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On Geodesics of Some Special Riemannian Spaces on $\mathbb{R}_0^n \times \mathbb{R}^1$

Doctoral Dissertation

Szeged, 1989

Contents

I. Preface

In this paper we will investigate the properties of geodesics in the case of some special Riemannian manifolds with interesting features.

$\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$

The sets of points of our Riemannian manifolds are the spaces $\mathbb{R}^n_0 \times \mathbb{R}^1$ parametrized on the natural way. The metric is given by the following equality

$$g_{(\, a,\alpha\,)}(\,(\, x,\xi\,)\, ,\, (\, y,\eta\,)\,) = <x,y> + \xi\cdot\eta\cdot U(\, |\, a\, |^2),$$

where $U:\mathbb{R}\longrightarrow\mathbb{R}$ is a smooth mapping, $<.\,,.>$ is the canonical inner product of $\mathbb{R}^n_0$, $a,x,y\in\mathbb{R}^n_0$ and α,ξ,η are arbitrary numbers in the space of real numbers. As we will show out, the most interesting peculiar feature in this space is the following one: the projection of geodesics onto $\mathbb{R}^n_0$ along $\mathbb{R}^1$ is a trajectory of a moving particle in a central symmetric force field with potential $h\,/\,U(\,|\,a\,|^2)$, where h is a suitable constant.

We turn to the details in three cases:

i) $U(\,x\,) = constant$ (trivial case)

ii) $U(\,x\,) = c/x$ (c constant)

iii) $U(\,x\,) = c\cdot\sqrt{x}$ (c constant).

As a result of the considerations it will be turned out that the equations of the geodesics are integrable in all the three cases. We have calculated equations of the geodesics in an explicit form. By the help of these expressions we have given necessary and sufficient conditions for the purpose to determine the shape of projection of geodesics onto $\mathbb{R}^n_0$ along $\mathbb{R}^1$. For the end we have constructed some pictures with the aid of computer program *PHASER* to illustrate our results.

I would like to thank *Dr.Nagy Péter* for proposing this problem on geodesics and making valuable suggestions on the form and content of this dissertation. I also thank his colleagues for their decisive help.

II. Preliminaries

In this section we are going to give a brief introduction to the concepts belonging to the theory of Riemannian manifolds which are considerable importance in our further investigations. We deal only with treat of the most necessary objects as manifold, tangent space, connection, Riemannian metric, Levi–Civita connection and geodesics. The aim of this chapter is to make our terminology quite clear.

During the preparation of these introductory sections we follow the treatment of the basic chapters of Helgason's excellent book [22]. However, in some details we apply ideas differ from Helgason's ones as e.g. in proving the minimality property of geodesics.

2.1. Riemannian manifolds

Definition. *A Hausdorff space is said to be n–dimensional topological manifold if it has countable base and all the points of it have a neighborhood homeomorphic to an open subset in $\mathbb{R}^n$.*

Let $\mathcal{M}^n$ denote an n–dimensional topological manifold and $\mathcal{U} \subseteq \mathcal{M}^n$ be a neighborhood of it. The mapping $\varphi : \mathcal{U} \longrightarrow \mathbb{R}^n$ is a homeomorphism. This homeomorphism φ (coordinate–mapping) with $\mathcal{U}$ (coordinate–neighborhood) is called an n–dimensional chart.

The system of the charts are named by coordinate–atlas.

Definition. *An n–dimensional topological manifold $\mathcal{M}^n$ is said to be a differentiable manifold of class C^k if for any two coordinate–mappings φ_1 and φ_2 of it the function $\varphi_1 \circ \varphi_2^{-1}$ is of class C^k (on an appropriate neighborhood).*

A map $f : \mathcal{M}^n \longrightarrow \mathcal{N}^k$ between two differentiable manifolds is said to be differentiable if for any coordinate–mappings $\varphi_{\mathcal{M}}$ of $\mathcal{M}^n$ and $\varphi_{\mathcal{N}}$ of $\mathcal{N}^k$ the function $\varphi_{\mathcal{N}} \circ f \circ \varphi_{\mathcal{M}}^{-1}$ is differentiable (on a suitable coordinate neighborhood). Such a mapping is called diffeomorphism if it is bijective on the one hand, and its inverse is also differentiable, on the other hand.

Let F be the space of the differentiable functions on $\mathcal{M}^n$. The map $D : F \longrightarrow \mathbb{R}$ is said to be a derivative at the point $P \in \mathcal{M}^n$ if it satisfies the following conditions for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in F$:

$$D(\alpha \cdot f + \beta \cdot g) = \alpha \cdot D(f) + \beta \cdot D(g),$$

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g).$$

The set of all derivatives forms a space denoted by $T_p(\mathcal{M}^n)$. This set is called the tangent space of $\mathcal{M}$ at the point P. Easy to see that the dimension of it is n. All these facts are straightforward consequences of the Taylor expansion of the function $f \circ \varphi^{-1}$, where φ is a coordinate–mapping on a neighborhood of the point P:

$$f \circ \varphi^{-1}(x_1, \dots, x_n) = f(P) + \sum_{i=1}^{n} \left[x^i - \varphi_i(P) \right] \frac{\partial (f \circ \varphi^{-1})}{\partial x_i} +$$

$$\sum_{i=1}^{n} \left[x^i - \varphi_i(P) \right] \cdot \left[x^j - \varphi_j(P) \right] \cdot g(x_1, \dots, x_n),$$

where $\varphi(P) = \left[\varphi_1(P), \varphi_1(P), \dots, \varphi_n(P) \right] = (x_1, \dots, x_n)$, and g is the remainder term. This gives, by the properties of derivative D that

$$D(f) = \sum_{i=1}^{n} D(x^i) \frac{\partial (f o \varphi^{-1})}{\partial x_i} \left[\varphi(P) \right]$$

which leads to

$$D(f) = \sum_{i=1}^{n} D(x^i) \frac{\partial}{\partial x_i} .$$

The notion of the Lie bracket is very important. It is defined by

$$[X,Y] = XoY - YoX$$

for any two elements of $T_P(\mathcal{M}^n)$. The tangent space $T_P(\mathcal{M}^n)$ is a Lie algebra with this Lie bracket.

The union of the tangent spaces $T_P(\mathcal{M}^n)$ for all points P of $\mathcal{M}^n$ is called the tangent bundle of the manifold and denoted by $T(\mathcal{M}^n)$.

The map $X : \mathcal{M}^n \longrightarrow T(\mathcal{M}^n)$ is named by vector field if $X(P) \in T_P(\mathcal{M}^n)$ for all points P on $\mathcal{M}^n$.

DEFINITION. *The map* $g : [a,b] \longrightarrow \mathcal{M}^n$ *is said to be curve in the manifold* $\mathcal{M}^n$ *if it is differentiable and injective. A derivative* D *at point* $P = g(t)$ *is the tangent vector of the curve* g *at point* P *and denoted by* $\dot{g}(t)$ *if*

$$D f = \left. \frac{d f(g(t))}{dt} \right|_{x=t}$$

for all function $f : \mathcal{M}^n \longrightarrow \mathbb{R}$. *(It is clear that* $\dot{g}(t) \in T_{g(t)}(\mathcal{M}^n)$ *since it is a derivative.)*

A differentiable manifold $\mathcal{M}^n$ is said to be a Riemannian one if there is an inner product g_P on $T_P(\mathcal{M}^n)$ at each points $P \in \mathcal{M}^n$ which is differentiable and positive definite. This tensor field g is called by Riemannian metric.

DEFINITION. *In a Riemannian manifold* $\mathcal{M}^n$ *the length of a curve* $\gamma : [a,b] \longrightarrow \mathcal{M}^n$ *is defined by*

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt.$$

In the language of tensors this means that $ds^2 = g_{i,j}\, d\gamma^i\, d\gamma^j$, where ds is the length element of the curve γ and γ^i is it's convolution with the i–th coordinate–mapping.

One speaks about indefinite Riemannian space if the inner product g is indefinite. In our spaces to be investigating this property depends on the sign of the constant *c* in *U* so we can not calculate the length of our geodesics (at least in the above sense).

2.2. Connections

Let $\mathcal{X}$ denote the set of differentiable vector fields on the differentiable manifold $\mathcal{M}^n$.

DEFINITION. *The differentiable map* $\nabla : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ *is called connection or covariant derivative if it has the following three properties:*

$$i)\quad \nabla_X(\alpha\cdot Y+\beta\cdot Z) = \alpha\cdot\nabla_X Y + \beta\cdot\nabla_X Z,$$

$$ii)\quad \nabla_{f\cdot X+g\cdot Y}Z = f\cdot\nabla_X Z + g\cdot\nabla_Y Z,$$

$$iii)\quad \nabla_X(f\cdot Y) = f\cdot\nabla_X Y + X(f)\cdot Y,$$

where $\alpha,\beta \in \mathbb{R}$, $X,Y,Z \in \mathcal{X}$, $f,g \in F$ *and* $X(f) \in F$ *defined by*

$$X(f)(P) = X(P)(f)$$

for all $P \in \mathcal{M}^n$.

These properties imply the determinant of the connection by the vector fields $\nabla_{\partial_i}\partial_j$, where $\partial_i = \dfrac{\partial}{\partial x^i}$ and $1 \le i,j \le n$ since $\left\{ \partial_i \right\}_{i=1}^{n}$ is base in $T_P(\mathcal{M}^n)$ at each points $P \in \mathcal{M}^n$.

It is obvious that we can expand the vector field $\nabla_{\partial_i}\partial_j$ in the base $\left\{\partial_i\right\}_{i=1}^n$, but this expansion gives us a very important object to the theory of connections.

DEFINITION. *The function coefficients* $\Gamma^k_{i,j} \in F$ *of the expansion*

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma^k_{i,j} \cdot \partial_k$$

are the so–called Christoffel symbols.

It is trivial that the Christoffel symbols also determine the connection since they determine the vector fields $\nabla_{\partial_i}\partial_j$. More precisely the properties *i)–iii)* of the covariant derivative (listed in the definition) and some easy straightforward calculation imply

$$\nabla_X Y = \sum_{i,j=1}^n X_i \cdot \left(\frac{\partial Y_j}{\partial x^i} \cdot \partial_j + Y_j \sum_{k=1}^n \Gamma^k_{i,j} \cdot \partial_k\right),$$

where

$$X = \sum_{k=1}^n X_k \cdot \partial_k\,,\ Y = \sum_{k=1}^n Y_k \cdot \partial_k \ \text{ and } \ X_k, Y_k \in F \ \ (1 \le k \le n).$$

This allows us to define the covariant derivative of a vector field $X \in \mathcal{X}$ with respect to a fixed vector $v \in T_P(\mathcal{M}^n)$.

Definition. *Using the above introduced notations the covariant derivative of* $X \in \mathcal{X}$ *with respect to the vector* $v \in T_P(\mathcal{M}^n)$ *is*

$$\nabla_v X = \sum_{i,j=1}^{n} v_i \cdot \left(\frac{\partial X_j}{\partial x^i} \cdot \partial_j + X_j \sum_{k=1}^{n} \Gamma^k_{i,j} \cdot \partial_k \right),$$

where $v = \sum_{k=1}^{n} v_k \cdot \partial_k$ *and all the functions and vector fields are taken at the point* $P \in \mathcal{M}^n$.

