Definability in the embeddability ordering of finite directed graphs

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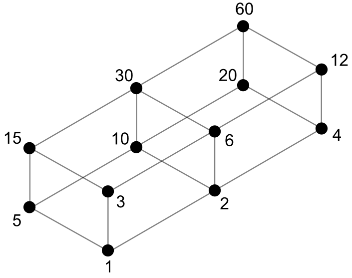
First-order definability in posets



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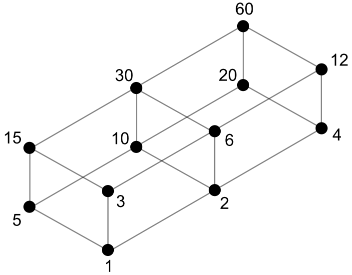

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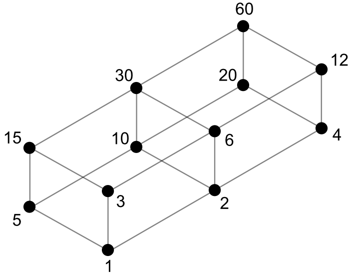

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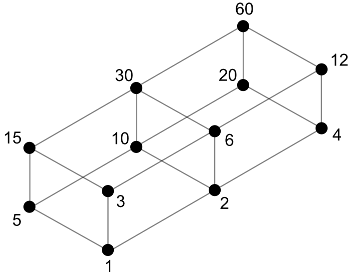

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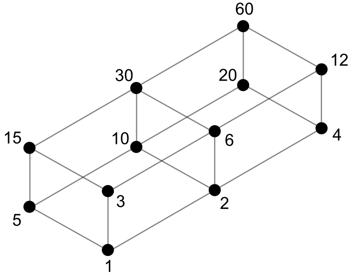

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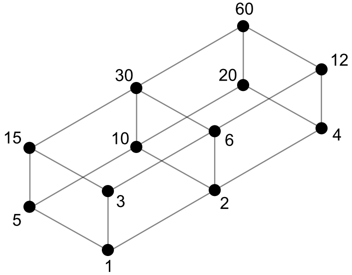

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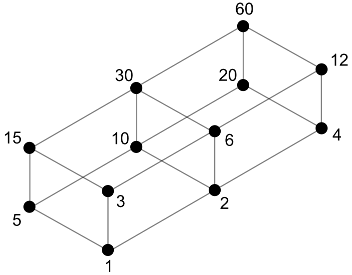

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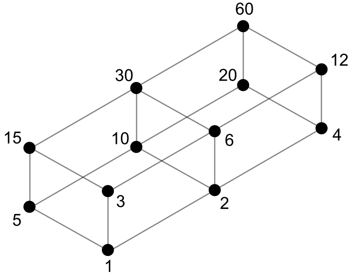

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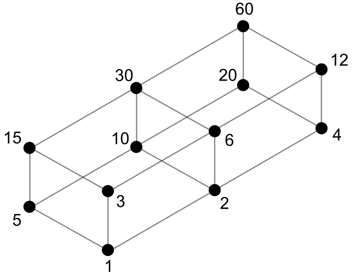

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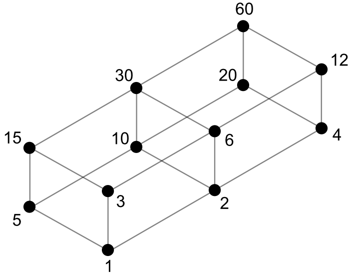

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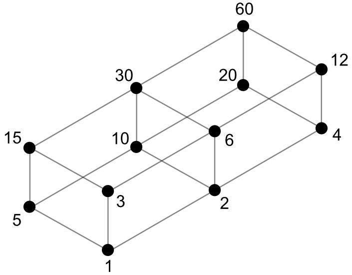

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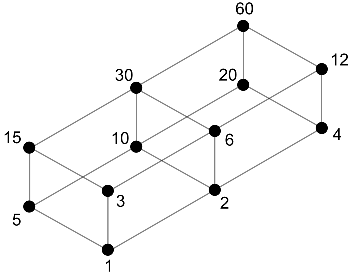

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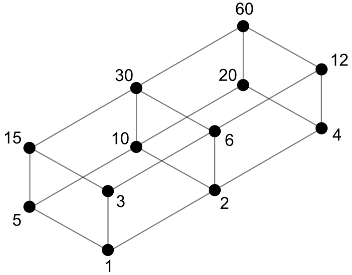

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Proof: an automorphism: $1 \mapsto 1$, $2 \mapsto 2$, $4 \mapsto 4$, $3 \mapsto 5$, $5 \mapsto 3$, $6 \mapsto 10$, $10 \mapsto 6$, $15 \mapsto 15$, $30 \mapsto 30$, $12 \mapsto 20$, $20 \mapsto 12$, $60 \mapsto 60$.

