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Definability in the substructure ordering of finite directed graphs

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Abstract We deal with first-order definability in the *substructure* ordering $(\mathscr{D};\sqsubseteq)$ of finite directed graphs. In two papers, the author has already investigated the first-order language of the *embeddability* ordering $(\mathscr{D};\leq)$. The latter has turned out to be quite strong, e.g., it has been shown that, modulo edge-reversing (on the whole graphs), it can express the full second-order language of directed graphs. Now we show that, with finitely many directed graphs added as constants, the first order language of $(\mathscr{D};\sqsubseteq)$ can express that of $(\mathscr{D};\leq)$.

The limits of the expressive power of such languages are intimately related to the automorphism groups of the orderings. Previously, analogue investigations have found the concerning automorphism groups to be quite trivial, e.g., the automorphism group of $(\mathscr{D};\leq)$ is isomorphic to $\mathbb{Z}_2$. Here, unprecedentedly, this is not the case. Even though we conjecture that the automorphism group is isomorphic to $(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2$, with a particular α in the semidirect product, we only prove it is finite.

Keywords First-order definability · Directed graph · Embeddability ordering · Substructure ordering · Automorphism group

1 Introduction and formulation of our main theorems

In 2009–2010 J. Ježek and R. McKenzie published a series of papers [1–4] in which they have examined (among other things) the first-order definability in the substructure orderings of finite mathematical structures with a given type, and determined the automorphism group of these orderings. They considered finite semilattices [1], ordered sets [4], distributive lattices [2] and lattices [3]. Similar investigations [5–9]

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have emerged since. The current paper is one of such, connected strongly to the author's papers [5,6] that dealt with the *embeddability* ordering of finite directed graphs. Now, instead of embeddability, we are examining the *substructure* ordering of finite directed graphs.

Let us consider a nonempty set V and a binary relation $E \subseteq V^2$. We call the pair $G = (V,E)$ a *directed graph* or just *digraph*. Let $\mathscr{D}$ denote the set of isomorphism types of finite digraphs. The elements of $V(= V(G))$ and $E(= E(G))$ are called the *vertices* and *edges* of G, respectively. A digraph G is said to be *embeddable* into G', and we write $G \leq G'$, if there exists an injective homomorphism $\varphi : G \to G'$, i.e. an injective map for which $(v_1,v_2) \in E(G)$ implies $(\varphi(v_1),\varphi(v_2)) \in E(G')$. A digraph G is a *substructure* of G', and we write $G \sqsubseteq G'$, if it is isomorphic to an induced substructure (on some subset of the vertices) of G' . Every substructure is embeddable but the converse is not true. The names of these two concepts often mix both orally and on paper when it is clear from the context which notion we are using the whole time. In the present paper, however, we must be very cautios as both concepts are used alternately throughout the whole paper. It is easy to see that both $\leq$ and $\sqsubseteq$ are partial orders on $\mathscr{D}$. Both partially ordered sets are naturally graded. The digraph G is on the nth level of $(\mathscr{D};\leq)$ or $(\mathscr{D};\sqsubseteq)$ if $|V(G)|+|E(G)| = n$ or $|V(G)| = n$, respectively. See Figures 1 and 2 for the bottoms of the Hasse diagrams of the two partial orders.

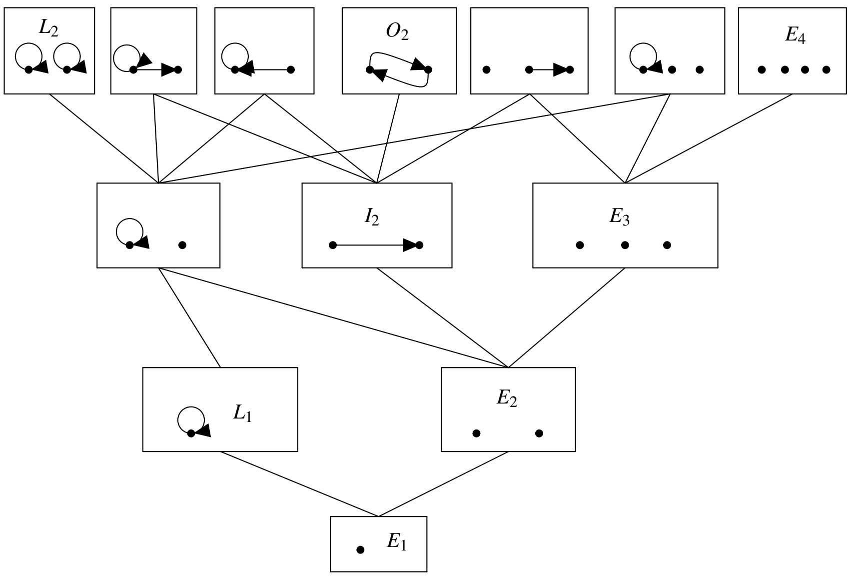


Fig. 1 The bottom part of the Hasse diagram of $(\mathscr{D};\leq)$.

Let $(\mathscr{A};\leq)$ be an arbitrary poset. An n-ary relation R is said to be (first-order) definable in $(\mathscr{A};\leq)$ if there exists a first-order formula $\Psi(x_1,x_2,\ldots,x_n)$ with free

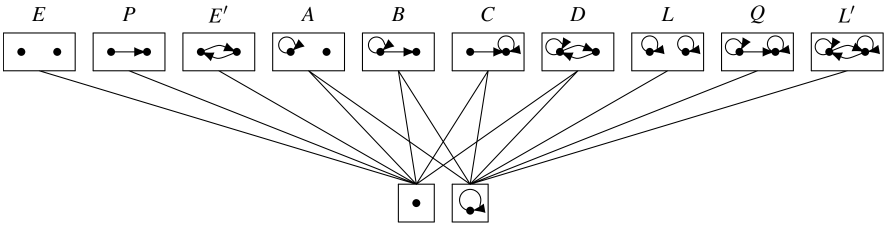


Fig. 2 The bottom part of the Hasse diagram of $(\mathscr{D};\sqsubseteq)$.

variables $x_1, x_2, \ldots, x_n$ in the language of partially ordered sets such that for any $a_1, a_2, \ldots, a_n \in \mathscr{A}$, $\Psi(a_1, a_2, \ldots, a_n)$ holds in $(\mathscr{A};\leq)$ if and only if $(a_1, a_2, \ldots, a_n) \in R$. A subset of $\mathscr{A}$ is definable if it is definable as a unary relation. An element $a \in \mathscr{A}$ is said to be definable if the set $\{a\}$ is definable.

Our main result is the following.

Theorem 1 *There exists a finite set of finite directed graphs $\{C_1, \ldots, C_k\}$ such that the binary embeddability relation,*

$$\{(G, G') : G \leq G'\},$$

is definable in the first-order language of $(\mathscr{D};\sqsubseteq, C_1, \ldots, C_k)$. Consequently, every relation definable in the first-order language of $(\mathscr{D};\leq)$ is definable in that of $(\mathscr{D};\sqsubseteq, C_1, \ldots, C_k)$.