This means that $\nabla_v X$ is a vector in the tangent space at point P. From this we can define the important notion of the derivative of a vector field $X \in \mathcal{X}$ along a curve γ on such a way that the derivated vector field X' is a vector field along the curve γ.

Definition. *The derivative of a vector field* $X \in \mathcal{X}$ *along a curve*

$$\gamma : [a,b] \to M^n$$

is

$$X' : [a,b] \longrightarrow \mathbb{R}^n \quad (t \longmapsto \nabla_{\dot\gamma} X(\gamma(t)))$$

A vector field $X \in \mathcal{X}$ is said to be parallel along the curve γ if $\nabla_{\dot\gamma} X = 0$ This condition is formulated according to the analogous concept of parallel vector fields on the surfaces.

The following differential equation for parallel vector fields along the curve γ above can be easily obtain from the previously exhibited formulas by some substitution:

$$\frac{d(X_i o\gamma)}{dt} + \sum_{k,j=1}^{n} \dot{\gamma}_k \cdot X_j o\gamma \cdot \Gamma^i_{k,j} \cdot \gamma \equiv 0$$

where $1 \leq i \leq n$.

This implies the existence of a unique parallel vector field $X \in \mathcal{X}$ for any curve $\gamma : [a,b] \longrightarrow \mathcal{M}^n$ and any vector $v \in T_{\gamma(a)}(\mathcal{M}^n)$ which satisfies the condition $v = X(a)$.

This makes the introduction of parallel translation possible and shows the descriptive meaning of the connection. In this sense the following theorem is very natural.

THEOREM. Let $\tau^{\gamma}_{\Delta,t}$ denote the parallel translation along the above curve $\gamma : [a,b] \longrightarrow \mathcal{M}^n$ from $\gamma(t+\Delta)$ to $\gamma(t)$. If $X \in \mathcal{X}$ then the following relation holds

$$\nabla_{\dot{\gamma}} X(\gamma(t)) = \lim_{\Delta \longrightarrow 0} \frac{\tau^{\gamma}_{\Delta,t}(X) - X(\gamma(t))}{\Delta} .$$

We do not prove this result since it is quite clear from the above mentioned motivations.

There are two fundamental tensors given by the covariant derivative:

torsion tensor: $T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$,

Riemannian curvature tensor: $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}$

where $X,Y,Z \in \mathcal{X}$.

We note the following fact : the torsion is zero if and only if the Christoffel symbols are symmetric in their lower indices. This statement is elementary, so the proof can be omitted.

The most important fundamental theorem about the connection on a Riemannian manifold is the following. It expresses the uniqueness of the connection satisfying some condition.

THEOREM. *On a Riemannian manifold* $(\mathcal{M}^n, g)$ *there exists a unique connection* ∇ *which is torsion free and satisfies*

$$Xg(Y,Z) = g(\nabla_X Y,Z) + g(Y,\nabla_X Z)$$

for arbitrary vector fields $X,Y,Z \in \mathcal{X}$.

PROOF. From the conditions it is immediately that

$$2\cdot g(\nabla_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$
$$+ g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

Since we know the right hand side of this equality for all vector fields $Z \in \mathcal{X}$ and g is definite we can determine $\nabla_X Y$ exactly which was to be proved. ⋈

DEFINITION. *The unique connection determined in the above theorem is called Levi–Civita or Riemannian connection.*

2.3. Geodesics

Let $\mathcal{M}^n$ be a differentiable manifold with the connection ∇. A curve γ is said to be geodesic line if

$$\nabla_{\dot\gamma} \dot\gamma \equiv 0,$$

that is, its tangent vector field is parallel along the curve γ.

The arguments in the previous section imply, on an easy way, the following differential equation

$$\frac{d}{dt} \dot\gamma_i + \sum_{j,k=1}^{n} \dot\gamma_k \cdot \dot\gamma_j \cdot \Gamma^i_{k,j} \circ \gamma = 0,$$

where

$$\dot\gamma(t) = \sum_{i=1}^{n} \dot\gamma_i(t) \cdot \frac{\partial}{\partial x^i}.$$

By the Picard–Lindelöf theorem we can conclude that for any point $P \in \mathcal{M}^n$ and any vector $v \in T_P(\mathcal{M}^n)$ there exists a unique geodesic passing through the point $P \in \mathcal{M}^n$ with direction $v \in T_P(\mathcal{M}^n)$ (with the speed v).

Since the Christoffel symbols are differentiable functions, there is a neighborhood $\mathcal{U}$ of the origin in $T_P(\mathcal{M}^n)$, where the so–called exponential map *Exp* has meaning. If $v \in \mathcal{U}$, then the exponential map *Exp* takes the point of the unique geodesic through the point $P \in T_P(\mathcal{M}^n)$ with the speed $v \in T_P(\mathcal{M}^n)$, parameterized by 1. For example, if $\gamma(0) = P$ and $\dot{\gamma}(0) = v$ then $Exp(v) = \gamma(1)$.

We shall calculate this exponential map in our special spaces in the following chapter.

To end this chapter we shall prove another important peculiarity of geodesics. Let g be a Riemannian metric on $\mathcal{M}^n$ and ∇ be the Levi–Civita connection. Furthermore let $\gamma : [0,d] \longrightarrow \mathcal{M}^n$ be a curve.

DEFINITION. *A differentiable function*

$$v : [-\varepsilon,\varepsilon] \times [0,d] \to \mathcal{M}^n$$

is called the variation of the curve γ *if*

$$v(s,0) \equiv \gamma(0), \quad v(s,d) \equiv \gamma(d), \quad v(0,t) \equiv \gamma(t).$$

Obviously, the curve $v_s(t) = v(s,t)$ joins the points $\gamma(0)$ and $\gamma(d)$. Let $L(s)$ be the length of the curve $v_s(t)$ then we have the following theorem.

THEOREM. *A curve between two points is extremal with respect to its length if and only if it is a geodesic line.*

PROOF. It is enough to see that $L'(0) = 0$ for any variation v if and only if for this curve: $\nabla_{\dot\gamma} \dot\gamma \equiv 0$.

For the sake of simplicity let γ be parametrised by its arclength. Then

$$L'(0) = \frac{d}{ds} \int_0^d \sqrt{g_{v_s(t)}(\dot v_s(t), \dot v_s(t))}\, dt \bigg|_{s=0}$$

$$= \int_0^d \frac{1}{2} \cdot \frac{d}{ds} \left[g_{v_s(t)}(\dot v_s(t), \dot v_s(t)) \right] \bigg|_{s=0} dt$$

$$= \int_0^d g_{v_0(t)}\left(\frac{d}{ds} \dot v_s(t) \bigg|_{s=0}, \dot v_0(t) \right) dt$$

$$= \int_0^d g_{\gamma(t)}\left(\frac{d}{ds} \frac{d}{dt} v(0,t), \frac{d}{dt}\gamma(t) \right) dt$$

$$= \int_0^d \frac{d}{dt} g_{\gamma(t)}\left(\frac{d}{ds} v(0,t), \frac{d}{dt}\gamma(t) \right) dt - \int_0^d g_{\gamma(t)}\left(\frac{d}{ds} v(0,t), \frac{d^2}{dt^2}\gamma(t) \right) dt$$

$$= \int_0^d g_{\gamma(t)}\left(\frac{d}{ds} v(0,t), \nabla_{\dot\gamma} \dot\gamma \right) dt.$$

Since the function $\frac{d}{ds}v(\,0,.\,)$ can be arbitrarily choosen, $L'(\,0\,) = 0$ implies that $\nabla_{\dot\gamma}\dot\gamma = 0$. At the same time $\nabla_{\dot\gamma}\dot\gamma = 0$ implies obviously that $L'(0) = 0$,which was to be proved.

Chapter B

III. Geodesics on $\mathbb{R}^n_0 \times \mathbb{R}^1$ equipped a special metric

In the following we investigate the geodesics in some special Riemannian manifolds defined on $\mathbb{R}^n_0 \times \mathbb{R}^1$, where $\mathbb{R}^n_0 = \mathbb{R}^n - \{ 0 \}$, whose metric is defined by the following properties:

i) The projection onto $\mathbb{R}^n_0$ along $\mathbb{R}^1$ of this Riemannian scalar product is the canonical Euclidean one.

ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}^n_0$.

iii) The projection $\mathbb{R}^1$ along $\mathbb{R}^n_0$ of the scalar product at $(a, \alpha) \in \mathbb{R}^n_0 \times \mathbb{R}^1$ is the canonical one multiplied by $U(| a |^2)$, where $U : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is smooth.

These properties determine uniquely the scalar product of vectors

$$(X,\xi) , (Y,\eta) \in T_{(a,\alpha)}(\mathbb{R}^n_0 \times \mathbb{R}^1)$$

and it can be written in the form

(1) $$g_{(a,\alpha)}((X,\xi),(Y,\eta)) = < X,Y > + \xi \cdot \eta \cdot U(| a |^2) ,$$

where $< X,Y > = \sum_{i=1}^{n} X_i \cdot Y_i$. For the sake of simplicity we shall write

$$< (X,\xi),(Y,\eta) >_* = g_{(a,\alpha)}((X,\xi),(Y,\eta)) ,$$

which simplification will not make any confusion since we know every time which point the tangent vector belongs to. We will regard α in (a,α) like the (n+1)–th coordinate.

3.1. General results about the geodesics, first case

One of our basic results is formulated in the following theorem.

THEOREM. *The Levi–Civita connection of the Riemannian metric* (1) *introduced above has the following Christoffel symbols:*

$$\Gamma^{k}_{i,j}(a,a) = \left\{ \begin{array}{ccl} 0 & \text{if} & i,j,k \le n \\ 0 & \text{if} & i,j \le n,\ k=n+1 \\ 0 & \text{if} & i,k \le n,\ j=n+1 \\ 0 & \text{if} & j,k \le n,\ i=n+1 \\ -\partial_k(U(r))/2 & \text{if} & k \le n,\ i,j=n+1 \\ \partial_i(U(r))/2U(r) & \text{if} & j,k=n+1 \\ \partial_j(U(r))/2U(r) & \text{if} & i,k=n+1 \\ 0 & \text{if} & i,j,k=n+1 \end{array} \right\},$$

where $1 \le i,j,k \le n+1$, $r = \langle a,a \rangle$ *and* ∂_s *is the derivative with respect to the s–th coordinate.*

PROOF. The Levi–Civita connection is torsion free thus we have $\Gamma^{k}_{i,j} = \Gamma^{k}_{j,i}$ (the symmetry of Christoffel symbols). The other defining equation for this connection is

$$(T,\tau)\left[g_{(a,a)}(\,(X,\xi),(Y,\eta)\,)\right] = <\nabla_{(T,\tau)}(X,\xi),(Y,\eta)>_{*} + <(X,\xi),\nabla_{(T,\tau)}(Y,\eta)>_{*},$$

where (T,τ) is a tangent vector. Let $\{ E_i \}_{i=1}^{n}$ be an orthonormal base in the tangent space, and

$$\partial_i = \begin{cases} (\,E_i,0\,)\ .\text{if}\ \ 1 \leq i \leq n \\ (\,0,1\,)\ \ \ \text{if}\ \ i=n+1 \end{cases}.$$

We obtain that

$$\partial_i g_{(a,a)}(\partial_j,\partial_k) = \sum_{s=1}^{n+1} \left[\Gamma^{s}_{i,j}((a,a))\cdot<\partial_s,\partial_k>_{*} + \Gamma^{s}_{i,k}((a,a))\cdot<\partial_j,\partial_s>_{*} \right],$$

where $1 \leq i,j,k \leq n+1$. The simple way on which this system can be solved is presented in the pattern below.

