Definability in substructure and embeddability orderings

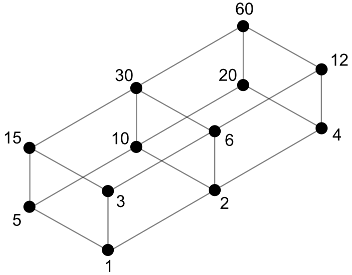
Ádám Kunos

Algebra Seminar, Bolyai Institute

Szeged, May 30, 2018

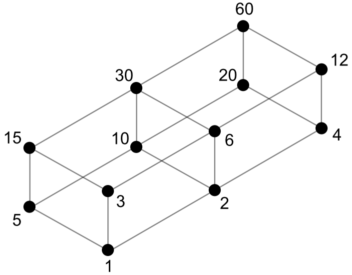
Supported by the UNKP-17-3 New National Excellence Program of the Ministry of Human Capacities.

First-order definability in posets



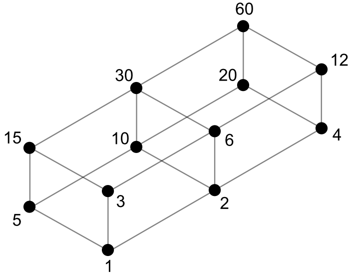
First-order definability in posets

$\{1\}$



First-order definability in posets

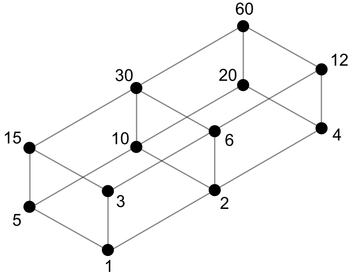
$\{1\} = \{x : (\forall y)(x \leq y)\}$



First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

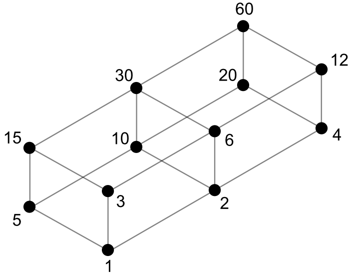


First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} =$

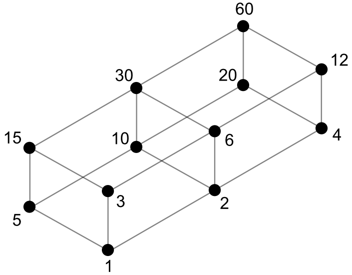


First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$

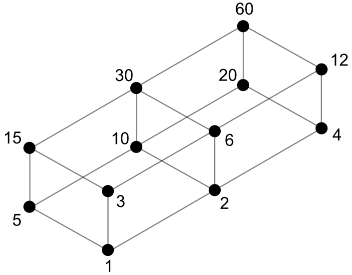


First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$



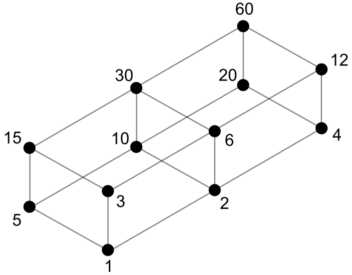
$\prec \, = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$

First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$$\{2, 3, 5\} = \{\text{the covers of } 1\} = \{x : 1 \prec x\}$$



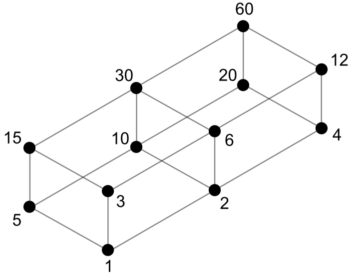
$$\prec = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$$

First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$
$\qquad = \{x : 1 \prec x\}$



$\prec = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$

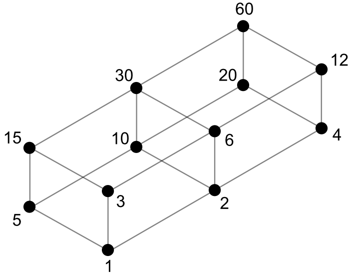
$\{3, 5\}$

First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$
$\quad = \{x : 1 \prec x\}$



$\prec = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$

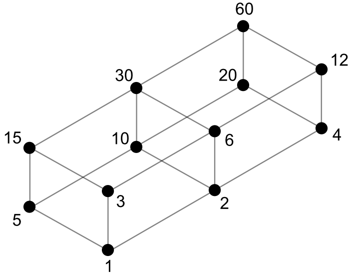
$\{3, 5\} = \{x : 1 \prec x, \ x \text{ has exactly two covers}\}$