In itself, this theorem is quite weightless, what fills it with content is that we already know [5, 6] that the first-order language of $(\mathscr{D};\leq)$ is surprisingly strong. The paper [6] has two parts. The first deals with definability in $(\mathscr{D};\leq)$, the second determines the automorphism group of $(\mathscr{D};\leq)$ (building on the first part, of course). The paper [5] extends the main result of the first part of [6], hence if one is only interested in definability, it is enough to read [5]. The main result there [5, Theorem 5] is some kind of a characterization of the first-order definable relations in $(\mathscr{D};\leq)$. To even state the result precisely, there is a 3-page-long preparation which we don't repeat here. We only provide some corollaries, demonstrating the power of definability in $(\mathscr{D};\leq)$. With Theorem 1, these corollaries transform immediately into statements for the first-order language of $(\mathscr{D};\sqsubseteq, C_1, \ldots, C_k)$. As this paper is about the substructure ordering, we formulate these versions, rather than the versions talking about $(\mathscr{D};\leq)$.

Corollary 2 *There exists a finite set of finite directed graphs $\{C_1, \ldots, C_k\}$ such that in the first-order language of $(\mathscr{D};\sqsubseteq, C_1, \ldots, C_k)$*

- *every single digraph G is definable,*
- *the set of weakly connected digraphs is definable, moreover,*
- *the full second-order language of digraphs becomes available.*

Again, for the full scope of Theorem 1, see [5, Section 2].

We remark that the notations of Theorem 1 and Corollary 2 may suggest that the set $\{C_1,\dots,C_k\}$ in the two statements can be the same. This is not necessarily true, even though there is a strong connection between the two sets. Depending on the set of Theorem 1, an additional digraph might have to be added the get the corresponding set of Corollary 2. This is due to the fact that the first-order language of $(\mathscr{D};\leq)$ does not yield the listed statements of Corollary 2 in itself. A constant (a particular digraph) needs to be added to the first-order language of $(\mathscr{D};\leq)$ to make these true. If this constant is not already there in the set of $\{C_1,\dots,C_k\}$ of Theorem 1 then its addition might be required to get that of Corollary 2. As the equality of the sets is not stated anywhere, this is not a problem. This affair is actually about $(\mathscr{D};\leq)$, which is not the subject of our investigation here. The interested reader should consult the first two sections of [5].

We wish to make another remark on the lists $\{C_1,\dots,C_k\}$ to avoid false expectations. Naturally, as we proceed with our proof the lists $\{C_1,\dots,C_k\}$ will be continously growing. The final list is revealed late in the paper, and that is why we now outline it in advance. To do so, we describe a family of our arguments used in the last, technical section of the paper. Some properties of digraphs can be told by saying something about the list of their, say, at most 4-element subgraphs (without multiplicity, naturally). For example one can tell if a digraph has loops based on the list of its 1-element subgraphs. Similarly, one can judge if it has a non-loop edge by the list of its (at most) 2-element subgraphs. Far more complicated properties can be told in this way, say, *locally*. We adopt this thinking in the last section of the paper. This will force our lists $\{C_1,\dots,C_k\}$ to be $\{$at most 4-element digraphs$\}$. This list is long but finite nevertheless.

The papers [1–4, 6, 9], beyond dealing with definability, determined the automorphism groups of the orderings in question. In every case, the automorphisms came naturally and the automorphism groups were either trivial or isomorphic to $\mathbb{Z}_2$. Despite all expectations, the partially ordered set $(\mathscr{D},\sqsubseteq)$ stands out in that aspect. There are automorphisms far from trivial. Unfortunately, we are not able to determine the automorphism group, we can only prove it is finite.

Theorem 3 *The automorphism group of* $(\mathscr{D},\sqsubseteq)$ *is finite.*

Even though we can not prove it, we formulate a conjecture for the automorphism group.

In Section 2, we prove Theorem 3, and tell our conjecture on the automorphism group in detail. Section 3 contains the proof of Theorem 1 without some technicalities. In Section 4, the reader finds the technicalities skipped in Section 3.

2 On the automorphism group of $(\mathscr{D};\sqsubseteq)$

First, we prove Theorem 3 using Theorem 1.

Proof (of Theorem 3) It is clear that the orbits of the automorphism group are finite as an automorphism can only move a digraph inside its level in $(\mathscr{D},\sqsubseteq)$. Let $o(G)$ denote the size of the orbit of the digraph G (which is therefore a positive integer).

We state that it suffices to present a finite set of digraphs such that the only automorphism fixing them all is the identity. To prove that, let $\{C_1,\dots,C_k\}$ be such a set and φ be an arbitrary automorphism. Observe that the images of C_i under φ determine φ completely, or in other words, the only automorphism agreeing with φ on $\{C_1,\dots,C_k\}$ is φ. Indeed, with the notations

$$S = \{\alpha \in \mathrm{Aut}(\mathscr{D};\sqsubseteq) : \alpha(C_i) = \varphi(C_i),\ \ i = 1,\dots,k\},$$

and $S' = \{\alpha\varphi^{-1} : \alpha \in S\}$, $|S| = |S'|$ holds, and $|S'| = 1$ for all elements of S' fix all of $\{C_1,\dots,C_k\}$. The fact that an automorphism is completely determined by its action on $\{C_1,\dots,C_k\}$ means that the automorphism group has at most $o(C_1)\cdot\ldots\cdot o(C_k)$ elements. That proves our statement.

Finally, we claim that $\{C_1,\dots,C_k\}$ of Theorem 1 suffices for the purpose above, namely the only automorphism fixing them all is the identity. Let φ be an automorphism that fixes all C_i. Let $G \in \mathscr{D}$ be arbitrary. We need to show that $\varphi(G) = G$. We know from Corollary 2 that there exists a formula $\phi_G(x)$ with one free variable, that defines G in first order language of $(\mathscr{D},\sqsubseteq,C_1,\dots,C_k)$. If we change all occurrences of C_i to $\varphi(C_i)$ in $\phi_G(x)$, then we get a formula $\phi_{\varphi(G)}(x)$ defining $\varphi(G)$. For φ fixes all C_is, $\phi_G(x) = \phi_{\varphi(G)}(x)$, implying $G = \varphi(G)$. $\square$

In the remaining part of the section, we present the automorphisms that we know of. Here, no claim is proven rigorously, they are all rather conjectures. Our intention is just to offer some insight on how the author sees the automorphism group at the moment. Before the (semi-)precise definition of our automorphisms, we feel it is useful to give a nontechnical glimpse at them. Automorphisms map digraphs to digraphs of $\mathscr{D}$. To define an automorphism φ, we need to tell how to get $\varphi(G)$ from G. All the automorphisms, that we know of at the moment, share a particular characteristic. They are all, say, *local* in the following sense. Roughly speaking, to get $\varphi(G)$ from G, one only needs to consider and modify G's at most two element subgraphs according to some given rule.

To make this clearer, we give an example. Let $\varphi(G)$ be the digraph that we get from G such that we change the direction of the edges on those two element subgraphs of G that have loops on both vertices. It is easy to see that this defines an automorphism, indeed. Perhaps, one would quickly discover the automorphism that gets $\varphi(G)$ by reversing all edges of G, but this is different. In this example, the modification of G happens only locally, namely on 2-element subgraphs. All the automorphisms, that we know of, share this property.

Now, we define our automorphisms φ_i (semi-)precisely. We do so by telling how to get $\varphi_i(G)$ from G. One of the most trivial automorphisms is

– φ_1: where there is a loop, clear it, and vica versa, to the vertices with no loop, insert one.

Observe that this automorphism operates with the 1-element subgraphs. Now we start to make use of the labels of Fig. 2.

– φ_2: change the subgraphs (isomorphic to) E to E' and vica versa.
– φ_3: change the subgraphs (isomorphic to) L to L' and vica versa.
– φ_4: reverse the edges in the subgraphs (isomorphic to) P.

- φ_5: reverse the edges in the subgraphs (isomorphic to) Q.