In the first column we have written the delimited cases according to values of indices which were just investigating. In the second column are the equations corresponding to the indices in the first column.

In the third column can be founded the solution of the respected equation in the second column. The solutions of the equations in the third and fourth rows are obtained as a solution of a system of linear (for the Γ –s) equations.

CASE	EQUATION	RESOLUTION
$j=k=n+1$	$\partial_i U(r)=2\cdot\Gamma^{n+1}_{i,n+1}\cdot U(r)$	$\Gamma^{n+1}_{i,n+1}=\frac{\partial_i U(r)}{2U(r)}$
$i=k=n+1,\ j\leq n$	$\Gamma^{n+1}_{j,n+1}\cdot U(r)=-\Gamma^{j}_{n+1,n+1}$	$\Gamma^{j}_{n+1,n+1}=-\frac{\partial_i U(r)}{2}$
$k=n+1,\ i,j\leq n$	$\Gamma^{n+1}_{i,j}\cdot U(r)=-\Gamma^{j}_{i,n+1}$	$\Gamma^{n+1}_{i,j}=0$
$i=n+1,\ j,k\leq n$	$\Gamma^{k}_{n+1,j}=-\Gamma^{j}_{n+1,k}$	$\Gamma^{j}_{i,n+1}=0$
$i,j,k\leq n$	$\Gamma^{k}_{i,j}=-\Gamma^{j}_{i,k}$	$\Gamma^{i}_{k,j}=0$

These resolutions show the statments in theorem. ⋈

COROLLARY 2. *The differential equation system of geodesics is*

$$\dot{\alpha} = h \,/\, U(r)$$

$$\ddot{a}_j = a_j h^2\, U'(r) \,/\, U^2(r) \qquad 1\leq j\leq n$$

where h *is a suitable constant* , $r = \langle a,a \rangle$, *and* ($a(s),\alpha(s)$) *is the geodesic whose coordinates are* $\{ a_j \}_{j=1}^{n}$ *and* α .

PROOF. The general differential equation for geodesics is

$$\ddot{x}_j + \sum_{s,i=1}^{n+1} \dot{x}_s \cdot \dot{x}_i \cdot \Gamma^j_{s,i}(x) = 0 ,$$

where $x(s) = (x_1(s), \ldots, x_{n+1}(s))$ is a geodesic. In our case, we get the following

$$\ddot{\alpha} + 2\dot{\alpha} \sum_{i=1}^{n} \dot{a}_i \cdot \frac{\partial_i (U(r))}{2\,U(r)} = 0 ,$$

$$\ddot{a}_j + (\dot{\alpha})^2 \cdot \frac{-\partial_i (U(r))}{2\,U(r)} = 0 \quad (1 \le j \le n),$$

where $(a(s), \alpha(s)) = (a_1(s), a_2(s), \ldots, a_n(s), \alpha(s))$ is a geodesic. Since

$$\sum_{i=1}^{n} \dot{a}_i \cdot \frac{\partial_i (U(r))}{2\,U(r)} = \frac{d}{ds} (U(r)) / U(r) ,$$

we obtain from the first equation that

$$U(r) \cdot \ddot{\alpha} + \frac{d}{ds} (U(r)) \cdot \dot{\alpha} = 0 ,$$

which implies the existence of a constant h satisfying

$$\dot{\alpha}\, U(r) = h .$$

This and a simple calculation give from our second differential equation that

$$\ddot{a} - a_j \cdot \frac{h^2\, U'(r)}{U^2(r)} = 0 \qquad (1 \le j \le n). \quad \bowtie$$

REMARK. *It is easy to realize in the last differential equation obtained that*

$$\ddot{a}_j + \partial_j \left[\frac{h^2}{2U(r)}\right] = 0 \qquad (1 \le j \le n).$$

We know from the theoretical mechanic that this equation represent the motion of a particle ,in the central force field with potential $h^2 / U(r)$. *This shows that the projection of geodesics has to be a conic section.* ⋈

We will deal in the following three cases. The first case, when the function U is constant, is trivial, because at this time we have only changed the unit in $\mathbb{R}^1$. Thus the geodesics are straight lines. The further cases are

$$(2) \qquad U(x) = \frac{c}{x}$$

$$(3) \qquad U(x) = c \cdot \sqrt{x} \,,$$

where $c \neq 0$.

$-c\|x\|^{-1}$

$c > 0$

$c\|x\|^{+2}$

3.2. Second case

In this case the determining function of the metric is $U(r) = c/r$. We have the following description of geodesics.

THEOREM 3. *Let* $(a(s),\alpha(s))$ *be a geodesic in* $\mathbb{R}^n_0 \times \mathbb{R}^1$ *with respect to the Riemannian metric* (1) . *Then the geodesics are*

i) *when* $c > 0$ *and* $h \neq 0$

$$a_j(s) = \frac{\sqrt{c \cdot h_j}}{h} \cdot \sin\left[s \cdot \frac{h}{\sqrt{c}} - \omega_j \right] ,$$

ii) *when* $c < 0$ *and* $h \neq 0$

$$a_j(s) = \frac{\sqrt{-c \cdot h_j}}{h} \cdot \mathrm{sh}\left[s \cdot \frac{h}{\sqrt{-c}} - \omega_j \right] ,$$

iii) *when* $h = 0$

$$a_j(s) = v_j(t) + w_j ,$$

where h_j, v_j, w_j, ω_j ($1 \leq j \leq n$) *are constant. The* $\alpha(s)$ *can be obtained from*

$$\dot{\alpha} = \frac{h}{c} \sum_{j=1}^{n} a_j^2(s) .$$

PROOF. The last statement of the theorem can be shown on substituting $U(r) = \frac{c}{r}$ into the first equation of the Corollary 2. The second equation of Corollary 2. gives

$$\ddot{a}_j + a_j \cdot \frac{h^2}{c} = 0 .$$

The multiplication of this by $2 \cdot \dot{a}_j$ leads to

$$\dot{a}_j^2 + a_j^2 \cdot \frac{h^2}{c} = h_j ,$$

where h_j is a constant. As it is well known, the general solution of this is that is stated in the theorem. ⋈

COROLLARY 4. *We denote the initial values at* $s = 0$ *of geodesic* (a, α) *by* $a_0 = a(0)$, $\alpha_0 = \alpha(0)$, $T = \dot{a}(0)$ *and* $\tau = \dot{\alpha}(0)$. *Then we get the following description of geodesics for* $c = 1$:

i) *if* $\tau = 0$ *then the geodesic is the line*

$$a(s) = T \cdot s + a_0 , \quad \alpha(s) \equiv \alpha_0$$

ii) *if* $\tau \neq 0$ *and* $T = 0$, *then the projection of geodesics onto* $\mathbb{R}^n_0$ *is the half segment* $(0, a_0]$, *the point* 0 *is a singular point.*

iii) *if* $\tau \neq 0$ *and* $T \neq 0$, *then, assuming* $|T| = 1$, *the projection of geodesics onto* $\mathbb{R}^n_0$ *is an ellipse with focal–point* 0 *in the 2–plane* $\mathcal{W}$ *spanned by the vectors* a_0 *and* T. *Let* E_1, E_2 *be orthogonal unit vectors in* $\mathcal{W}$ *such that*

$$a_0 = a^0_1 \cdot E_1 \,,\; T = \cos\gamma \cdot E_1 + \sin\gamma \cdot E_2 \,.$$

The equation of this projected ellipse is

$$\frac{(\tau^2 + \cos 2\gamma)}{\sin^2\gamma} \cdot y^2 + x^2 - xy = |a_1|^2 \,,$$

where x, y *are coordinates in* $\mathcal{W}$ *assigned to* $\{E_1, E_2\}$. *Especially, if* $\gamma = 0$, *then the projection is the half segment* $\left(0, \frac{\tau^2+1}{\tau} \cdot a_0\right]$ *and the point* 0 *is singular.*

PROOF. Let $\{E_1, \ldots, E_n\}$ be an orthonormal base in $\mathbb{R}^n$ such that

$$a_0 = a^0_1 \cdot E_1 \,,\; T = \cos\gamma \cdot E_1 + \sin\gamma \cdot E_2 \,.$$

and E_{n+1} is a base vector of $\mathbb{R}^1$. Then the Picard–Lindelöf theorem and our Corollary 2. give that $a_j \equiv 0$ for $3 \leq j \leq n$ since $\dot{a}_j = 0$ and $a_j = 0$. Also Corollary 2. gives that if $\tau = 0$ then $h = 0$ and so $\ddot{a}_j = 0$, which proves our first statement.

If $\tau \neq 0$, then $h = \tau / a_1^0$ and so

$$a_1 = \sqrt{h_1} \cdot \frac{a_1^0}{\tau} \cdot \sin\left[s \cdot \frac{\tau}{a_1^0} - \omega_1 \right] ,$$

$$a_2 = \sqrt{h_2} \cdot \frac{a_1^0}{\tau} \cdot \sin\left[s \cdot \frac{\tau}{a_1^0} - \omega_2 \right] ,$$

$$a_3 = a_4 = \ldots = a_n = 0 .$$

If $T = 0$, then ω_1 and ω_2 must be $\pi/2$ and so

$$a_1 = a_1^0 \cdot \cos\left[s \cdot \frac{\tau}{a_1^0} \right] , \quad a_2 = a_3 = \ldots = a_n = 0 .$$

This shows our second statement.

Now on we assume $T \neq 0$. In this case we immediately get from initial conditions that

$$a_1 = a_1^0 \cdot \cos\left[s \cdot \frac{\tau}{a_1^0} \right] + \cos\gamma \cdot \frac{a_1^0}{\tau} \cdot \sin\left[s \cdot \frac{\tau}{a_1^0} \right] ,$$

$$a_2 = \sin\gamma \cdot \frac{a_1^0}{\tau} \cdot \sin\left[s \cdot \frac{\tau}{a_1^0} \right] ,$$

$$a_3 = a_4 = \ldots = a_n = 0 .$$

These equations are the parametric representations of the ellipse stated in iii). If $\gamma = 0$, these equations imply

$$a_1 = \frac{a_1^0}{\tau}\sqrt{\tau^2+1}\ \sin\left[s\cdot\frac{\tau}{a_1^0} + \lambda \right] \text{ and } a_2 \equiv 0\ ,$$

where $\cos\lambda = 1 / \sqrt{\tau^2+1}$. This gives our last statement in the corollary. ⋈

COROLLARY 5. *If* $T \neq 0$ *and* $\tau \neq 0$, *then the projection of geodesic* (a, α) *is an ellipse inscribed into the rectangle with vertices*

$$\left[\pm\frac{a_1^0}{\tau}\sqrt{\tau^2+\cos^2\gamma}\ ,\ \pm\frac{a_1^0}{\tau}\cdot\sin\gamma \right].$$

The tangent points of the ellipse and this rectangle are

$$\pm\left[\frac{a_1^0}{\tau}\cdot\cos\gamma\ ,\ \frac{a_1^0}{\tau}\cdot\sin\gamma \right].$$

and

$$\pm\left[\frac{a_1^0}{\tau}\sqrt{\tau^2+\cos^2\gamma}\ ,\ \frac{a_1^0}{\tau}\cdot\frac{\sin\gamma\cdot\cos\gamma}{\sqrt{\tau^2+\cos^2\gamma}} \right].$$

PROOF. Using the equation (4) we get for the tangent point with $\dot{x} \neq 0$ and $\dot{y} = 0$ the coordinates

$$x_0 = \pm\frac{a_1^0}{\tau}\cdot\cos\gamma \text{ and } y_0 = \pm\frac{a_1^0}{\tau}\cdot\sin\gamma\ .$$

These give the points with tangent line parallel to the x–axis. Similarly the conditions $\dot{x} = 0$, $\dot{y} \neq 0$ give the coordinates

$$x_0 = \pm \frac{a_1^0}{\tau} \sqrt{\tau^2 + \cos^2\gamma} \text{ and } y_0 = \pm \frac{a_1^0}{\tau} \cdot \frac{\sin\gamma \cdot \cos\gamma}{\sqrt{\tau^2 + \cos^2\gamma}}$$

of the points having tangent lines parallel to the y–axis. ⋈

3.3. Third case

In this case the determining function of the metric is $U(\mathrm{r}) = \mathrm{c}\cdot\sqrt{\mathrm{r}}$. We have the following description of the geodesics.