First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$
$\quad = \{x : 1 \prec x\}$



$\prec = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$

$\{3, 5\} = \{x : 1 \prec x, \ x \text{ has exactly two covers}\}$

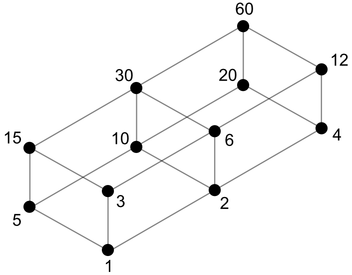
$\{3\} = \{x : ???\}$

First-order definability in posets

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$
$= \{x : 1 \prec x\}$



$\prec = \{(x, y) : x \leq y \ \wedge \ x \neq y \wedge (\forall z)(x \leq z \leq y \ \Rightarrow \ z = x \ \vee \ z = y)\}$

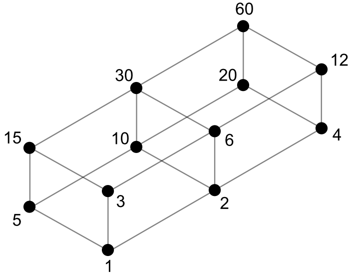
$\{3, 5\} = \{x : 1 \prec x, \ x \text{ has exactly two covers}\}$

$\{3\} = \{x :$???$\}$ Conjecture: NO suitable formula

$\{1\} = \{x : (\forall y)(x \leq y)\}$

$\{60\} = \{x : (\forall y)(y \leq x)\}$

$\{2, 3, 5\} = \{\text{the covers of } 1\}$
$= \{x : 1 \prec x\}$



$\prec = \{(x, y) : x \leq y \;\wedge\; x \neq y \wedge (\forall z)(x \leq z \leq y \;\Rightarrow\; z = x \;\vee\; z = y)\}$

$\{3, 5\} = \{x : 1 \prec x,\; x \text{ has exactly two covers}\}$

$\{3\} = \{x : ???\}$ Conjecture: NO suitable formula

Proof: an automorphism: $1 \mapsto 1$, $2 \mapsto 2$, $4 \mapsto 4$, $3 \mapsto 5$, $5 \mapsto 3$, $6 \mapsto 10$, $10 \mapsto 6$, $15 \mapsto 15$, $30 \mapsto 30$, $12 \mapsto 20$, $20 \mapsto 12$, $60 \mapsto 60$.

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

Substructure, $\sqsubseteq$
$G \sqsubseteq G'$ if and only if G is isomorphic to an induced substructure of G'.

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

Substructure, $\sqsubseteq$
$G \sqsubseteq G'$ if and only if G is isomorphic to an induced substructure of G'.

Embeddability, $\leq$
$G \leq G'$ if and only if there exists $\varphi : G \to G'$ injective graph homomorphism,

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

Substructure, $\sqsubseteq$
$G \sqsubseteq G'$ if and only if G is isomorphic to an induced substructure of G'.

Embeddability, $\leq$
$G \leq G'$ if and only if there exists $\varphi : G \to G'$ injective graph homomorphism, that is $(u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(G')$.

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

Substructure, $\sqsubseteq$
$G \sqsubseteq G'$ if and only if G is isomorphic to an induced substructure of G'.

Embeddability, $\leq$
$G \leq G'$ if and only if there exists $\varphi : G \to G'$ injective graph homomorphism, that is $(u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(G')$.

Examples

Directed graphs, substructure versus embeddability

$\mathcal{D}$: isomorphism types of finite directed graphs, or shortly digraphs, i. e. finite sets with a binary relation on them

