \(1 \mapsto 1, 2 \mapsto 2, 4 \mapsto 4, 3 \mapsto 5, 5 \mapsto 5, 6 \mapsto 3 \mapsto 10, 10 \mapsto 6, 15 \mapsto 15, 30 \mapsto 30, 12 \mapsto 12, 60 \mapsto 60\).
First-order definability in posets

\[ \{1\} = \{ x : (\forall y)(x \leq y) \} \]
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\{2, 3, 5\} = \{\text{the covers of } 1\}
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\prec = \{(x, y) : x \leq y \land x \neq y \land (\forall z)(x \leq z \leq y \Rightarrow z = x \lor z = y)\}
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Proof: an automorphism: 1 \mapsto 1, 2 \mapsto 2, 4 \mapsto 4, 3 \mapsto 5, 5 \mapsto 3, 6 \mapsto 10, 10 \mapsto 6, 15 \mapsto 15, 30 \mapsto 30, 12 \mapsto 20, 20 \mapsto 12, 60 \mapsto 60.
Main concept: $A \leq B$ iff $A$ is isomorphic to a substructure of $B$. 

Directed graphs

\[ \mathcal{D} \text{: Isomorphism types of finite directed graphs (digraphs)} \]
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Example:

\[ \leq \text{ is reflexive, transitive, antisymmetric, so } (\mathcal{D}; \leq) \text{ is a poset.} \]
The “bottom” of the poset \((D; \leq)\)
The map $G \mapsto G^T$ (reversing the arrows) is a nontrivial automorphism of $(\mathcal{D}; \leq)$. 
Some results

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Theorem

In $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is definable for arbitrary $G \in \mathcal{D}$. 

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**Corollary**

The poset $(\mathcal{D}; \leq)$ has only one nontrivial automorphism, namely $G \mapsto G^T$. Therefore it's automorphism group is isomorphic to $\mathbb{Z}_2$. 

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\[ \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} \]

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In $(\mathcal{D}; \leq)$, the set \{ $G, G^T$ \} is definable for arbitrary $G \in \mathcal{D}$. In $(\mathcal{D}; \leq, A)$, every $G \in \mathcal{D}$ is definable.
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$CD$: a small category with:

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- \(f = (A, \alpha, B), \ g = (B, \beta, C): \ fg = (A, \beta \circ \alpha, C)\)
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Four constants:

- \(E_1 \in O^{CD}\): \(V(E_1) = \{1\},\ E(E_1) = \emptyset\)
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\(CD' = CD + \text{these four constants}\)
Some languages

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- $L_{CD'}$: first-order language of categories + the 4 constants
- $L(D;\leq,A)$: first-order language of posets + $A$

Observation: $L_{CD'}$ can capture isomorphism and embeddability of digraphs.

A morphism $f \in CD(A, B)$ is injective if $\forall X \in O_{CD} \forall g, h \in CD(X, A) : gf = hf \iff g = h$.

This means all (n-ary) relations first-order denable in $(D;\leq)$ are first-order denable in $CD'$ as well.

We denote this fact by $\text{Def}[(D;\leq)] \subseteq \text{Def}[CD']$.

It is easy to show that $\text{Def}[(D;\leq,A)] \subseteq \text{Def}[CD']$ holds as well.
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Def[$L\rightarrow]\subseteq\text{Def}[CD']$

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\((x, y) \in E(G)\) holds iff

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\exists h \in CD(I_2, G) : f_1 h = f, f_2 h = g.
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Def[\(L\rightarrow\)] \(\subseteq\) Def[\(CD'\)]

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\]
Example. Let $B$ and $C$ be digraphs shown below.

Let us consider the following (heterogeneous) relation:

$$R = \{\ldots\} \subseteq B \times C.$$

We represent $R$ in the following way:

$$B \xrightarrow{e_3} C.$$

So $R$ can be represented as $(E_3, \ldots)$, where $E_3$ are two morphisms.

Def $[L^2] \subseteq \text{Def}[CD']$
Def[$L_{\rightarrow}^2$] $\subseteq$ Def[$CD'$]

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Final results

We’ve seen:

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Example

The set of connected and weakly connected digraphs are both first-order definable in \((\mathcal{D}; \leq, A)\).
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Theorem

$\text{Def}[(D; \leq, A)] \supseteq \text{Def}[CD']$

Every isomorphism-invariant relation that is first-order definable in $CD'$ is first-order definable in $(D; \leq, A)$ (after factoring by isomorphism).
Thank you for your attention!