THEOREM 6. *Let* $(\ a(\mathrm{s}),\alpha(\mathrm{s})\ )$ *be a geodesic in* $\mathbb{R}^{\mathrm{n}}_0 \times \mathbb{R}^1$ *with respect to the Riemannian metric* (1) . *We denote its initial values at* $\mathrm{s} = 0$ *by* $a_0 = a(\,0\,)$, $\alpha_0 = \alpha(\,0\,)$, $\mathrm{T} = \dot{a}(\,0\,)$, $\tau = \dot{\alpha}(\,0\,)$. *Let* E_1 , $\mathrm{E}_2 \in \mathbb{R}^{\mathrm{n}}_0$ *be orthogonal unit vectors in* $\mathcal{W}$ *which is spanned by* a_0 *and* T . *Choose* E_1 , E_2 *satisfying the following relation:*

$$a_0 = a_1\cdot\mathrm{E}_1\,,\quad \mathrm{T} = \mathrm{T}_1\cdot\mathrm{E}_1 + \mathrm{T}_2\cdot\mathrm{E}_2\,.$$

If $\mathrm{T}_2 \neq 0$ *we get the following description of geodesics:*

The geodesics do not leave the space spanned by $\mathcal{W}$ *and* $\mathbb{R}^1$. *Furthermore, if we denote the projection of* T *to* $\mathbb{R}^1$ *along* $\mathbb{R}^{\mathrm{n}}_0$ *by* T_3 , *there are three possibilities:*

i) *if* $|\,(\,\mathrm{T},\tau\,)\,|^2_* = \mathrm{T}_1^2 + \mathrm{T}_2^2 + \mathrm{c}\cdot|\,a_1^0\,|\cdot\mathrm{T}_3^2 < 0$, *then the projection of the geodesic onto* $\mathcal{W}$ *is ellipse,* *ii)* *if* $|(\mathrm{T},\tau)|_*^2 = \mathrm{T}_1^2 + \mathrm{T}_2^2 + c\cdot|a_1^0|\cdot\mathrm{T}_3^2 = 0$, *then the projection of the geodesic onto* $\mathcal{W}$ *is parabola,*

iii) *if* $|(\mathrm{T},\tau)|_*^2 = \mathrm{T}_1^2 + \mathrm{T}_2^2 + c\cdot|a_1^0|\cdot\mathrm{T}_3^2 > 0$, *then the projection of the geodesic onto* $\mathcal{W}$ *is hyperbola.*

The equation of the projected geodesic in polar–coordinate is

$$p(\varphi)=\frac{2\cdot|a_1^0|^3\cdot\mathrm{T}_2^2}{-c\cdot\mathrm{T}_3^2\cdot|a_1^0|^3 + v\cdot\cos(\varphi-\omega)},$$

where

$$v=\mathrm{sgn}(c)\cdot\sqrt{4\cdot\mathrm{T}_1^2\cdot\mathrm{T}_2^2\cdot|a_1^0|^4 + (2\cdot\mathrm{T}_2^2\cdot|a_1^0|^2 + c\cdot\mathrm{T}_3^2\cdot|a_1^0|^3)^2},$$

$$\omega=\arcsin\left[\frac{2\cdot\mathrm{T}_1\cdot a_1^0\cdot\mathrm{sgn}(c)}{U}\right].$$

and $p = |a|$, $\cos\varphi = \langle a,\mathrm{E}_1\rangle/|a|$.

PROOF. Let $\{\mathrm{E}_1,\dots,\mathrm{E}_n\}$ be orthonormal base in $\mathbb{R}^n$ such that

$$a_0 = a_1^0\cdot\mathrm{E}_1,\quad \mathrm{T} = \mathrm{T}_1\cdot\mathrm{E}_1 + \mathrm{T}_2\cdot\mathrm{E}_2$$

and E_{n+1} be unit vector in $\mathbb{R}^1$. From Corollary 2. we get the following differential equation for the geodesic $(a(s),\alpha(s))$:

$$\ddot{a}_j - a_j\cdot\frac{h^2}{2\cdot c\cdot|a_1|^3} = 0 \qquad (1\leq j\leq n) \tag{5}$$

(6) $$\dot{\alpha} = \frac{h}{c \cdot |a_1|},$$

where $h = \tau \cdot c \cdot |a_0|$. In our coordinate system $a_j(0) = \dot{a}_j(0) = 0$ for $3 \leq j \leq n$ hence by the Picard–Lindelöf theorem we conclude, that $a_j(0) \equiv 0$. So it is enough to investigate the case if $n = 2$.

Let $p(s) = |\ a(s)\ |$ and take the polar coordinate system in $\mathbb{R}_0^2$, i.e.

$$a_1(s) = p(s) \cdot \cos(\ \varphi(s)\ )\ ,\quad a_2(s) = p(s) \cdot \sin(\ \varphi(s)\ ).$$

From the differential equations (5) we have

(5') $$2 \cdot c \cdot p^2 \cdot (\ \ddot{p} \cdot \cos\varphi - 2 \cdot \dot{p} \cdot \dot{\varphi} \cdot \sin\varphi - p \cdot \dot{\varphi}^2 \cdot \cos\varphi - p \cdot \ddot{\varphi} \cdot \sin\varphi\ ) = h^2 \cdot \cos\varphi\ ,$$

(5'') $$2 \cdot c \cdot p^2 \cdot (\ \ddot{p} \cdot \sin\varphi - 2 \cdot \dot{p} \cdot \dot{\varphi} \cdot \cos\varphi - p \cdot \dot{\varphi}^2 \cdot \sin\varphi - p \cdot \ddot{\varphi} \cdot \cos\varphi\ ) = h^2 \cdot \sin\varphi\ .$$

Take the linear combination of these equations by $(\ \sin\varphi, -\cos\varphi\ )$ and $(\ \cos\varphi, \sin\varphi\ )$ to obtain the following ones.

$$2 \cdot \dot{p} \cdot \dot{\varphi} + p \cdot \ddot{\varphi} = 0$$

$$2 \cdot c \cdot p^2 \cdot (\ \ddot{p} - p \cdot \dot{\varphi}^2\ ) = h^2$$

After multiplying the first one by p , a simple integration gives

(7) $$p^2 \cdot \dot{\varphi} = q\ ,$$

where q is a suitable constant. On substituting this into the second equation and dividing it by p^2 , multiplying it by $\dot{p}$ we can integrate it, that yields to

$$c\cdot\dot{p}^2 + c\cdot\frac{q^2}{p^2} + \frac{h^2}{p} = q' , \tag{8}$$

where q' is constant.

Using the equations (7), (8) it is easy to get, that

$$\frac{dp}{d\varphi} = \pm\frac{p}{q}\cdot\sqrt{\frac{p^2\cdot q' - p\cdot h^2 - c\cdot q^2}{c}} .$$

On substituting $p = 1/\varrho$ into this equation it appears in the following integrable form

$$\mp 1 = \frac{q\cdot\frac{d\varrho}{d\varphi}}{\frac{q'}{c} - \frac{h^2}{c}\cdot\varrho - q^2\cdot\varrho^2} ,$$

hence

$$p(\varphi) = \frac{-2\cdot a_1\cdot q^2}{h^2 - \cos(\, k \mp \varphi\,)\cdot\sqrt{4\cdot a_1\cdot q^2\cdot q' + h^4}} , \tag{9}$$

where k is constant given by the integration of the previous equation.

It must be noted here, that q is zero if and only if $\dot{\varphi}\cdot p^2 \equiv 0 \Leftrightarrow \dot{\varphi} \equiv 0$ i.e. φ is constant. Hence the geodesic is a straight line passing through the origin and the equation (9) is not true. Easy calculation shows, that $\dot{\varphi} = T_2/|a_1^0|$ and $\varphi(0) = 0$. Thus $q = 0$ if and only if $T_2 = 0$, i.e. the projection of geodesic onto $\mathbb{R}_0^2$ along $\mathbb{R}^1$ is straight line if and only if its starting speed T is parallel to

a_0 . In this case (8) shows that $\dot{p} = \sqrt{q'/c - h^2/cp}$. This equation together with (6) gives

$$\frac{d\varphi}{dp} = \frac{\dot{\alpha}}{\dot{p}} = \frac{h}{\sqrt{c \cdot p^2 \cdot q' - c \cdot p \cdot h^2}} .$$

There are three possibilities now:

i) if $c \cdot q' > 0$, then

$$\varphi(p) = q'' + \frac{h}{\sqrt{c \cdot q'}} \cdot \ln\left[p - \frac{h^2}{2 \cdot q'} + \sqrt{p^2 - h^2 \cdot p/q'} \right] ,$$

ii) if $c \cdot q' < 0$, then

$$\varphi(p) = q'' + \frac{h}{\sqrt{-c \cdot q'}} \cdot \arcsin\left[\frac{4 \cdot p \cdot q'^2 - 2 \cdot q' \cdot h^2}{h^4} \right] ,$$

iii) if $c \cdot q' = 0 \Leftrightarrow q' = 0$. Since $c \cdot \dot{p}^2 = -h^2/p$, c has to be negative, and so $dp/d\varphi = \sqrt{-c \cdot p}$, which leads to

$$p(\varphi) = (\alpha \cdot \sqrt{-c} + q'')^2/4$$

The most interesting case is the second one, where p is bounded and the geodesic vibrates in the interval $\left[\frac{h^2}{2q'} - \frac{h^4}{4q'^2} , \frac{h^2}{2q'} + \frac{h^4}{4q'^2} \right]$. We will not deal with these cases further.