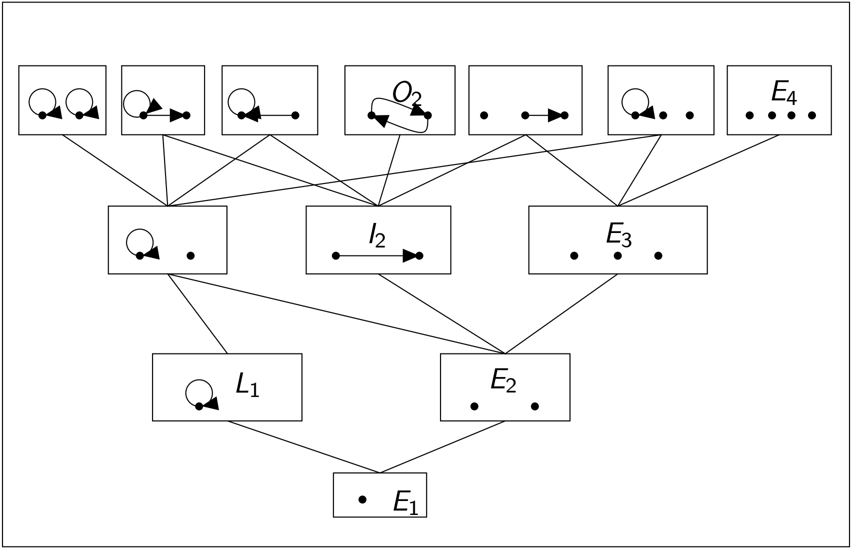
Substructure, $\sqsubseteq$
$G \sqsubseteq G'$ if and only if G is isomorphic to an induced substructure of G'.

Embeddability, $\leq$
$G \leq G'$ if and only if there exists $\varphi : G \to G'$ injective graph homomorphism, that is $(u, v) \in E(G) \Rightarrow (\varphi(u), \varphi(v)) \in E(G')$.

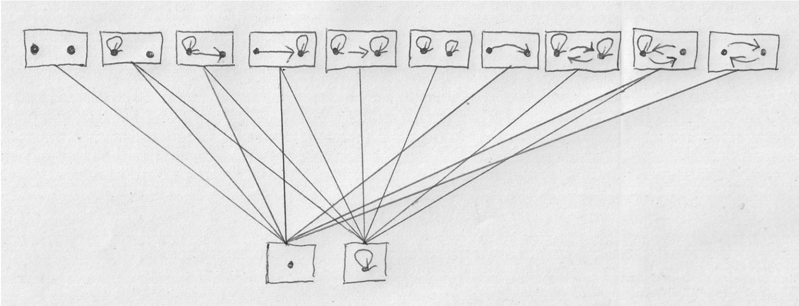
Examples

$(\mathcal{D}; \sqsubseteq)$ and $(\mathcal{D}; \leq)$ are completely different partial orders.

EMBEDDABILITY: the bottom of the poset $(\mathcal{D}; \leq)$



SUBSTRUCTURE: the bottom of the poset $(\mathcal{D}; \sqsubseteq)$



First-order definability in **substructure** orderings

1. J. Ježek and R. McKenzie, *Definability in substructure orderings, I: finite semilattices.* Algebra Universalis **61**, 2009, 59-75.
2. J. Ježek and R. McKenzie, *Definability in substructure orderings, II: finite ordered sets.* Order **27**, 2010, 115-145.
3. J. Ježek and R. McKenzie, *Definability in substructure orderings, III: finite distributive lattices.* Algebra Universalis **61**, 2009, 283-300.
4. J. Ježek and R. McKenzie, *Definability in substructure orderings, IV: finite lattices.* Algebra Universalis **61**, 2009, 301-312.

Results:

1: Every semilattice is definable.

2: The set $\{P, P^d\}$ is definable.

3: The set $\{D, D^d\}$ is definable.

4: The set $\{L, L^d\}$ is definable.

Where I started

EMBEDDABILITY: $(\mathcal{D}; \sqsubseteq)$
Why?...

Where I started

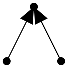
EMBEDDABILITY: $(\mathcal{D}; \sqsubseteq)$
Why?...

Theorem (K, 2015)

In $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is definable for arbitrary $G \in \mathcal{D}$.

Where I started

EMBEDDABILITY: $(\mathcal{D}; \sqsubseteq)$
Why?...

A: 

Theorem (K, 2015)

In $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is definable for arbitrary $G \in \mathcal{D}$.

EMBEDDABILITY: $(\mathcal{D}; \sqsubseteq)$
Why?...

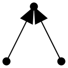
A: 

Theorem (K, 2015)

In $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is definable for arbitrary $G \in \mathcal{D}$. In $(\mathcal{D}; \leq, A)$, every $G \in \mathcal{D}$ is definable.

Where I started

EMBEDDABILITY: $(\mathcal{D}; \sqsubseteq)$
Why?...