Let S_4 denote the symmetric group over the four-element set $\{A,B,C,D\}$, and $\pi \in S_4$. We define

- φ_π: We change the subgraphs (isomorphic to) $X \in \{A,B,C,D\}$ to $\pi(X)$ (such that the loops remain in place).

Observe that, with the exception of φ_1, the automorphisms defined above do not touch loops (when getting $\varphi_i(G)$ from G). We conjecture that these automorphisms generate the whole automorphism group.

Finally, we investigate the structure of the group of our conjecture. Let I denote the set of possible indexes of our φs, namely

$$I = \{1,\dots,5\} \cup \{\pi \in S_4\}.$$

Let $\langle\rangle$ stand for subgroup generation. Let $S = \langle \varphi_i : i \in I\rangle$ denote the group of our conjecture. It seems that S splits into the internal semidirect product

$$S = \langle \varphi_i : i \in I \setminus \{1\}\rangle \rtimes \langle \varphi_1 \rangle.$$

Furthermore, the second factor appears to be a(n internal) direct product

$$\langle \varphi_2\rangle \times \langle \varphi_3\rangle \times \langle \varphi_4\rangle \times \langle \varphi_5\rangle \times \langle \varphi_\pi : \pi \in S_4\rangle.$$

Here, at the last factor, the subgroup generation is just a technicality as, clearly, the φ_πs constitute a subgroup themselves. These observations all need a proper checking, but they give rise to the conjecture that S is isomorphic to

$$(\mathbb{Z}_2^4 \times S_4) \rtimes_\alpha \mathbb{Z}_2,$$

where S_4, again, denotes the symmetric group over the set $\{A,B,C,D\}$, and α is the following. Obviously, $\alpha(0) = \mathrm{id} \in \mathrm{Aut}(\mathbb{Z}_2^4 \times S_4)$. To define $\alpha(1)$, let $p,q,r,s \in \{0,1\}$ and $\pi \in S_4$. Then

$$\alpha(1) : (p,q,r,s,\pi) \mapsto (q,p,s,r,(BC)\pi(BC)),$$

where (BC) is just the usual cycle notation of the permutation of S_4 that takes B to C and vica versa. Note that the group of our conjecture has 768 elements. Even though we cannot prove that there are no more automorphisms beyond the ones in S, we conjecture so.

In the remaining part of the section, we present the automorphisms that we know of. Here, no claim is proven rigorously, they are all rather conjectures. Our intention is just to offer some insight on how the author sees the automorphism group at the moment. All pairs of vertices fall into one of the following three categories in directed graphs, based on the number of loops they have. A pair of vertices is *loop-free* if there is no loop in it, it is *loop-full* if both vertices have loops, and it is *mixed* if one vertex does have a loop, while the other does not. Similarly, we categorize with regard to the number of non-loop edges. A pair of vertices is *disconnected* if there is no edge between the two vertices, *strongly connected* if there are edges in both directions, and *weakly connected* if there is edge only in one direction. We are ready to formulate some automorphisms. We do so by telling how to get $\varphi_i(G)$ from G.

φ_1: Put loops on loop-free vertices, and clear the loops from loop-full ones.
φ_2: On pairs of vertices that are weakly connected and loop-free, change the direction of the edges.
φ_3: On pairs of vertices that are weakly connected and loop-full, change the direction of the edges.
φ_4: On pairs of vertices that are weakly connected and mixed, change the direction of the edges.
φ_5: On loop-free pairs, change disconnected pairs to strongly connected ones and vica versa.
φ_6: On loop-full pairs, change disconnected pairs to strongly connected ones and vica versa.
φ_7: On mixed pairs, change disconnected pairs to strongly connected ones and vica versa (with the positions of the loops staying the same).

Obviously, arbitrary compositions of these are again automorphisms. All the listed automorphisms are of order two. Unfortunately, they do not commute, e.g., $\varphi_1\varphi_2 \neq \varphi_2\varphi_1$. Let $\langle\rangle$ stand for subgroup generation. For the subgroup $S := \langle\varphi_i : 1 \leq i \leq 7\rangle$ of the automorphism group, the seven-element generator set, it is given by, is not even minimal as, for example, $\varphi_1\varphi_2\varphi_1 = \varphi_3$. The automorphism $\varphi_1\varphi_2$ is of order 4. We have seen now that the automorphism group is far from $\mathbb{Z}_2^7$, which may be the first guess after seeing the seven automorphisms listed above. Still, we think that the automorphism group has $128(= 2^7)$ elements. It seems that S is the internal direct product of its subgroups $\langle\varphi_1,\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle$ and $\langle\varphi_4,\varphi_7\rangle$. Furthermore, the factor $\langle\varphi_1,\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle$ is the internal semidirect product of $\langle\varphi_1\rangle$ acting on $\langle\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle$:

$$\langle\varphi_1,\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle = \langle\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle \rtimes \langle\varphi_1\rangle,$$

and $\langle\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle$ factors into the internal direct product

$$\langle\varphi_2,\varphi_3,\varphi_5,\varphi_6\rangle = \langle\varphi_2\rangle \times \langle\varphi_3\rangle \times \langle\varphi_5\rangle \times \langle\varphi_6\rangle.$$

These observations all need a proper checking, but they give rise to the conjecture that S is isomorphic to

$$(\mathbb{Z}_2^4 \rtimes_\alpha \mathbb{Z}_2) \times \mathbb{Z}_2^2, \text{ where } (a,b,c,d) \xmapsto{\alpha(1)} (b,a,d,c).$$

It would be nice to see a digraph with a 128-element orbit under the action of S. We nominate the digraph of Fig. 3 for this.

Even though we cannot prove that there are no more automorphisms beyond the ones in S, we conjecture so.

3 The proof of Theorem 1 without some technicalities

As long and technical as it may seem, the whole proof of Theorem 1 is based on a simple idea, which we outline here. We get substructures of a directed graph by leaving out vertices, while, to get embeddable digraphs, we can leave out vertices

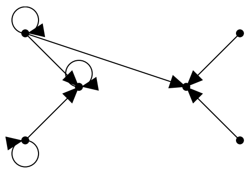


Fig. 3

and edges both. We want to define the latter, so we need to be able to leave out edges somehow. Our main idea is the following. In a digraph G, if there is an edge $(u,v) \in E(G)$, then we add a vertex and two edges to "support" the edge (u,v). Namely, we add w to the set of vertices, and the edges (u,w) and (w,v) to the set of edges. After the addition, we say that the edge (u,v) is "supported". The idea is that the supportedness of an edge can be terminated by leaving out a vertex, in the previous example w, what we can do by taking substructures. Roughly, what we should do is: support all edges, take a substructure, and in one more step, leave only the supported edges in. Of course, there seems to be many problems with this (if told in such a simplified way). Firstly, how can we distinguish between the supporting vertices and the original ones? This appears to be an essential part of the plan. Secondly, the plan ended with "leave only the supported edges in" which just looks running into the original problem again: we cannot leave edges out. Even though the plan seems flawed for these reasons, it is manageable. The whole section is no more than building the apparatus and carrying it out.

Definition 4 In this section, we use three particular automorphism:

- the *loop-exchange automorphism*, denoted by l, which is φ_1 (of the previous section),
- the *edge-reverse (transposition) automorphism*, denoted by t, which is $\varphi_2\varphi_3\varphi_4$, and
- the *complement automorphism*, denoted by c, which is $\prod_{i=1}^{7}\varphi_i$.

The edge-reverse automorphism just reverses all edges in a digraph, while the complement automorphism replaces $E(G)$ with $V(G)^2 \setminus E(G)$.