From the border conditions one can show by a straightforward but tedious calculation that

$$\text{(10)} \qquad p(\varphi) = \frac{|\,a_0\,| \cdot T_2^2}{u - v \cdot \cos(\,\varphi - \omega\,)} \,,$$

where

$$u = -\tau^2 \cdot |a_0| \cdot \tfrac{c}{2} \,,$$

$$v = -\mathrm{sgn}(c) \cdot \sqrt{T_1^2 \cdot T_2^2 + (\,T_2^2 + \tau^2 \cdot |a_0| \cdot \tfrac{c}{2}\,)^2} \,,$$

$$\omega = \arcsin\left[\frac{T_1 \cdot T_2}{-\mathrm{sgn}(\,c\,) \cdot v}\right] .$$

It is well know, that this equation defines conic sections. To determine its shape we have to investigate its eccentricity

$$\text{(11)} \qquad \varepsilon = \frac{v}{u} = \frac{\sqrt{T_1^2 \cdot T_2^2 + (\,T_2^2 + \tau^2 \cdot |a_1| \cdot \tfrac{c}{2}\,)^2}}{\tau^2 \cdot |\,a_1| \cdot \tfrac{c}{2}} .$$

A quick calculation shows, that

$$\varepsilon^2 = 1 + \frac{4 \cdot T_2^2}{c^2 \cdot a_1^2 \cdot \tau^2} \cdot (\,T_1^2 + T_2^2 + c \cdot |\,a_1\,| \cdot T_3^2\,) \,,$$

which implies on easy way, that ε more than, less than or equal to 1 according to the sign of $|\,(\,T,\tau\,)\,|_*^2 = T_1^2 + T_2^2 + c \cdot |\,a_0\,| \cdot T_3^2$, which was to be proved. ⋈

COROLLARY 7. *If* $c > 0$, *all the projections of geodesics are hyperbolas which have two asymptotic straight lines through the origin with the direction* $\omega - \arccos(1/\varepsilon)$ *and* $\omega + \arccos(1/\varepsilon)$. *The nearest point of these asymptotic lines to the origin is* $(\, \omega \, , \, | \, a_0 \, | \cdot T_2^2 \, / \, (\, u - v \,)\,)$. *Thus the origin is not contained by inside of the hyperbola.*

PROOF. If $c > 0$, then $| \, (T, \tau) \, |^2 > 0$ and so $\varepsilon > 1$. Thus the equation of the projection of geodesics is

$$p(\varphi) = \frac{| \, a_0 \, | \cdot T_2^2}{(-v) \cdot \cos(\varphi - \omega) \; - \; (-u)} \; ,$$

where $-v, -u > 0$ and $\varepsilon = \frac{-v}{-u} > 1$. It is clear that $p(\varphi)$ is minimal if $\cos(\varphi - \omega)$ is maximal. This proves the second statement of the Corollary.

On the other hand the denominator can not be zero, and $p(\varphi)$ tends to infinite if φ tends to $\omega - \arccos(u \,/\, v)$ or $\omega + \arccos(u \,/\, v)$. This completes the proof. $\bowtie$

COROLLARY 8. *The projection of geodesic is a circle if and only if* $c < 0$, T *is perpendicular to* a_0 *and* $| \, T \, |^2 + | \, (T, \tau) \, |_*^2 = 0$. *The radius of this circle is* $2 \cdot | \, T \, |^2 \, / \, (-c \cdot \tau^2)$. *Its center is the origin.*

PROOF. The projection is a circle if and only if $v = 0$. Since $c \neq 0$, this gives

$$T_1 \cdot T_2 = 0 \quad \text{and} \quad T_2^2 + \tfrac{c}{2} \cdot \tau^2 \cdot | \, a_0 \, | = 0 \, . \quad \bowtie$$

COROLLARY. *If the projection of geodesic is ellipse, and for its eccentricity* $\varepsilon \neq 0$ *, then its long axis has direction* ω *and length* $\dfrac{2\cdot|\ a_0\ |\cdot T_2^2\cdot u}{u^2 - v^2}$ *. It has two focal points: the origin and* $\left[\omega\ ,\ \dfrac{2\cdot|\ a_0\ |\cdot T_2^2\cdot u}{u^2 - v^2}\right]$ *. Its short axis has length* $2\cdot|\ a_0\ |\cdot T_2^2$ *.*

PROOF. It is clear that the nearest and the most far point of projection are on the long axis. We can get these points and their distances from the origin, when $\cos(\varphi-\omega) = \pm 1$. The length of long axis is the sum of their distances. The difference of these distances gives the distance of second focal points from the origin. If x is the half of short axis, the Pythagoras theory gives

$$\frac{|\ a_0\ |\cdot T_2^2\cdot u}{u^2 - v^2} = \left(\frac{|a_0|^2\cdot T_2^4\cdot v^2}{(\ u^2 - v^2\ )^2} + x^2 \right)^{\frac{1}{2}} .$$

The solution of this equation completes the proof. ⋈

COROLLARY 9. *If the projection of geodesic is parabola, then it is open in direction* ω *. Its nearest point is* $(\ \omega + \pi\ ,\ -T_2^2\ /\ (\ c\cdot\tau^2\ )\ )$ *and its focal point is the origin.*

PROOF. This corollary can be easily obtained on substituting $v = \varepsilon\cdot u = u$ into (10). ⋈

COROLLARY 10. *If the projection of geodesic is hyperbola and* $c < 0$ *, then its focal point is the origin. It has two asymptotic straight lines with direction*

$$\omega + \arccos(1/\varepsilon) \quad \text{and} \quad \omega - \arccos(1/\varepsilon) \; .$$

PROOF. The proof of our first corollary shows the way on which we can get this one. ⋈

THEOREM 11. *If* $\tau > (<) \ 0$ *then* α *strictly increasing (decreasing) and* α *depends on* $p = | \, a \, |$ *according to the following differential equation*

$$(12) \qquad \frac{d\alpha}{dp} = \frac{\mathrm{sgnsin}(\varphi - \omega) \cdot \tau \cdot | \; a_0 \; | \cdot T_2}{\sqrt{p^2(v^2-u^2)+2|a_0|T_2^2 \cdot u \cdot p - |a_0|^2 \cdot T_2^4}} \; ,$$

where we have used the notations of our first theorem.

PROOF. On investigating (7) at the startpoint, we get $q = T_2 \cdot | \, a_0 \, |$. From (10) we conclude

$$p \cdot v \cdot \sin(\varphi - \omega) = \mathrm{sgnsin}(\varphi - \omega) \cdot \sqrt{p^2(v^2-u^2)+2|a_0|T_2^2 \cdot u \cdot p - |a_0|^2 \cdot T_2^4} \; .$$

Let $\alpha(t) = \alpha\left[p(\varphi(t))\right]$. The theorem will be implicated by (6),(7),(10) on the following way:

$$\tau|a_0| = \frac{da}{dt}\cdot p = \frac{da}{dp}\cdot\frac{dp}{d\varphi}\cdot\frac{d\varphi}{dt}\cdot p$$

$$= \frac{da}{dt}\cdot\frac{-p\cdot v\cdot \sin(\varphi-\omega)}{|a_0|\cdot T_2^2}\cdot p\cdot\frac{q}{p^2}\cdot p$$

$$= \frac{da}{dp}\cdot \mathrm{sgn}\sin(\varphi-\omega)\cdot\frac{\sqrt{p^2(v^2-u^2)+2|a_0|T_2^2\cdot u\cdot p-|a_0|^2\cdot T_2^4}}{T_2}\ .$$

The monotonicity of a follows from (6) directly, since

$$\mathrm{sgn}\left[\frac{h}{c\cdot|a|}\right] = \mathrm{sgn}\tau\ .\quad \bowtie$$

COROLLARY 12. *If the projection of geodesic is ellipse, then*

$$p(a) = \frac{c\cdot a_0^2\cdot\tau^2}{|(T,\tau)|_*^2} - \frac{|a_0|\cdot v}{|(T,\tau)|_*^2}\cdot\sin\left[\frac{\sqrt{|T|^2+\tau^2}}{\tau\cdot|a_0|\cdot \mathrm{sgn}(\sin(\varphi-\omega))} - \mathrm{const}\right],$$

where const is such a number, that $p(a) = |a_0|$.

PROOF. Since the projection is ellipse, $v^2-u^2<0$. We can rewrite (12) in the form

$$\frac{da}{dp}=\frac{T_2\cdot\tau\cdot|a_0|\cdot\operatorname{sgn}(\sin(\varphi-\omega))}{\sqrt{u^2-v^2}\cdot\left[\dfrac{|a_0|^2\cdot T_2^4\cdot v^2}{(u^2-v^2)^2}-\left[p-\dfrac{|a_0|\cdot T_2^2\cdot u}{u^2-v^2}\right]^2\right]^{\frac{1}{2}}}\;.$$

The integration of this formula implies the Corollary. ⋈

COROLLARY 13. *If the projection of geodesic is parabola, then*

$$p(a)=\frac{c\cdot\tau^2}{4}\cdot(a_0-a)+|a_0|\;.$$

PROOF. In this case, $v^2=u^2$ thus (12) appears in the form

$$\frac{da}{dp}=\frac{\operatorname{sgn}\sin(\varphi-\omega)\cdot|a_0|\cdot\tau}{\sqrt{2\cdot|a_0|\cdot u\cdot p-|a_0|^2\cdot T_2^2}}\;.$$

On integrating this equation, we get the Corollary. ⋈

COROLLARY 14. *If the projection of geodesic is hyperbola, then*

$$p(a)=\frac{\text{const}}{2}\cdot e^{\frac{a\cdot|(T_2,\tau)|_*}{\mu}}+\frac{|a_0|^2\cdot v^2}{|(T_2,\tau)|_*^4}\cdot e^{\frac{-a\cdot|(T_2,\tau)|_*}{\mu}}\cdot\frac{1}{2\cdot\text{const}}+\frac{|a_0|^2\cdot v^2\cdot c}{2\cdot|(T_2,\tau)|_*^2}\;,$$

where $\mu=\tau\cdot|a_0|\cdot\operatorname{sgn}(\sin(\varphi-\omega))$ *and* const *is such a number, that* $p(a_0)=|a_0|$.

PROOF. Since $v^2 > u^2$, we have from (12), that

$$\frac{\mathrm{d}\alpha}{\mathrm{dp}} = \frac{T_2 \cdot \tau \cdot |a_0| \cdot \mathrm{sgn}(\sin(\varphi - \omega))}{\sqrt{v^2 - u^2} \cdot \left[\frac{|a_0|^2 \cdot T_2^4 \cdot v^2}{(u^2 - v^2)^2} + \left[p + \frac{|a_0| \cdot T_2^2 \cdot u}{v^2 - u^2} \right]^2 \right]^{\frac{1}{2}}} .$$

The integration gives α like a function of p, from which the corollary follows. ⋈

REMARK. *All the above give the result, that we would be able to write down the geodesics completely in the cylindrical coordinate system* (p, φ, α) *if we choose* α *for the parameter.*

3.4. Pictures about the third case made by computer

We have used to make pictures the program *PHASER* . Since it can solve (of course, only numerically) , only first order differential equations, we had to modify our differential equation system by growing it dimension as follows.