A: 

Theorem (K, 2015)

In $(\mathcal{D}; \leq)$, the set $\{G, G^T\}$ is definable for arbitrary $G \in \mathcal{D}$. In $(\mathcal{D}; \leq, A)$, every $G \in \mathcal{D}$ is definable.

Corollary (K, 2015)

The poset $(\mathcal{D}; \leq)$ has only one nontrivial automorphism, namely $G \mapsto G^T$. Therefore it's automorphism group is isomorphic to $\mathbb{Z}_2$.

Can we go further?

We already know that a finite set $H \subseteq \mathcal{D}$ is first-order definable in $(\mathcal{D}; \leq)$ if and only if $\forall G \in \mathcal{D}: \; G \in H \Leftrightarrow G^T \in H$.

Can we go further?

We already know that a finite set $H \subseteq \mathcal{D}$ is first-order definable in $(\mathcal{D}; \leq)$ if and only if $\forall G \in \mathcal{D}: \; G \in H \Leftrightarrow G^T \in H$.
This settles the definability of **finite** subsets.

Can we go further?

We already know that a finite set $H \subseteq \mathcal{D}$ is first-order definable in $(\mathcal{D}; \leq)$ if and only if $\forall G \in \mathcal{D}: \; G \in H \Leftrightarrow G^T \in H$.
This settles the definability of **finite** subsets.

Can we say anything about the definability of **infinite** subsets at this point?

Can we go further?

We already know that a finite set $H \subseteq \mathcal{D}$ is first-order definable in $(\mathcal{D}; \leq)$ if and only if $\forall G \in \mathcal{D}:\ G \in H \Leftrightarrow G^T \in H$.
This settles the definability of **finite** subsets.

Can we say anything about the definability of **infinite** subsets at this point? Not really...

Can we go further?

We already know that a finite set $H \subseteq \mathcal{D}$ is first-order definable in $(\mathcal{D}; \leq)$ if and only if $\forall G \in \mathcal{D}: \ G \in H \Leftrightarrow G^T \in H$.
This settles the definability of **finite** subsets.

Can we say anything about the definability of **infinite** subsets at this point? Not really...

For example, is the set of weakly connected digraphs first-order definable in $(\mathcal{D}; \leq)$?

First- and second-order languages of digraphs

Even the first-order language of digraphs is stronger than what we have so far.

First- and second-order languages of digraphs

Even the first-order language of digraphs is stronger than what we have so far. However a basic model-theoretic argument shows that the set of weakly connected digraphs are NOT definable in this language.

First- and second-order languages of digraphs

Even the first-order language of digraphs is stronger than what we have so far. However a basic model-theoretic argument shows that the set of weakly connected digraphs are NOT definable in this language.

The (full) second-order language of digraphs is much stronger: you can quantify over relations (of arbitrary arity), too.

First- and second-order languages of digraphs

Even the first-order language of digraphs is stronger than what we have so far. However a basic model-theoretic argument shows that the set of weakly connected digraphs are NOT definable in this language.

The (full) second-order language of digraphs is much stronger: you can quantify over relations (of arbitrary arity), too. Weakly connectedness is an easily definable property here.

First- and second-order languages of digraphs

Even the first-order language of digraphs is stronger than what we have so far. However a basic model-theoretic argument shows that the set of weakly connected digraphs are NOT definable in this language.

The (full) second-order language of digraphs is much stronger: you can quantify over relations (of arbitrary arity), too. Weakly connectedness is an easily definable property here.

Theorem (K, 2018+)

The first-order language of $(\mathcal{D}; \leq, A)$ can express the second-order language of directed graphs.

A small category

$\mathcal{CD}$: a small category with:

- objects$=$ $O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$

A small category

$\mathcal{CD}$: a small category with:

- objects$=$ $O^{\mathcal{CD}}$: digraphs with vertices $\{1, \dots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B \text{ homomorphism}\}$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B$ homomorphism$\}$
- $\mathrm{id}_A \in \hom(A, A)$

A small category

$\mathcal{CD}$: a small category with:

- objects$=$ $O^{\mathcal{CD}}$: digraphs with vertices $\{1,\dots,n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A,B) = \{(A,\alpha,B) : \alpha : A \to B \text{ homomorphism}\}$
- $\mathrm{id}_A \in \hom(A,A)$
- $f = (A,\alpha,B)$, $g = (B,\beta,C)$: $fg = (A, \beta\circ\alpha, C)$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \dots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B$ homomorphism$\}$
- $\mathrm{id}_A \in \hom(A, A)$
- $f = (A, \alpha, B)$, $g = (B, \beta, C)$: $fg = (A, \beta \circ \alpha, C)$