Some basic definitions follow

Definition 5 For digraphs $G, G' \in \mathscr{D}$, let $G \dot\cup G'$ denote their disjoint union, as usual.

Definition 6 Let E_n $(n = 1,2,\dots)$ denote the "empty" digraph with n vertices and F_n $(n = 1,2,\dots)$ denote the "full" digraph with n vertices:

$$V(E_n) = \{v_1, v_2, \dots, v_n\},\ \ E(E_n) = \emptyset,$$

$$V(F_n) = \{v_1, v_2, \dots, v_n\},\ \ E(F_n) = V(F_n)^2.$$

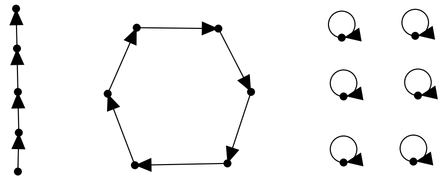
Definition 7 Let I_n $(n = 1,2,\dots)$, O_n $(n = 3,4,\dots)$, and L_n $(n = 1,2,\dots)$ be the following (Fig. 4.) digraphs:

$$V(I_n) = V(O_n) = V(L_n) = \{v_1, v_2, \dots, v_n\},$$

$$E(I_n) = \{(v_1,v_2),(v_2,v_3),\dots,(v_{n-1},v_n)\},$$
$$E(O_n) = \{(v_1,v_2),(v_2,v_3),\dots,(v_{n-1},v_n),(v_n,v_1)\},$$
$$E(L_n) = \{(v_1,v_1),(v_2,v_2),\dots,(v_n,v_n)\}.$$

The digraphs I_n are called *lines*, and the digraphs O_n are called *circles*.

Note $E_1 = I_1$.



I_5 O_6 L_6

Fig. 4

Definition 8 A directed graph is called an *IO-graph* if it satisfies the following conditions. The only one-element substructure of it is E_1. If X is a two-element substructure then it is either E_2 or I_2. If X is a three-element substructure then X is E_3, or $I_2 \dot\cup E_1$, or I_3, or O_3. Let the set of IO-graphs be denoted by IO.

Lemma 9 *The set IO is definable.*

Proof Observe that the set IO is already given by a first-order definition, using the one, two, and three element digraphs as constants. □

Observe that the set IO is closed under taking substructures. The following lemma motivates our notation IO.

Lemma 10 *A directed graph is an IO-graph if and only if it is a disjoint union of lines and/or circles.*

Proof Straightforward induction on the number of vertices suffices, using the closedness mentioned prior to the lemma. □

Lemma 11 *The set* $\{O_n : n \geq 3\}$ *is definable.*

Proof It is clear that all elements of the set are IO-graphs, we just need to choose which. It is easy to see that, in IO, those that have a unique lower-cover (within IO) are:

$$\underbrace{G \dot\cup \cdots \dot\cup G}_{k \text{ copies}}, \text{ where } G \in \{E_1, I_2\} \cup \{O_n : n \geq 3\},$$

for $k \geq 1$ except when $X = E_1$, then $k > 1$. In this set, the desired digraphs are exactly those that are minimal (in this particular set) and have I_3 or O_3 as a substructure. □

Definition 12 A digraph is called *loop-full* if all vertices have loops on them, and *loop-free* if none. The *loop-full part* of a digraph is the maximal loop-full substructure of it, and the *loop-free part* is the maximal loop-free substructure.

Lemma 13 *The relation*

$$\{(G,F,G\,\dot\cup\,F) : G,F \in \mathscr{D},\ G \text{ is loop-full and } F \text{ is loop-free}\}$$

is definable.

Proof The relation consists of those triples (X,Y,Z) for which

- X is the loop-full part of Z,
- Y is the loop-free part of Z, and
- there is no two element substructure of Z that consists exactly one loop and has a non-loop edge in it.

□

Definition 14 Let $L_\rightarrow$ denote the digraph with

$$V(L_\rightarrow) = \{v_1,v_2\}, \text{ and } E(G) = \{(v_1,v_1),(v_1,v_2)\}.$$

Definition 15 Let G be a loop-full digraph with $V(G) = \{v_1,\dots,v_n\}$. Then $l(G)$ is loop-free. Let the set of its vertices be $l(G) = \{v_1',\dots,v_n'\}$ with

$$\text{for } i \neq j:\ (v_i',v_j') \in E(l(G)) \Leftrightarrow (v_i,v_j) \in E(G).$$

Let $G \rightarrow l(G)$ denote the digraph for which

$$V(G \rightarrow l(G)) = V(G) \cup V(l(G)), \text{ and}$$

$$E(G \rightarrow l(G)) = E(G) \cup E(l(G)) \cup \{(v_i,v_i') : 1 \leq i \leq n\}.$$

Lemma 16 *The relation*

$$\{(G,l(G),G \rightarrow l(G)) : G \in \mathscr{D},\ G \text{ is loop-full}\}$$

is definable.

Proof Let us consider the triples (X,Y,Z) for which

- X is the loop-full part of Z, and Y is the loop-free part of Z,
- $X \,\dot\cup\, E_1 \not\sqsubseteq Z$, and $Y \,\dot\cup\, L_1 \not\sqsubseteq Z$ (both are definable by Lemma 13),
- on two points, the only substructure having exactly one loop and at least one non-loop edge is $L_\rightarrow$, and
- no digraph of the first two pictures of Fig. 5 is a substructure. We consider the dashed edges possibilities, either we draw them (individually) or not. In this way, there are 6 (isomorphism types) encoded into the first two pictures of Fig. 5. We exclude them all.

Now we have ensured that the edges $L_\rightarrow$ constitute a bijection between the vertices of X and Y in Z. It only remains to force this bijection to be edge and non-edge preserving as well. This can be done by requiring the additional the property

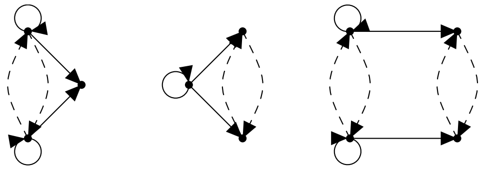


Fig. 5

– Consider the third picture of Figure 5 as before, the dashed edges are possibilities. We forbid those from being substructures in which the dashed edges are not symmetrically drawn on the two (loop-full and loop-free) sides.

$\square$

We are going to need some basic arithmetic later. We define addition in the following lemma.

Lemma 17 *The following relation is definable:*

$$\{(E_n, E_m, E_{n+m}) : n, m \geq 1\}.$$

Proof The set $\{E_n\}$ is definable as it consists of E_1 plus those digraphs which have only E_2 as a two-element substructure. $E_n \dot\cup (L_m \to E_m)$ is the digraph X for which

- $E_n \dot\cup L_m \sqsubseteq X$ (using Lemma 13),
- $L_m \to E_m \sqsubseteq X$ (using Lemma 16),
- the second digraph of Fig. 5, without the dashed edges, is not a substructure,
- $E_{n+1} \dot\cup L_m \not\sqsubseteq X$ (E_{n+1} is just the cover of E_n in $\{E_n\}$),
- on two vertices, the only subgraph having a non-loop edge is $L_\to$,
- the maximal loop-full subgraph of X is L_m, and
- the maximal loop-free subgraph of X is of the form E_i.