Our original system is

$$\ddot{a}_1 = a_1 \cdot \frac{\tau^2 \cdot c \cdot |a_0|^2}{2 \cdot |a|^3}$$

$$\ddot{a}_2 = a_2 \cdot \frac{\tau^2 \cdot c \cdot |a_0|^2}{2 \cdot |a|^3}$$

$$\dot{\alpha} = \tau \frac{|a_0|}{|a|} .$$

On substituting $x_1 = a_1$, $x_2 = a_2$, $x_3 = \alpha$, $x_4 = \dot{a}_1$, $x_5 = \dot{a}_2$ we obtain the five dimensional system.

$$x_1' = x_4 \; ; \; x_2' = x_5 \; ; \; x_3' = \tau |a_0| \,/\, \sqrt{x_1^2 + x_2^2} \; ;$$

$$x_4' = x_1 \cdot c \cdot \tau^2 |a_0|^2 \,/\, 2 \sqrt{x_1^2 + x_2^2}^{\,3} \; ;$$

$$x_5' = x_2 \cdot c \cdot \tau^2 |a_0|^2 / 2\sqrt{x_1^2 + x_2^2}^{\,3} .$$

The initial conditions are

$$x_1(0) = b \, , \, x_2(0) = d \, , \, x_3(0) = 0 \, , \, x_4(0) = T_1 \, , \, x_5(0) = T_2 \, .$$

Hence the geodesic will start at $(b, d, 0)$ with speed (T_1, T_2, τ).

In the realization of these equations on the computer program *PHASER* we have taken -10 for the value of parameter c, $1/2$ for c_1 and $3/2$ for c_2.

All the figures were made by printing the results of *PHASER* from the screen directly, which has caused some torsion of the picture appeared as a contraction along the y–axis.

In the following the figures are presented in the sort according to the corollaries at the middle of the previous section.

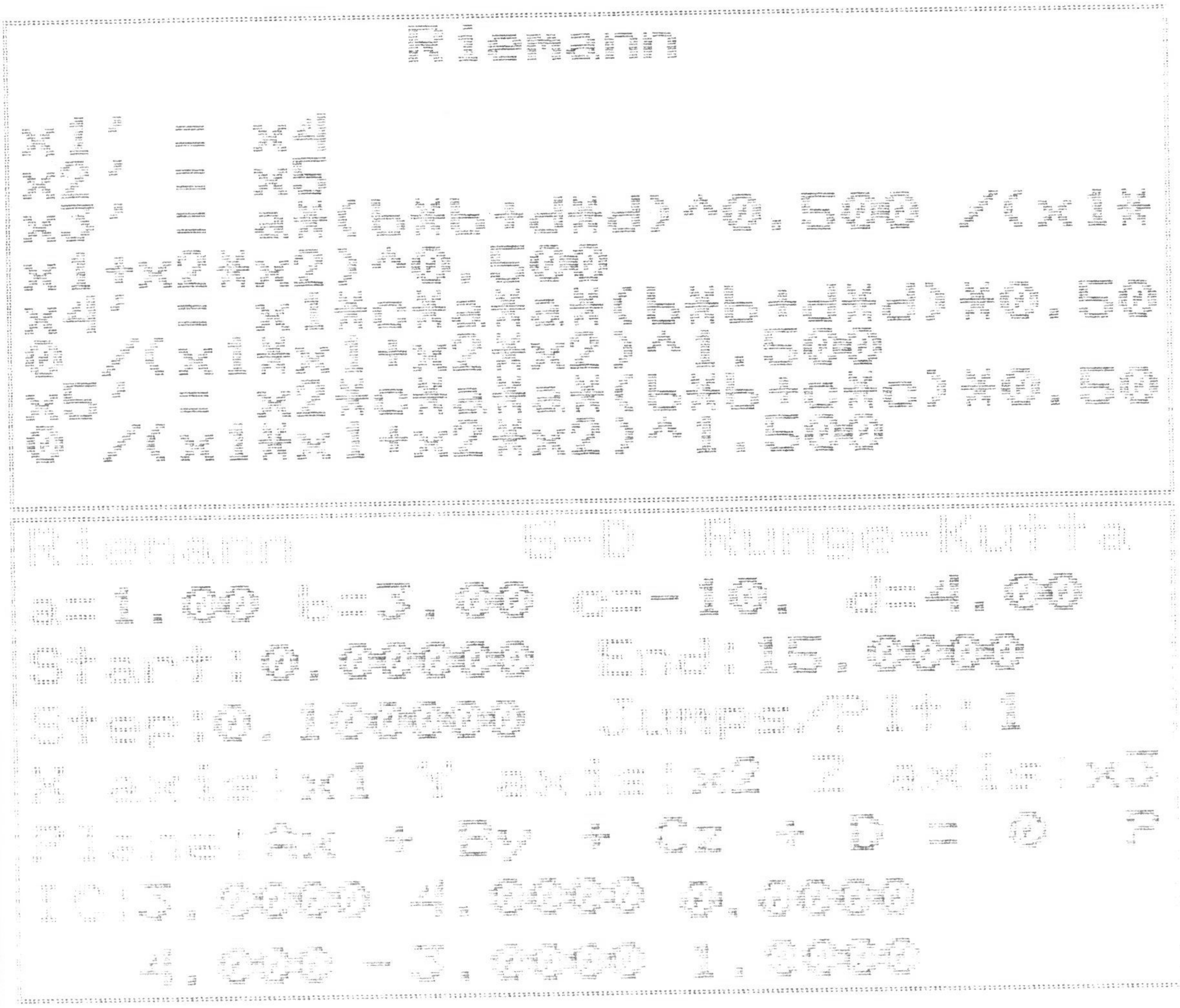

FIGURE 1.a : *The projection of geodesic is circle.*

The datas printed on this figure are the measures in Corollary 8. The role of the scalar τ *is played here by* a . *As it can be easily seen, the conditions of Corollary 8 are satisfied by the parameters* a,b,c,d .

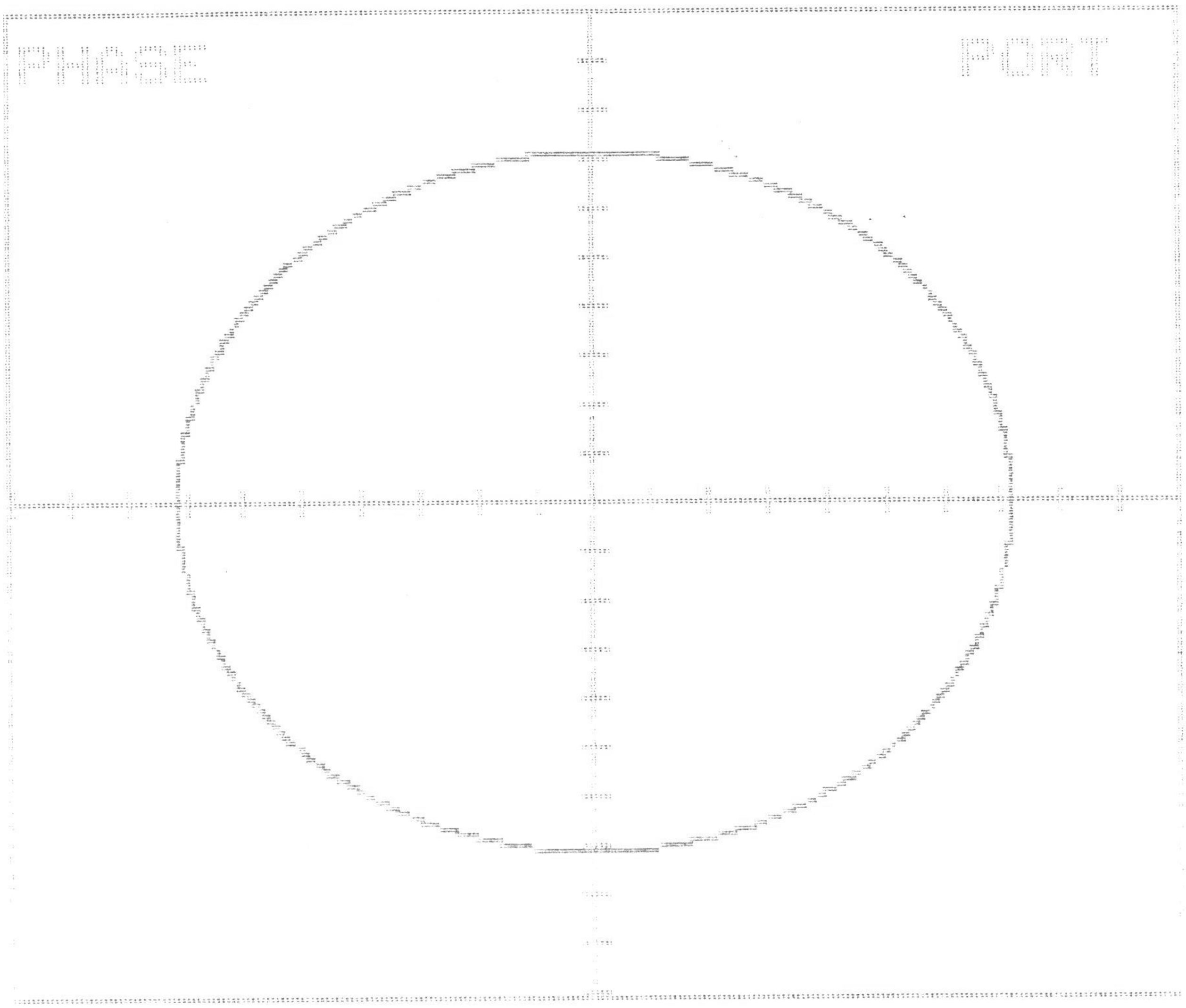

FIGURE 1.b : *The projection of geodesic is circle.*

On this figure one can see the projection of the geodesic in the case determined in Corollary 8. As it can be easily computed from the datas of figure 1.a, the radius of this circle is as it was determined in the Corollary 8.