Four constants:

- $\mathbf{E}_1 \in O^{\mathcal{CD}}: \quad V(\mathbf{E}_1) = \{1\},\ E(\mathbf{E}_1) = \emptyset,$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B$ homomorphism$\}$
- $\mathrm{id}_A \in \hom(A, A)$
- $f = (A, \alpha, B)$, $g = (B, \beta, C)$: $fg = (A, \beta \circ \alpha, C)$

Four constants:

- $\mathbf{E}_1 \in O^{\mathcal{CD}}: \quad V(\mathbf{E}_1) = \{1\},\ E(\mathbf{E}_1) = \emptyset,$
- $\mathbf{I}_2 \in O^{\mathcal{CD}}: \quad V(\mathbf{I}_2) = \{1, 2\},\ E(\mathbf{E}_1) = \{(1, 2)\},$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B$ homomorphism$\}$
- $\mathrm{id}_A \in \hom(A, A)$
- $f = (A, \alpha, B)$, $g = (B, \beta, C)$: $fg = (A, \beta \circ \alpha, C)$

Four constants:

- $\mathbf{E}_1 \in O^{\mathcal{CD}}: \quad V(\mathbf{E}_1) = \{1\},\ E(\mathbf{E}_1) = \emptyset,$
- $\mathbf{I}_2 \in O^{\mathcal{CD}}: \quad V(\mathbf{I}_2) = \{1, 2\},\ E(\mathbf{E}_1) = \{(1, 2)\},$
- $\mathbf{f}_1 \in \hom(\mathbf{E}_1, \mathbf{I}_2): \quad \mathbf{f}_1 = (\mathbf{E}_1, \{1 \mapsto 1\}, \mathbf{I}_2),$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B \text{ homomorphism}\}$
- $\mathrm{id}_A \in \hom(A, A)$
- $f = (A, \alpha, B)$, $g = (B, \beta, C)$: $fg = (A, \beta \circ \alpha, C)$

Four constants:

- $\mathbf{E}_1 \in O^{\mathcal{CD}}: \quad V(\mathbf{E}_1) = \{1\},\ E(\mathbf{E}_1) = \emptyset,$
- $\mathbf{I}_2 \in O^{\mathcal{CD}}: \quad V(\mathbf{I}_2) = \{1, 2\},\ E(\mathbf{E}_1) = \{(1, 2)\},$
- $\mathbf{f}_1 \in \hom(\mathbf{E}_1, \mathbf{I}_2): \quad \mathbf{f}_1 = (\mathbf{E}_1, \{1 \mapsto 1\}, \mathbf{I}_2),$
- $\mathbf{f}_2 \in \hom(\mathbf{E}_1, \mathbf{I}_2): \quad \mathbf{f}_2 = (\mathbf{E}_1, \{1 \mapsto 2\}, \mathbf{I}_2).$

A small category

$\mathcal{CD}$: a small category with:

- objects$= O^{\mathcal{CD}}$: digraphs with vertices $\{1, \ldots, n\}$
- morphisms: $A, B \in O^{\mathcal{CD}}$:
  $\hom(A, B) = \{(A, \alpha, B) : \alpha : A \to B$ homomorphism$\}$
- $\mathrm{id}_A \in \hom(A, A)$
- $f = (A, \alpha, B)$, $g = (B, \beta, C)$: $fg = (A, \beta \circ \alpha, C)$

Four constants:

- $\mathbf{E}_1 \in O^{\mathcal{CD}}: \quad V(\mathbf{E}_1) = \{1\},\ E(\mathbf{E}_1) = \emptyset,$
- $\mathbf{I}_2 \in O^{\mathcal{CD}}: \quad V(\mathbf{I}_2) = \{1, 2\},\ E(\mathbf{E}_1) = \{(1, 2)\},$
- $\mathbf{f}_1 \in \hom(\mathbf{E}_1, \mathbf{I}_2): \quad \mathbf{f}_1 = (\mathbf{E}_1, \{1 \mapsto 1\}, \mathbf{I}_2),$
- $\mathbf{f}_2 \in \hom(\mathbf{E}_1, \mathbf{I}_2): \quad \mathbf{f}_2 = (\mathbf{E}_1, \{1 \mapsto 2\}, \mathbf{I}_2).$

$\mathcal{CD}' = \mathcal{CD} +$ these four constants

The language $L_{\mathcal{C}\mathcal{D}'}$

$L_{\mathcal{C}\mathcal{D}'}$: first-order language of categories + the 4 constants

The language $L_{\mathcal{CD}'}$

$L_{\mathcal{CD}'}$: first-order language of categories + the 4 constants

$L_{\mathcal{CD}'}$ can capture isomorphism and embeddability of digraphs.