The E_i of the last condition is E_{n+m}. $\square$

Lemma 18 *The following relation is definable:*

$$\{(G, F) : G \text{ and } F \text{ have the same number of vertices}\}. \quad (1)$$

Proof We "determine" the number of vertices for the loop-full and the loop-free parts of the graphs separately and add them using Lemma 17. Let G_1 denote the loop-full part of G, and G_2 denote the loop-free part. Let X denote the digraph with the following properties:

- The loop-full part of X is G_1, and the loop-free part is E_i for some i.
- On two points, the only substructure having exactly one loop and at least one non-loop edge is $L_\to$.
- $G_1 \dot\cup E_1 \not\sqsubseteq X$, and $E_i \dot\cup L_1 \not\sqsubseteq X$.
- Just as in the proof of Lemma 16, no digraph of the 6 digraphs of the first two pictures of Fig. 5 is a substructure. (No matter, we wouldn't even need all 6 in this case.)

Observe that in X, the edges $L_{\to}$ constitute a bijection between G_1 and E_i, consequently i in the first condition is $|V(G_1)|$.

Now we proceed analogously for the loop-free part, G_2. We do not write all the conditions down again, as they are just the ones above converted with the automorphism l. This way, we get L_j with $j = |V(G_2)|$. We already have E_i and L_j defined, such that $i + j = |V(G)|$. To conclude, we use the relation of Lemma 16 to get E_j and Lemma 17 to obtain the desired E_{i+j}, marking the number of vertices of G.

Finally, $(G,F) \in (1)$ holds if and only if, by doing the same, we get the same $E_{i'+j'}$ marking the number of vertices. □

We define some more arithmetic in the following lemma, namely multiplication.

Lemma 19 *The following relation is definable:*

$$\{(E_n, E_m, E_{nm}) : n, m \geq 1\}.$$

Proof The relation $\{(E_i, F_i) : i = 1, 2, \dots\}$ is definable as, beyond (E_1, F_1), for $i > 1$, F_i is the only digraph having the same vertices as E_i that has only F_2 as a two element substructure. Let X be a digraph that is maximal with the following properties:

1. $E_1 \not\sqsubseteq X$ to ensure that the relation $E(X)$ is reflexive.
2. $l(I_2) \not\sqsubseteq X$ to ensure that the relation $E(X)$ is symmetric.
3. The digraph of Fig. 6 is not a substructure of X to ensure that the relation $E(X)$ is transitive.
4. L_n is the maximal L_i subgraph.
5. F_m is the maximal F_i subgraph.

The conditions 1-3 force $E(X)$ to be an equivalence. Condition 4 tells the equivalence has at most n classes and condition 5 requires the classes to have at most m elements. It is easy to see that such an equivalence relation has a base set of at most nm elements, hence $|V(X)| = nm$. Thus, we can finish with Lemma 18. □

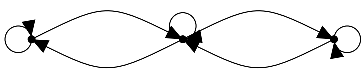


Fig. 6

Lemma 20 *Disjoint union of IO graphs is definable, i.e. the following relation is definable:*

$$\{(G_1, G_2, G_1 \dot\cup G_2) : G_1, G_2 \in IO\}.$$

Proof Using G_1 and G_2, we want to define

$$G_1 \dot\cup (l(G_2) \to G_2), \tag{2}$$

whose loop-free part is the sought $G_1 \dot\cup G_2$. For this, let X satisfy the following conditions.

- $|V(X)| = |V(G_1)| + 2|V(G_2)|$ (using Lemmas 18 and 17).
- $G_1 \dot{\cup} l(G_2) \sqsubseteq X$.
- $l(G_2) \to G_2 \sqsubseteq X$.

It easy to see that these three conditions ensure that (2) is embeddable (not substructure!) into X: there can be edges between the subgraphs G_1 and G_2 which we need to exclude. If there is an edge from G_2 to G_1 (in this particular direction), then the first graph of Fig. 7 is a substructure, without the dashed edges. Analogously, if an edge goes from G_1 to G_2, then the second digraph of Fig. 7 is a substructure, without the dashed edges. Thus we need to exclude these two subgraphs. Let Y satisfy the following conditions.

- $|V(Y)| = |V(G_2)| + 2$, and $Y \sqsupseteq l(G_2)$.
- I_2 and $L_\to$ are substructures of Y.
- The digraph of Fig. 8 is not a substructure of Y.

These three conditions does not define the two digraphs of Fig. 7 without the dashed edges, they rather define the set of those with the dashed edges meant as possibilities, as usual. However none of the dashed edges can actually appear in our X so by excluding all such, we do not do more than by excluding only the two without the dashed edges. Finally, (2) is the loop-free part of X . □

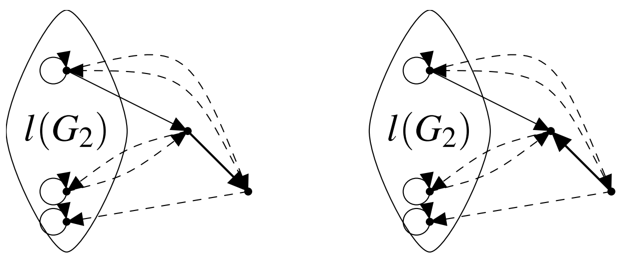


Fig. 7

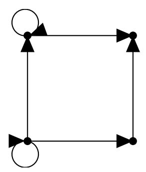


Fig. 8

Lemma 21 *The following set is definable.*

$$\{G:\ G \text{ is a disjoint union of circles of different sizes}\}.$$

Proof The set of digraphs that are disjoint unions of circles contains those *IO* graphs that have unique upper-covers (in the set IO). In this set, the digraphs of the form $O_i \dot\cup O_i$ are those that have a unique circle substructure O_i and have twice as many vertices as O_i. We have defined two sets of digraphs, the set of the lemma is just the set of those digraphs of the first set that have no substructures from the second. □

Lemma 22 *The following relation is definable.*

$$\{(O^*, G \dot\cup O^*) : G \in \mathscr{D} \text{ and } O^* \text{ is a disjoint union of } |V(G)|\text{-many circles of different sizes such that the smallest has at least } |V(G)|+1 \text{ vertices}\}. \tag{3}$$

Proof First, we define a relation counting the number of circles in O^*, actually we formulate it without the restriction on the sizes of the circles:

$$\{(E_i, O) : O \text{ is a disjoint union of } i \text{ circles}\}. \tag{4}$$

The set of O's of this relation was defined in the first sentence of the proof of Lemma 21. Let O' denote such a substructure of O that has no circle in it and has a maximal number of vertices with this property. Then $i + |V(O')| = |V(O)|$ holds for the i of (4), thus we can conclude with the addition relation defined earlier.

Let O^* be an element of the set defined in Lemma 21 and i be the number of its circles. Let X satisfy:

- $|V(X)| = |V(O^*)| + i$.
- The smallest circle in O^* has at least $i+1$ vertices.
- $O^* \sqsubseteq X$.
- X does not have a substructure Y for which
  - $|V(Y)| = |V(O^*)| + 1$, and $Y \sqsupseteq O^*$,
  - Y is loop-free, and
  - Y is not an *IO*-graph.
- X does not have a substructure Y for which
  - $|V(Y)| = |V(O^*)| + 1$, and $Y \sqsupseteq O^*$,
  - Y has a loop in it, and
  - Y has one of $L_\rightarrow$ or $t(L_\rightarrow)$ or $c(L_1 \dot\cup E_1)$ as a substructure.

With these properties, X is of the required form $G \dot\cup O^*$. □

Remark 1 Let us remark here that for the smoothest continuation of the proof, we should have had $(G, G \dot\cup O^*)$ instead of $(O^*, G \dot\cup O^*)$ (with the same assumptions) in (3). The definability of this, however, seems to be out of reach (at least for the author) at this point. That is why we proceed in the following, somewhat inelegant, way.