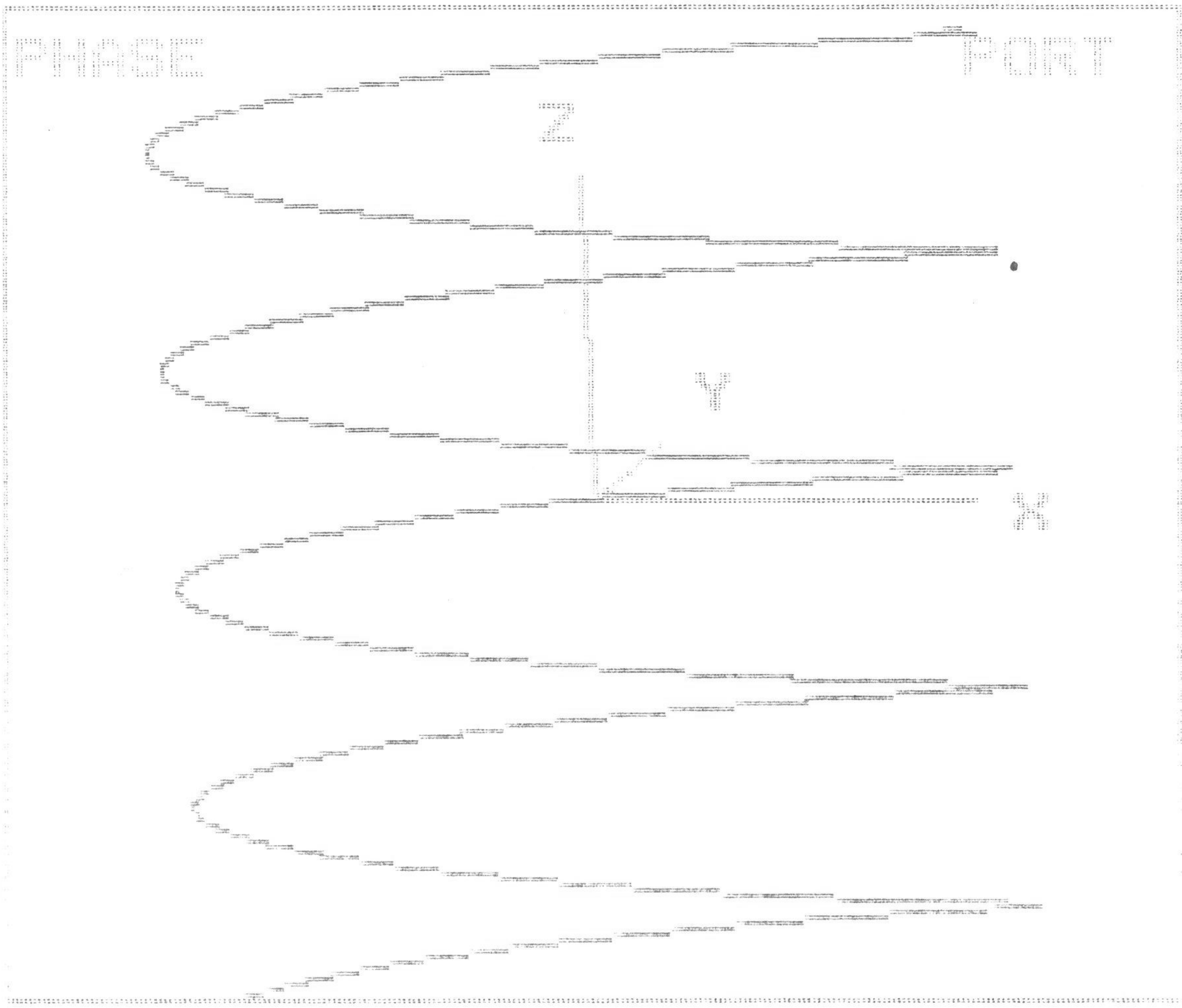

FIGURE 1.c : *The projection of geodesic is circle.*

We can see on this figure the image of the geodesic in the three dimension space. As it can be easily seen the third coordinate a grows strictrly monotonuously as it was determined in the Theorem 11.

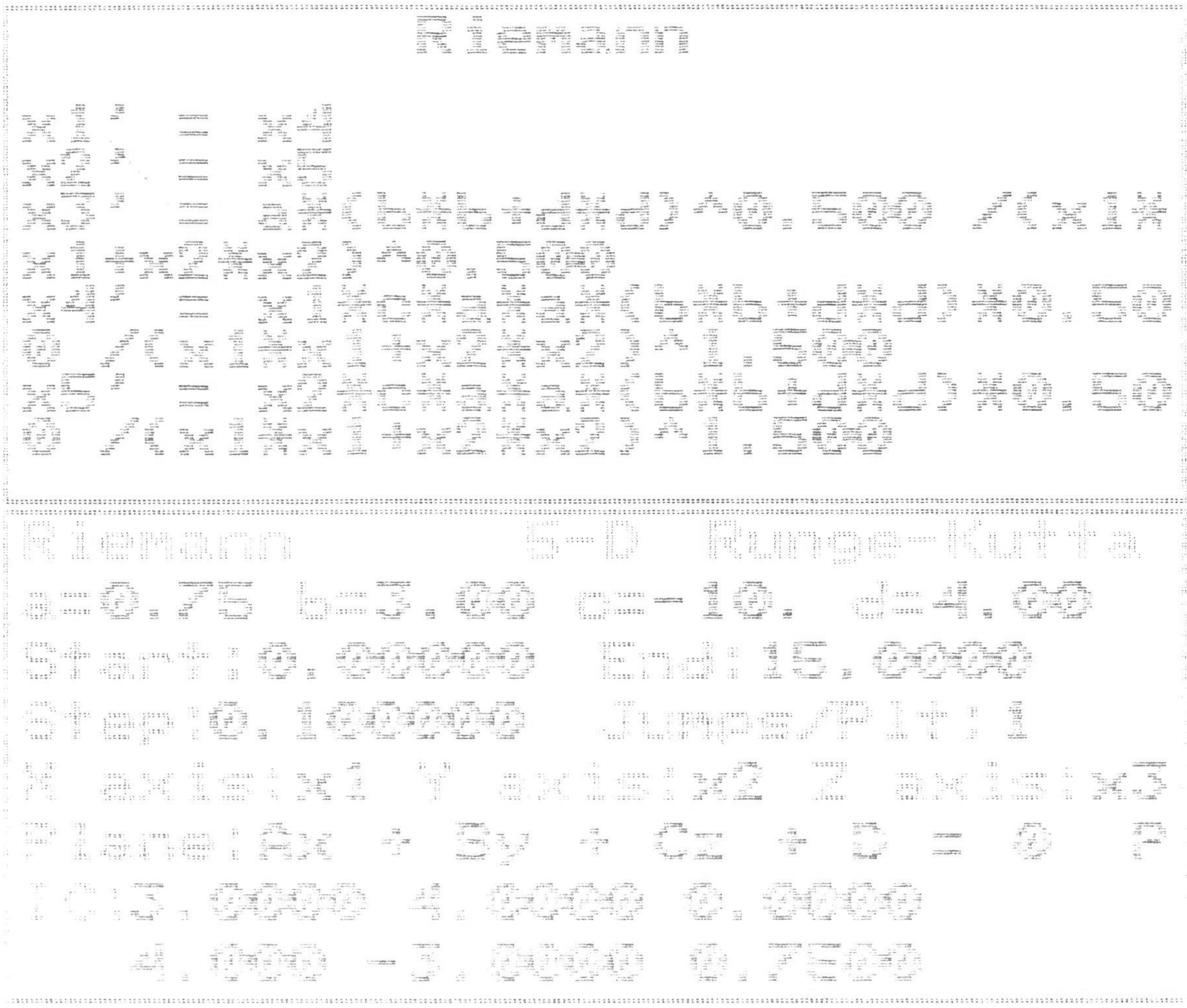

FIGURE 2.a : *The projection of geodesic is ellipse.*

The datas printed on this figure are the measures in corollary following Corollary 8. The role of the scalar τ *is played here by* a . *As it can be easily seen, the conditions of Corollary 8 are satisfied by the parameters* a,b,c,d .

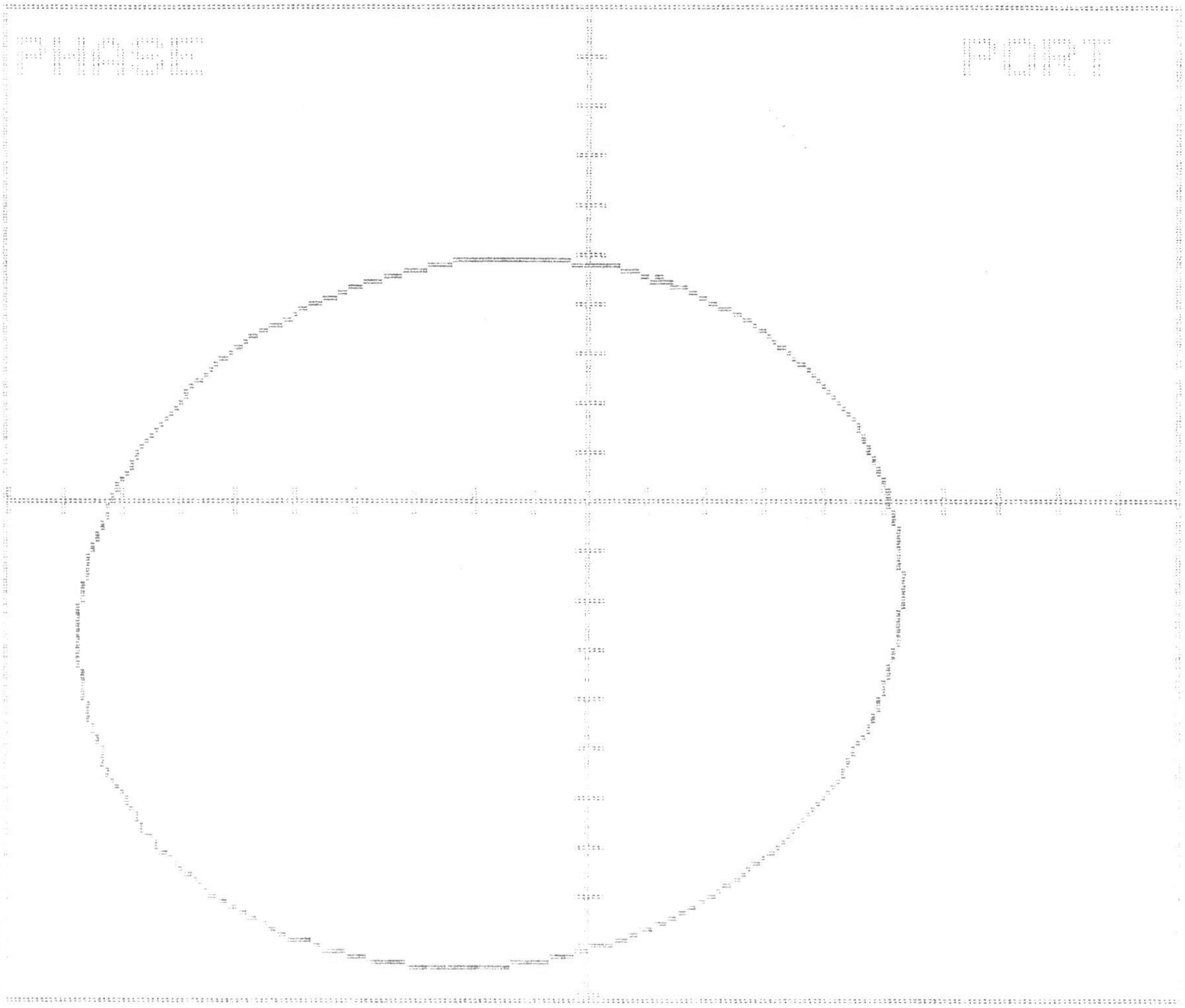

FIGURE 2.b : The projection of geodesic is ellipse.

On this figure one can see the projection of the geodesic in the case determined in corollary following Corollary 8. As it can be easily computed from the datas of figure 2.a, the long axis of this ellipse is as it was determined in that corollary and the origin is just a focal point.