The language $L_{\mathcal{CD}'}$

$L_{\mathcal{CD}'}$: first-order language of categories + the 4 constants

$L_{\mathcal{CD}'}$ can capture isomorphism and embeddability of digraphs.

A morphism $f \in CD(A, B)$ is

- injective iff: $\forall X \in O^{\mathcal{CD}}\ \forall g, h \in \mathrm{hom}(X, A): \quad gf = hf \Leftrightarrow g = h$,
- surjective iff: $\forall X \in O^{\mathcal{CD}}\ \forall g, h \in \mathrm{hom}(B, X): \quad fg = fh \Leftrightarrow g = h$.

The language $L_{\mathcal{CD}'}$

$L_{\mathcal{CD}'}$: first-order language of categories + the 4 constants

$L_{\mathcal{CD}'}$ can capture isomorphism and embeddability of digraphs.

A morphism $f \in CD(A, B)$ is

- injective iff: $\forall X \in O^{\mathcal{CD}}\ \forall g, h \in \mathrm{hom}(X, A) : \quad gf = hf \Leftrightarrow g = h$,
- surjective iff: $\forall X \in O^{\mathcal{CD}}\ \forall g, h \in \mathrm{hom}(B, X) : \quad fg = fh \Leftrightarrow g = h$.

This means all (n-ary) relations first-order definable in $(\mathcal{D}; \leq)$ are first-order definable in $\mathcal{CD}'$ as well.

$L_{\mathcal{C}\mathcal{D}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas.

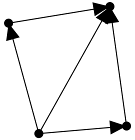
$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$,

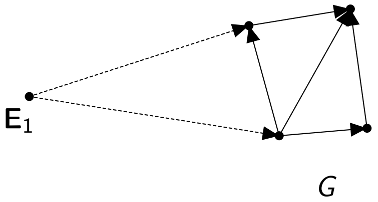


G

$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, $\hom(\mathbf{E}_1, G)$ is naturally bijective with G.

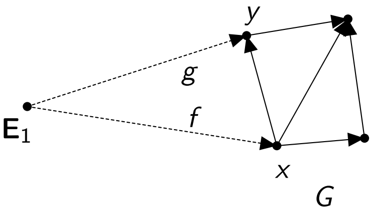


$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, $\hom(\mathbf{E}_1, G)$ is naturally bijective with G. Let

$$f = (\mathbf{E}_1, \{1 \mapsto x\}, G), \;\; g = (\mathbf{E}_1, \{1 \mapsto y\}, G) \;\; (x, y \in V(G)).$$



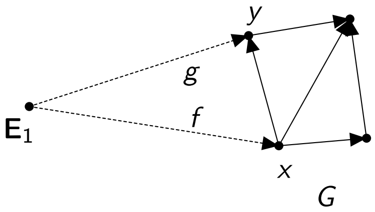
$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, hom$(\mathbf{E}_1, G)$ is naturally bijective with G. Let

$$f = (\mathbf{E}_1, \{1 \mapsto x\}, G), \;\; g = (\mathbf{E}_1, \{1 \mapsto y\}, G) \;\; (x, y \in V(G)).$$

$(x, y) \in E(G)$ holds iff



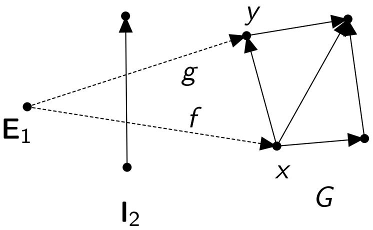
$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, $\hom(\mathbf{E}_1, G)$ is naturally bijective with G. Let

$$f = (\mathbf{E}_1, \{1 \mapsto x\}, G), \ \ g = (\mathbf{E}_1, \{1 \mapsto y\}, G) \ \ (x, y \in V(G)).$$