Lemma 23 *There exists a definable relation R for which*

$$\begin{aligned} &\{(G, O^*, G \dot\cup O^*) : (O^*, G \dot\cup O^*) \in (3)\} \subseteq R \subseteq \\ &\{(G, O^*, G_+ \dot\cup O^*) : (O^*, G_+ \dot\cup O^*) \in (3),\ |V(G)| = |V(G_+)|,\ G \leq G_+\}. \end{aligned} \tag{5}$$

Observe that the last condition in the formula, $G \leq G_+$, has embeddability (not substructureness) in it.

Proof We define a sufficient R as a set of triples (G,O^*,X) for which the following hold.

1. $(O^*,X) \in (3)$.
2. $|V(X)| = |V(O^*)| + |V(G)|$.
3. $G \sqsubseteq X$.
4. Let $G_{IO}^{\max}$ be an IO-substructure of G that has a maximal number of vertices. Note that this implies $\sqsubseteq$-maximality as well. We require $G_{IO}^{\max} \dot\cup O^* \sqsubseteq X$ (with Lemma 20).

First off, the left-side containment of (5) is clear. The right-side containment is less obvious. Let G_{IO}^{w} denote the subgraph of G that consist of those weakly connected components of G that are IO-graphs, and let G' denote "the rest" ($G = G_{IO}^{\mathrm{w}} \dot\cup G'$). At first glance, it might look like if condition 3 was enough to force $X = G \dot\cup O^*$. Unfortunately, this is not the case though, as condition 3 is not able to force G_{IO}^{w} outside O^*, because $G_{IO}^{\mathrm{w}} \sqsubseteq O^*$ is possible. On the other hand, $G' \dot\cup O^* \sqsubseteq X$ is ensured by condition 3, as O^* can only have IO-graph substructures. It is not hard to see that the last condition makes up for the deficiency we just saw, i. e. it "forces G_{IO}^{w} out of O^*". However, $X = G \dot\cup O^*$ is still not necessary as there can be "unwanted" edges between G_{IO}^{w} and G' in X, but the right-side containment of (5) lets this happen. □

Some technical tools follow. We introduce digraphs that we denote using the symbol ♂. The motivation is the shape of the digraphs, as usual. Note, that the same notations were used in the papers [5, 6] in a slightly different way.

Definition 24 Let $V(O_n) = \{v_1,\dots,v_n\}$ and let us define two digraphs with

$$V(♂_n) := V(O_n) \cup \{u_1,u_2\},\ E(♂_n) := E(O_n) \cup \{(v_1,u_1),(u_1,u_2)\}, \text{ and}$$

$$V(♂_n^L) := V(♂_n),\ E(♂_n^L) := E(♂_n) \cup \{(u_2,u_2)\}.$$

Now let m be a different positive integer from n and define $♂_m$ and $♂_m^L$ analogously with $V(♂_m) = V(♂_m^L) = \{v_1',\dots,v_m',u_1',u_2'\}$.

Now we are going to deal with pairs of the digraphs just defined, which leaves us $4 = 2 \times 2$ cases with respect to the presence of the loops. To avoid the tiresomeness of listing all 4 possibilities all the time, we resort to the following notation. We say, let $(\square,\triangledown) \in \{\emptyset,L\}^2$, and for example, in the case $(\square,\triangledown) = (L,\emptyset)$, we mean $(♂_n^L,♂_m)$ by $(♂_n^\square,♂_m^\triangledown)$, naturally.

Let $(\square,\triangledown) \in \{\emptyset,L\}^2$. We introduce two more types of digraphs with

$$V(♂_n^\square \to ♂_m^\triangledown) := V(♂_n) \cup V(♂_m),\ E(♂_n^\square \to ♂_m^\triangledown) := E(♂_n^\square) \cup E(♂_m^\triangledown) \cup \{(u_2,u_2')\}, \text{and}$$

$$V(♂_n^\square \leftrightarrow ♂_m^\triangledown) := V(♂_n) \cup V(♂_m),\ E(♂_n^\square \leftrightarrow ♂_m^\triangledown) := E(♂_n^\square \to ♂_m^\triangledown) \cup \{(u_2',u_2)\}.$$

Lemma 25 *The following relation is definable for all* $(\square,\triangledown) \in \{\emptyset,L\}^2$.

$$\{(E_i,E_j,♂_i^\square,♂_i^\square \dot\cup ♂_j^\triangledown,♂_i^\square \to ♂_j^\triangledown,♂_i^\square \leftrightarrow ♂_j^\triangledown) : i,j > 3, i \neq j\} \qquad (6)$$

The proof is put in the last section for its technical nature.

The following definition is not a technicality any more as it is a construction of great importance in the remaining half of the proof.

Definition 26 Let G be a digraph on n vertices with $V(G)=\{v_1,\dots,v_n\}$, and let $(O^*, G\dot\cup O^*)\in(3)$ with $V(O^*)=\{u_i^j : 1\le j\le n, 1\le i\le i_j\}$ such that the mth circle O_{k_m} of O^* consists of the vertices $\{u_i^m : 1\le i\le i_m\}$. Let $C(O^*)=\{O_{k_1},\dots,O_{k_n}\}$ denote the set of the circles of O^* and let $\alpha : C(O^*)\to V(G)$ be a bijective map. We introduce the notation $G\overset{\alpha}{\leftarrow}O^*$ for the digraph with

$$V(G\overset{\alpha}{\leftarrow}O^*)=V(G\dot\cup O^*)\cup\{w_1,\dots,w_n\},\ \text{and}$$
$$E(G\overset{\alpha}{\leftarrow}O^*)=E(G\dot\cup O^*)\cup\{(u_1^j,w_j):1\le j\le n\}\cup\{(w_j,\alpha(O_{k_j})):1\le j\le n\}.$$

Lemma 27 *The following relation is definable.*

$$\{(O^*, G\dot\cup O^*, G\overset{\alpha}{\leftarrow}O^*) : (O^*, G\dot\cup O^*)\in(3),\ \alpha : C(O^*)\to V(G)\}. \tag{7}$$

Proof As we already defined (3), we only need to define the digraphs $G\overset{\alpha}{\leftarrow}O^*$ (using O^* and $G\dot\cup O^*$). The relation of the lemma consists of those triples $(O^*, G\dot\cup O^*, X)$ for which:

- Let $V(G\dot\cup O^*)=V(O^*)+n$. Then $V(X)=V(G\dot\cup O^*)+n$.
- $G\dot\cup O^*\sqsubseteq X$.
- $O_i\le O^*$ implies $♂_i\sqsubseteq X$ or $♂_i^L\sqsubseteq X$.
- $O_i,O_j\le O^*$ $(i\ne j)$ implies $♂_i^\square\dot\cup ♂_j^\triangledown\sqsubseteq X$, or $♂_i^\square\to ♂_j^\triangledown\sqsubseteq X$, or $♂_j^\square\to ♂_i^\triangledown\sqsubseteq X$, or $♂_i^\square\leftrightarrow ♂_j^\triangledown\sqsubseteq X$ for some $(\square,\triangledown)\in\{\emptyset,L\}^2$.

□

In the following definition, we introduce the soul of our proof: the edge-supporting construction. Before starting to study the long definition, it is worth to read the simplified idea of it, back at the beginning of this section.