First-order definability in substructure orderings

- J. Ježek and R. McKenzie, *Definability in substructure orderings, I: finite semilattices.* Algebra Universalis **61**, 2009, 59-75.
- J. Ježek and R. McKenzie, *Definability in substructure orderings, II: finite ordered sets.* Order **27**, 2010, 115-145.
- J. Ježek and R. McKenzie, *Definability in substructure orderings, III: finite distributive lattices.* Algebra Universalis **61**, 2009, 283-300.
- J. Ježek and R. McKenzie, *Definability in substructure orderings, IV: finite lattices.* Algebra Universalis **61**, 2009, 301-312.

Main concept: $A \leq B$ iff A is isomorphic to a substructure of B.

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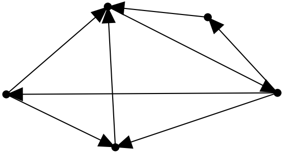

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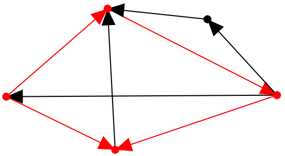

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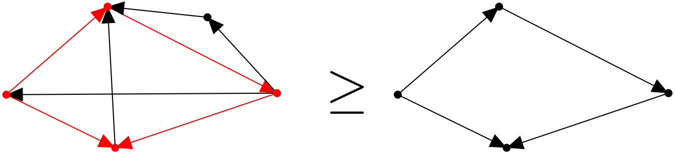

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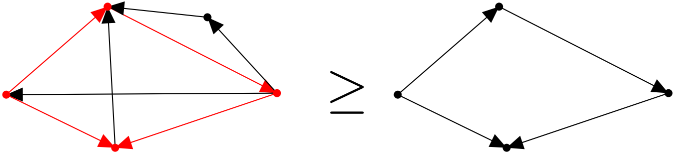
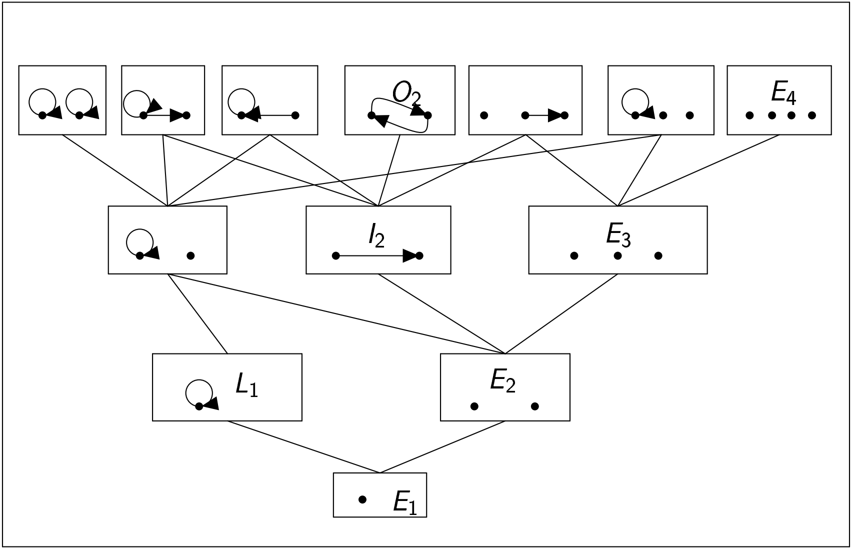


$\leq$ is reflexive, transitive, antisymmetric, so $(\mathcal{D}; \leq)$ is a poset.

The "bottom" of the poset $(\mathcal{D}; \leq)$



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Corollary

The poset $(\mathcal{D}; \leq)$ has only one nontrivial automorphism, namely $G \mapsto G^T$. Therefore it's automorphism group is isomorphic to $\mathbb{Z}_2$.