$(x, y) \in E(G)$ holds iff



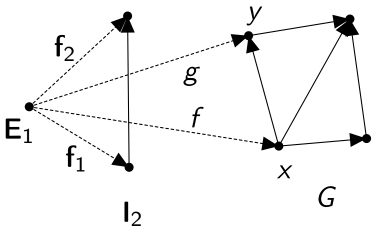
$L_{\mathcal{CD}'}$ is strong

Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, hom$(\mathbf{E}_1, G)$ is naturally bijective with G. Let

$$f = (\mathbf{E}_1, \{1 \mapsto x\}, G), \;\; g = (\mathbf{E}_1, \{1 \mapsto y\}, G) \;\; (x, y \in V(G)).$$

$(x, y) \in E(G)$ holds iff



$L_{\mathcal{CD}'}$ is strong

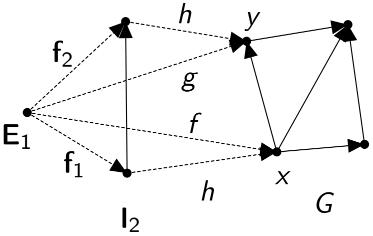
Within $(\mathcal{D}; \leq, A)$ the "inner structure" of the digraphs is unavailable by first order formulas. Surprisingly, in $\mathcal{CD}'$ we can capture the inner structure of digraphs, meaning the first-order language of digraphs can be expressed.

For any $G \in O^{\mathcal{CD}}$, $\hom(\mathbf{E}_1, G)$ is naturally bijective with G. Let

$$f = (\mathbf{E}_1, \{1 \mapsto x\}, G), \;\; g = (\mathbf{E}_1, \{1 \mapsto y\}, G) \;\; (x, y \in V(G)).$$

$(x, y) \in E(G)$ holds iff

$$\exists h \in \hom(\mathbf{I}_2, G) : \;\; \mathbf{f}_1 h = f, \; \mathbf{f}_2 h = g.$$



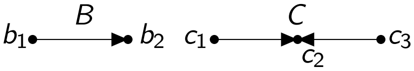
$L_{\mathcal{CD}'}$ is even stronger

$L_{\mathcal{C}\mathcal{D}'}$ is even stronger

Example.

$L_{\mathcal{CD}'}$ is even stronger

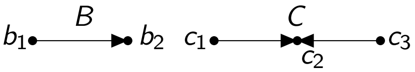
Example. Let B and C the digraphs shown below.



$L_{\mathcal{C}\mathcal{D}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

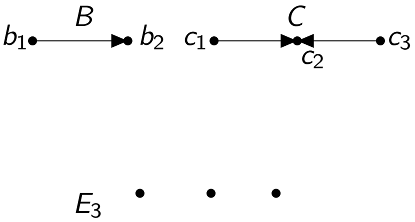


$L_{\mathcal{CD}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:

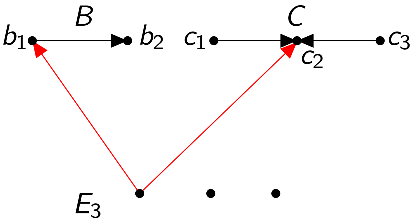


$L_{\mathcal{C}\mathcal{D}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:

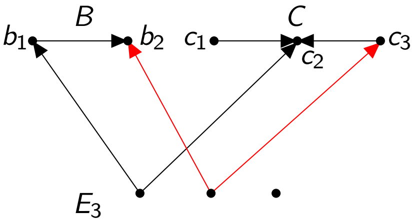


$L_{\mathcal{CD}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:

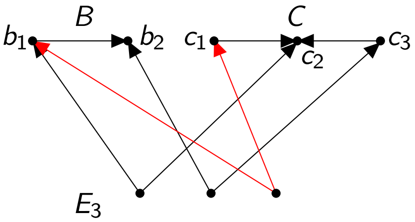


$L_{\mathcal{C}\mathcal{D}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:

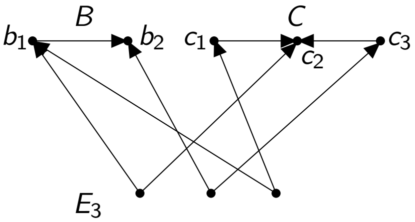


$L_{\mathcal{C}\mathcal{D}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:



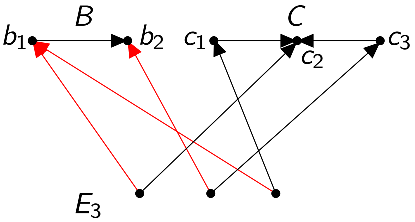
So R can be represented as (E_3, p_1, p_2), where p_1, p_2 are two morphisms.