Definition 28 In this definition, we introduce the *edge-supporting construction*. Let G be a digraph with

$$V(G)=\{v_1,\dots,v_n\},\ \text{and}\ E(G)=\{e_1,\dots,e_k\}.$$

Note that $k\le n^2$ is necessary. Let p_1 and p_2 be two maps from $E(G)$ to $\{v_1,\dots,v_n\}$ defined by the rule

$$\forall e\in E(G) : e=(v_{p_1(e)},v_{p_2(e)}).$$

Let us introduce a digraph G_s with

$$V(G_s):=V(G)\cup\{v_1^s,\dots,v_k^s\},\ \text{and}\ E(G_s):=E(G)\cup\bigcup_{i=1}^{k}\{(v_{p_1(e_i)},v_i^s),(v_i^s,v_{p_2(e_i)})\}.$$

Let

$$O^*=O_{l_1}\dot\cup O_{l_2}\dot\cup\cdots\dot\cup O_{l_n}\ \text{such that}\ n^2+n<l_1<l_2<\cdots<l_n.$$

Let D_s be a set of integers with

$$|D_s|=k(=|E(G)|),\ \text{and}\ x\in D_s\Rightarrow x>l_n. \tag{8}$$

Let s be a bijective map from D_s, satisfying (8), to $\{v^s_1, \ldots, v^s_k\}$. Let

$$O^*_s := O^* \,\dot\cup \bigcup_{x \in D_s}^{\cdot} O_x \;\text{ with }\; V(O^*_s) = \{u^j_i : j \in \{l_1, \ldots, l_n\} \cup D_s,\ 1 \le i \le j\}.$$

Let $\alpha : C(O^*) \to V(G)$ be a bijective map. We define the digraph $(G \overset{\alpha}{\leftarrow} O^*)_s$ by

$$(G \overset{\alpha}{\leftarrow} O^*)_s := G_s \overset{\beta}{\leftarrow} O^*_s, \text{ where } \beta|_{C(O^*)} := \alpha,\ \beta|_{\{O_x : x \in D_s\}} := \{(O_x, s(x)) : x \in D_s\},$$

and say its an *edge-supporting digraph for G*.

Remark 2 Note that the definition of the edge-supporting digraphs includes a condition for the size of the circles of O^*. That condition is very important here, and was not present in (7). We need to be cautious about this later on.

Lemma 29 *The following relation is definable.*

$$\{(O^*, G \overset{\alpha}{\leftarrow} O^*, (G \overset{\alpha}{\leftarrow} O^*)_s) : (G \overset{\alpha}{\leftarrow} O^*)_s \textit{ is an edge-supporting digraph for } G\} \quad (9)$$

Proof The relation in question consists of those triples (X,Y,Z) for which the highlighted conditions hold. There are explanations inserted between the conditions.

– There exists a triple $(X,W,Y) \in (7)$, meaning (X,Y) is of the form $(O^*, G \overset{\alpha}{\leftarrow} O^*)$.

Thus, instead of (X,Y), we use $(O^*, G \overset{\alpha}{\leftarrow} O^*)$ from now on in the proof. To ensure the structure of O^* (see Remark 2), first, we determine the number of vertices of G with

– $|V(O^*)| + 2n = |V(G \overset{\alpha}{\leftarrow} O^*)|$,

meaning G has n vertices. Now we are ready to shape O^*.

– $O_i \sqsubseteq O^*$ implies $i > n^2 + n$.

We turn to defining Z of the triple we started with.

– There exists a triple $(W_1, W_2, Z) \in (7)$, meaning (W_1, Z) is of the form $(O^*_s, G_s \overset{\beta}{\leftarrow} O^*_s)$.

At this point, O^*_s, G_s, and β are just notations yet, we need additional conditions to make them be like in Definition 28.

– $O^* \sqsubseteq O^*_s$,
– $O_i \sqsubseteq O^*_s$ implies $i \ge l_1$, where l_1 is the size of the smallest circle of O^*, as before.
– $G \overset{\alpha}{\leftarrow} O^* \sqsubseteq Z$.
– If $O_i \sqsubseteq O^*$ and $\male^L_i \sqsubseteq Z$, then there exists $k > l_n$ for which $\male^L_i \to \male_k \sqsubseteq Z$ and $\male_k \to \male^L_i \sqsubseteq Z$ both hold. Additionally, if l is different from i,k, and $O_l \sqsubseteq O^*_s$, then there exists $\diamond \in \{\emptyset, L\}$ for which $\male_k \,\dot\cup\, \male^\diamond_l$ holds.
– If $O_i, O_j \sqsubseteq O^*$, $i \neq j$, and $\male^\square_i \to \male^\triangledown_j \sqsubseteq Z$ with $(\square, \triangledown) \in \{\emptyset, L\}^2$, then there exists $k > l_n$ for which $\male^\square_i \to \male_k \sqsubseteq Z$ and $\male_k \to \male^\triangledown_j \sqsubseteq Z$ both hold. Additionally, if l is different from i,j,k, and $O_l \sqsubseteq O^*_s$, then there exists $\diamond \in \{\emptyset, L\}$ for which $\male_k \,\dot\cup\, \male^\diamond_l$ holds.

- If $O_i, O_j \sqsubseteq O^*$, $i \neq j$, and $♂_i^{\square} \leftrightarrow ♂_j^{\triangledown} \sqsubseteq Z$ with some $(\square, \triangledown) \in \{\emptyset, L\}^2$, then there exist two different $k_1, k_2 > l_n$ for which all of

$$♂_i^{\square} \to ♂_{k_1},\ ♂_{k_1} \to ♂_j^{\triangledown},\ ♂_j^{\triangledown} \to ♂_{k_2},\ \text{and}\ ♂_{k_2} \to ♂_i^{\square}$$

  are substructures of Z. Additionally, if l is different from i, j, k_i, and $O_l \sqsubseteq O_s^*$, then there exists $\diamond \in \{\emptyset, L\}$ for which $♂_{k_i} \dot\cup ♂_l^{\diamond}$ holds for $i = 1, 2$.
- If $O_k \sqsubseteq O_s^*$ and $k > l_n$, then k is one of the ks or k_is of the previous three conditions.

It is not hard to see that these conditions provide the structure we need. $\square$

We can now handle the problem described in Remark 1. The next lemma does just that.

Lemma 30 *The following relation is definable.*

$$\{(G, O^*, G \overset{\alpha}{\leftarrow} O^*) : \text{ the circles of } O^* \text{ have more than } |V(G)|^2 + |V(G)| \text{ vertices.}\} \tag{10}$$

Proof It is sufficient to define the relation

$$\{(G, O^*, G \dot\cup O^*) : \text{ the circles of } O^* \text{ have more than } |V(G)|^2 + |V(G)| \text{ vertices.}\}$$

as if we have this, we can easily finish the proof with (7). We start with the R of Lemma 23. For a pair (G, O^*), we need to find the triple $(G, O^*, G_+ \dot\cup O^*)$ of R such that G_+ has the least possible number of edges. With (7) and (9), the relation

$$\{(G_+ \dot\cup O^*, (G_+ \overset{\alpha}{\leftarrow} O^*)_s) : (G_+ \overset{\alpha}{\leftarrow} O^*)_s \text{ is an edge-supporting digraph for } G_+\}$$

is defined easily. To conclude, pick a pair from this relation whose second component has a least number of vertices possible. The first element of this pair is $G \dot\cup O^*$. $\square$

We are finally ready to prove our main theorem.