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$\mathcal{CD}' = \mathcal{CD} +$ these four constants

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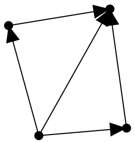
For any $G \in O^{\mathcal{CD}}$,



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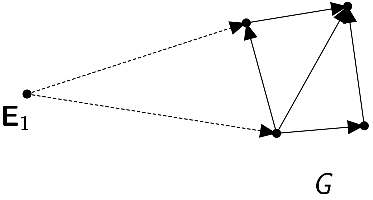

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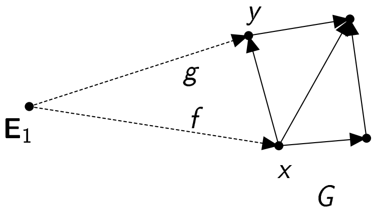

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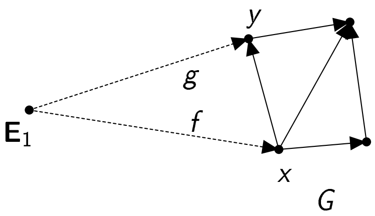

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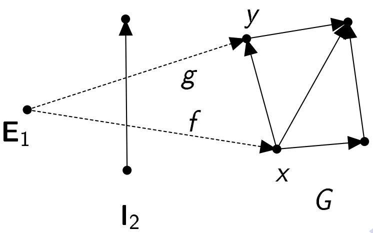

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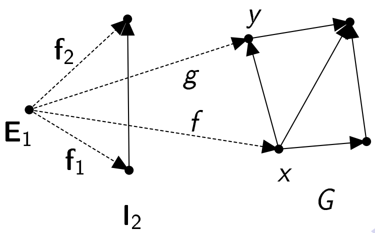

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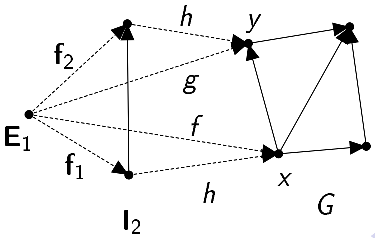

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Example.

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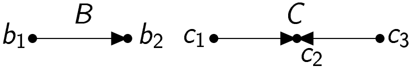
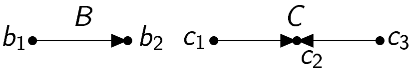


Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$



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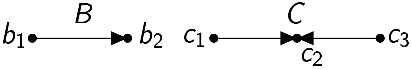

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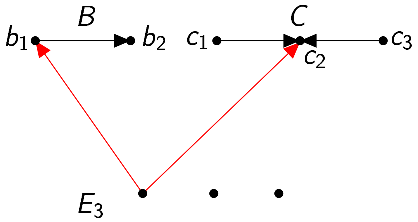

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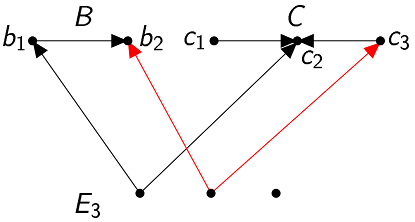

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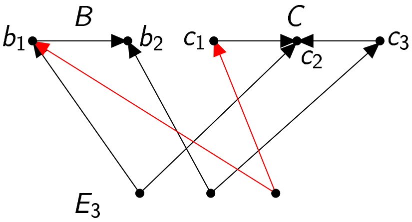

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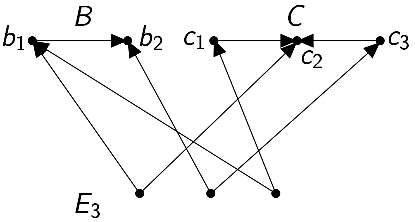

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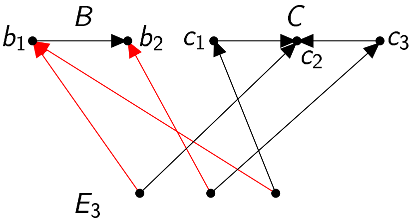

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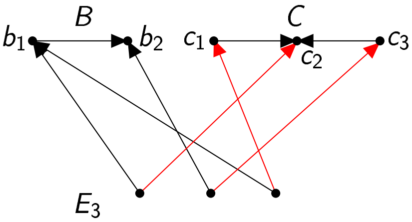

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Final results

We've seen:

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Theorem ($\mathrm{Def}[(\mathcal{D}; \leq, A)] \supseteq \mathrm{Def}[\mathcal{CD}']$)

Every isomorphism-invariant relation that is first-order definable in $\mathcal{CD}'$ is first-order definable in $(\mathcal{D}; \leq, A)$ (after factoring by isomorphism).

Thank you for your attention!