$L_{\mathcal{CD}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:

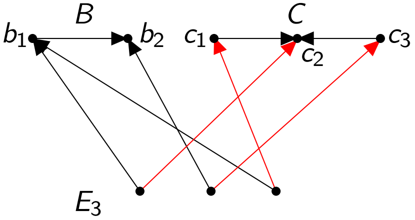


So R can be represented as (E_3, p_1, p_2), where p_1, p_2 are two morphisms.

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:



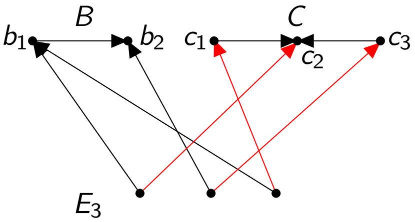
So R can be represented as (E_3, p_1, p_2), where p_1, p_2 are two morphisms.

$L_{\mathcal{CD}'}$ is even stronger

Example. Let B and C the digraphs shown below. Let us consider the following (heterogeneous) relation:

$$R = \{(b_1, c_2), (b_2, c_3), (b_1, c_1)\} \subseteq B \times C.$$

We repsesent R in the following way:



So R can be represented as (E_3, p_1, p_2), where p_1, p_2 are two morphisms. $L_{\mathcal{CD}'}$ is even stronger than the second-order language of digraphs.

A characterisation

We have already seen that (n-ary) relations first-order definable in $(\mathcal{D}; \leq)$ are first-order definable in $\mathcal{CD}'$ as well.

A characterisation

We have already seen that (n-ary) relations first-order definable in $(\mathcal{D}; \leq)$ are first-order definable in $\mathcal{CD}'$ as well. The converse is nontrivial, but true:

A characterisation

We have already seen that (n-ary) relations first-order definable in $(\mathcal{D}; \leq)$ are first-order definable in $\mathcal{CD}'$ as well. The converse is nontrivial, but true:

Theorem (K, 2018+)

The first-order language of $(\mathcal{D}; \leq, A)$ is "as strong as" $L_{\mathcal{CD}'}$.

A characterisation

We have already seen that (n-ary) relations first-order definable in $(\mathcal{D}; \leq)$ are first-order definable in $\mathcal{CD}'$ as well. The converse is nontrivial, but true:

Theorem (K, 2018+)

The first-order language of $(\mathcal{D}; \leq, A)$ is "as strong as" $L_{\mathcal{CD}'}$.

So far we have roughly discussed:

1. Á. Kunos, *Definability in the embeddability ordering of finite directed graphs.* Order **32/1**, 2015, 117-133.
2. Á. Kunos, *Definability in the embeddability ordering of finite directed graphs, II.*, submitted to Order

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and complement of digraphs,

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and complement of digraphs, but I found the

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and complement of digraphs, but I found the "loop-exchange automorphism" as well.

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and complement of digraphs, but I found the "loop-exchange automorphism" as well.
Is the automorphism group isomorphic to $\mathbb{Z}_2^3$?

The substructure ordering: $(\mathcal{D}; \sqsubseteq)$

A completely new partial order.

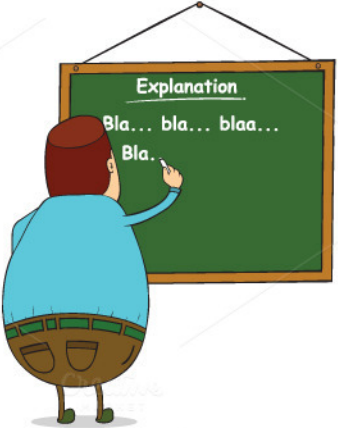
Even the automorhphism groups differ.

I conjectured $\mathbb{Z}_2^2$, induced by taking transposition and complement of digraphs, but I found the "loop-exchange automorphism" as well.
Is the automorphism group isomorphic to $\mathbb{Z}_2^3$?

Two approaches:

- build from scratch again
- try to use the existing result(s)

Board time



Thank you for your attention!