Proof (of Theorem 1) With (10) and (9) one can easily define the relation

$$\{(G, O^*, (G \overset{\alpha}{\leftarrow} O^*)_s) : (G \overset{\alpha}{\leftarrow} O^*)_s \text{ is an edge-supporting digraph for } G\}.$$

Fix a triple $(G, O^*, (G \overset{\alpha}{\leftarrow} O^*)_s)$ of this relation and let n be the number of vertices of G. We need to show that the set of digraphs embeddable into G is definable. Let $X \sqsubseteq (G \overset{\alpha}{\leftarrow} O^*)_s$ and let $(G_X, O_X^*, G_X \overset{\gamma}{\leftarrow} O_X^*)$ be a triple of the relation (10) for which the following conditions hold. (We have to be careful (see Remark 2), the listed conditions do not contradict the assumption of (10).)

- $O_i \sqsubseteq O_X^*$ holds if and only if both $O_i \sqsubseteq O^*$, and $♂_i^{\square} \sqsubseteq X$ for some $\square \in \{\emptyset, L\}$ hold.
- If $O_i, O_j \sqsubseteq O_X^*$, $i \neq j$, and $(\square, \triangledown) \in \{\emptyset, L\}^2$, then
  - $♂_i^{\square} \dot\cup ♂_j^{\triangledown} \sqsubseteq G_X \overset{\gamma}{\leftarrow} O_X^*$ holds if and only if one of the following three holds:
    - $♂_i^{\square} \dot\cup ♂_j^{\triangledown} \sqsubseteq X$, or

- $\sigma_i^{\square} \to \sigma_j^{\triangledown} \sqsubseteq X$, but the edge is not supported in X, i. e. there exists no $k > l_n$ (where l_n is the size of the largest circle of O^*, as before) for which $\sigma_i^{\square} \to \sigma_k \sqsubseteq X$ and $\sigma_k \to \sigma_j^{\triangledown} \sqsubseteq X$ both hold, or
- $\sigma_i^{\square} \leftrightarrow \sigma_j^{\triangledown} \sqsubseteq X$, but none of the two edges is supported in X.

– $\sigma_i^{\square} \to \sigma_j^{\triangledown} \sqsubseteq G_X \xleftarrow{\gamma} O_X^*$ holds if and only if one of the following two holds
  - $\sigma_i^{\square} \to \sigma_j^{\triangledown} \sqsubseteq X$, and the edge is supported in X, or
  - $\sigma_i^{\square} \leftrightarrow \sigma_j^{\triangledown} \sqsubseteq X$, but only the "$i \to j$" edge is supported in X.

– $\sigma_i^{\square} \leftrightarrow \sigma_j^{\triangledown} \sqsubseteq G_X \xleftarrow{\gamma} O_X^*$ holds if and only if $\sigma_i^{\square} \leftrightarrow \sigma_j^{\triangledown} \sqsubseteq X$ and both edges are supported in X.

It is clear that $G_X \leq G$ and all embeddable digraphs can be obtained this way. □

4 The remaining technicalities

Definition 31 The sum of the number of (both in- and out-)edges for a vertex, not counting the loops, is called the *loop-free degree* of the vertex.

Lemma 32 *Let $0 \leq p$ and $1 \leq q$ be two fixed integers. We can define, with finitely many constants added to $(\mathscr{D}, \sqsubseteq)$, the set of digraphs that contain at most p many vertices with loop-free degree at least q each.*

Before the easy proof, note that we can only use this lemma if we have a fixed constant, say $K = 4$, for the whole paper, such that all usage of the lemma restricts to $p, q \leq K$. Otherwise there would be no guarantee we are using finitely many constants at all. Fortunately, $K = 4$ will just do for the whole paper.

Proof Observe that the digraph G has more than p many vertices with at least q loop-free degree each, if and only if it has an at most $(p+1)q$ element "certificate" substructure with the same property. Hence, by forbidding all those (finitely many) certificates, we define the set we need. □

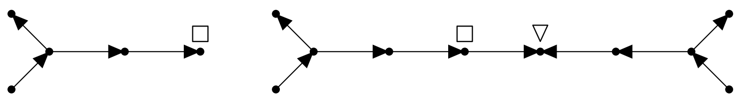


Fig. 9

Proof (of Lemma 25) Let us consider E_i and E_j given. We define the other components of the relation.

We start with $\sigma_i^{\square}$ which is just the digraph X for which

– $|V(X)| = i + 2$.
– $O_i \sqsubseteq X$.

- We use Lemma 32 with $p=1$, and $q=3$, i. e. X has at most one vertex with loop-free degree at least 3.
- We use Lemma 32 with $p=0$, and $q=4$ as well.
- The first digraph of Fig. 9 is a substructure. The $\square$ symbol is understood naturally, if $\square = L$, then there is a loop there, if $\square = \emptyset$, then there is not.
- Depending on $\square$,
  - if $\square = \emptyset$, then $O_i \dot\cup E_1 \sqsubseteq X$, that is the only cover of O_i among the IO-graphs,
  - if $\square = L$, then $O_i \dot\cup L_1 \sqsubseteq X$, that is definable with Lemma 13.

We now start to deal with $\male_i^{\square} \dot\cup \male_j^{\triangledown}$. $O_i \dot\cup O_j$ is the digraph with $i+j$ vertices that is a disjoint union of circles and both O_i and O_j are substructures. $\male_i^{\square} \dot\cup \male_j^{\triangledown}$ is the digraph X for which

- $|V(X)| = |V(\male_i^{\square})| + |V(\male_j^{\triangledown})|$.
- $\male_i^{\square} \sqsubseteq X$, and $\male_j^{\triangledown} \sqsubseteq X$.
- We use Lemma 32 with $p=2$, $q=3$ and with $p=0$, $q=4$.
- Depending on $(\square, \triangledown)$,
  - if $(\square, \triangledown) = (\emptyset, \emptyset)$, then $O_i \dot\cup O_j \dot\cup E_2 \sqsubseteq X$, which is just the digraph Y for which
    - $|V(Y)| = i+j+2$, and $O_i \dot\cup O_j \sqsubseteq X$,
    - Y has the maximal substructure E_k (among the ones with the previous property).
  - if $(\square, \triangledown) = (L, \emptyset)$ or $(\emptyset, L)$, then $O_i \dot\cup O_j \dot\cup E_1 \dot\cup L_1 \sqsubseteq X$, which is just the digraph Y for which
    - $|V(Y)| = i+j+2$, and $O_i \dot\cup O_j \sqsubseteq X$,
    - $O_i \dot\cup O_j \dot\cup E_1$, which is the only IO-graph cover of $O_i \dot\cup O_j$, is a subgraph,
    - $O_i \dot\cup O_j \dot\cup L_1$ is a subgraph, and
    - on two elements, there is no subgraph with both a loop and a loop-free edge.
  - if $(\square, \triangledown) = (L, L)$ then $O_i \dot\cup O_j \dot\cup L_2 \sqsubseteq X$.

Now we turn to $\male_i^{\square} \to \male_j^{\triangledown}$, which is just the digraph X for which

- $|V(X)| = |V(\male_i^{\square})| + |V(\male_j^{\triangledown})|$.
- $\male_i^{\square} \sqsubseteq X$, and $\male_j^{\triangledown} \sqsubseteq X$.
- We use Lemma 32 with $p=2$, $q=3$ and with $p=0$, $q=4$.
- The second digraph of Fig. 9 is substructure of X.

Finally, $\male_i^{\square} \leftrightarrow \male_j^{\triangledown}$ is defined with the analogues of the conditions for $\male_i^{\square} \to \male_j^{\triangledown}$. $\square$

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