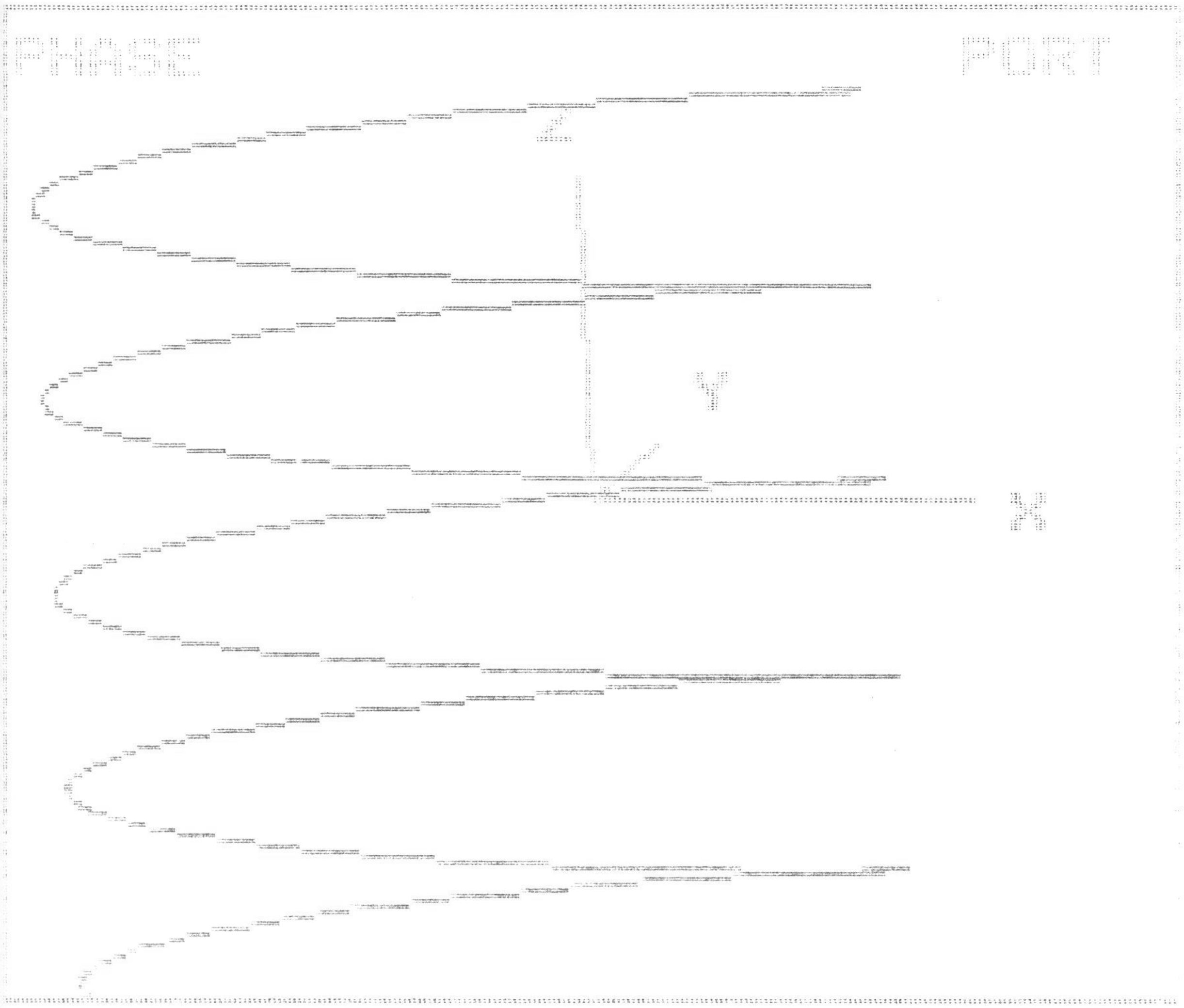

FIGURE 2.c : *The projection of geodesic is ellipse.*

We can see on this figure the image of the geodesic in the three dimension space. As it can be easily seen the third coordinate a grows strictrly monotonuously as it was determined in the Theorem 11.

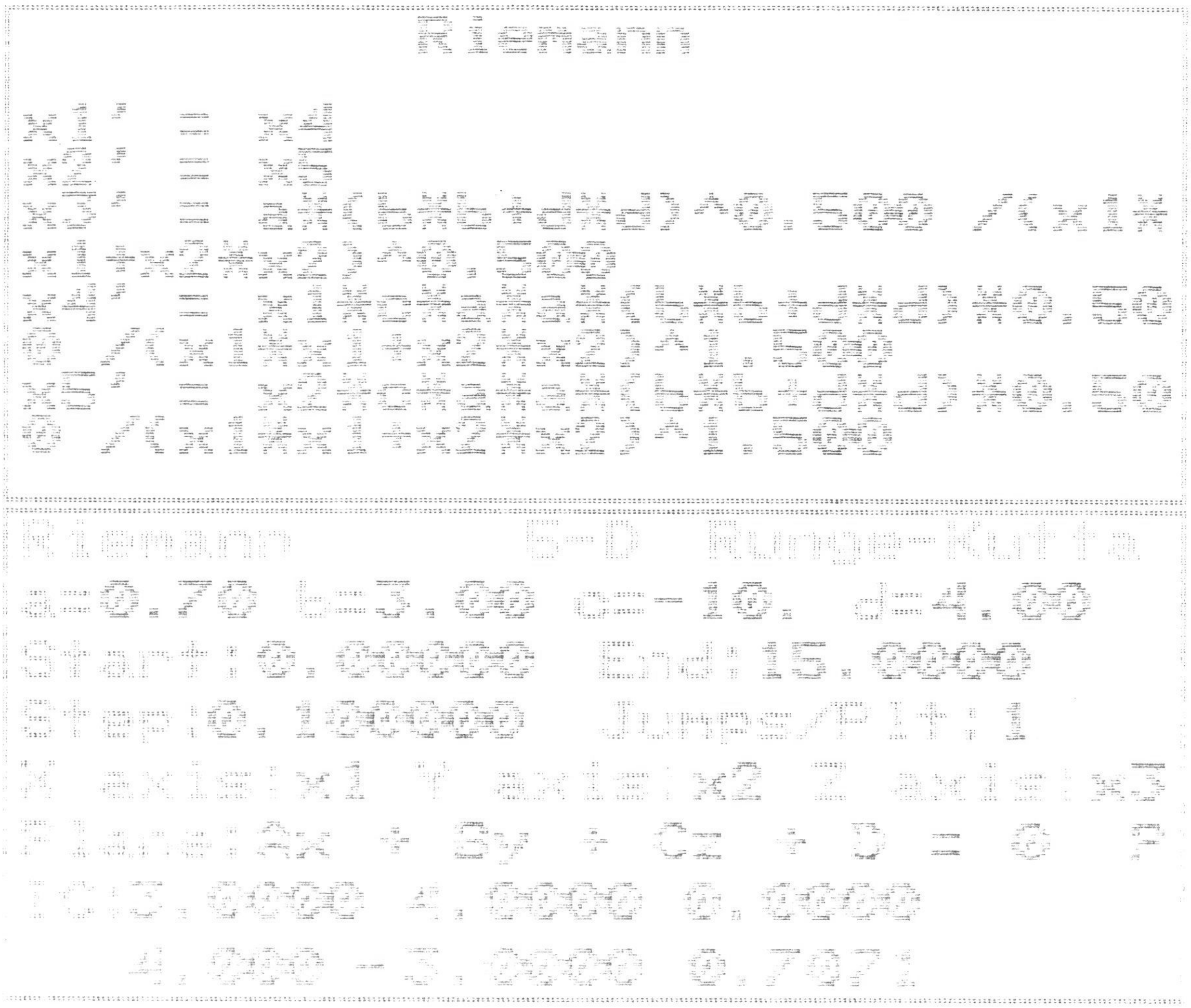

FIGURE 3.a : *The projection of geodesic is parabola.*

The datas printed on this figure are the measures in Corollary 9. The role of the scalar τ *is played here by* a . *As it can be easily seen, the conditions of Corollary 9 are satisfied by the parameters* a,b,c,d .

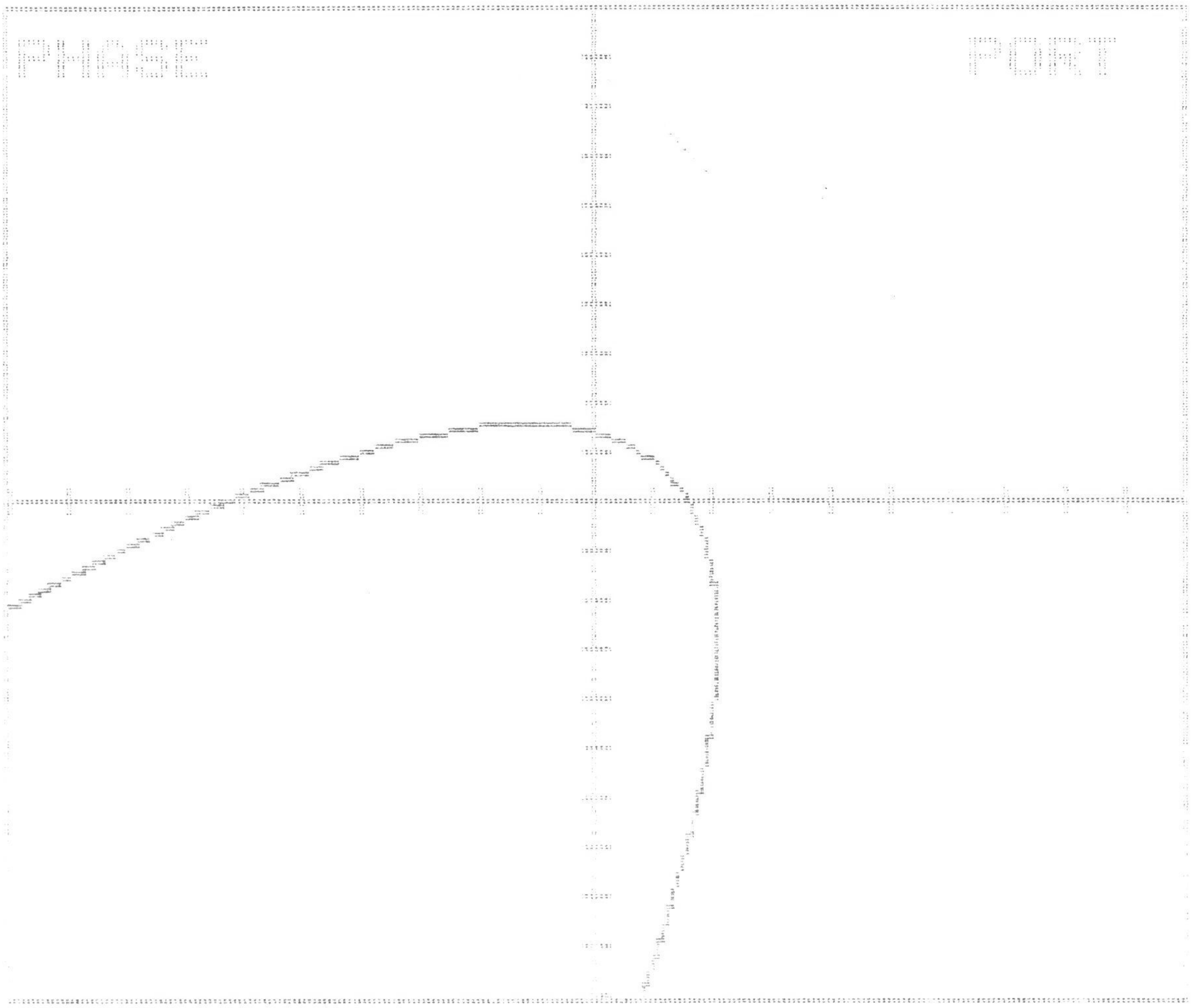

FIGURE 3.b : The projection of geodesic is parabola.

On this figure one can see the projection of the geodesic in the case determined in Corollary 9. As it can be easily computed from the datas of figure 3.a, the distance of the nearest point of this parabola to the origin is as it was determined in the Corollary 9.

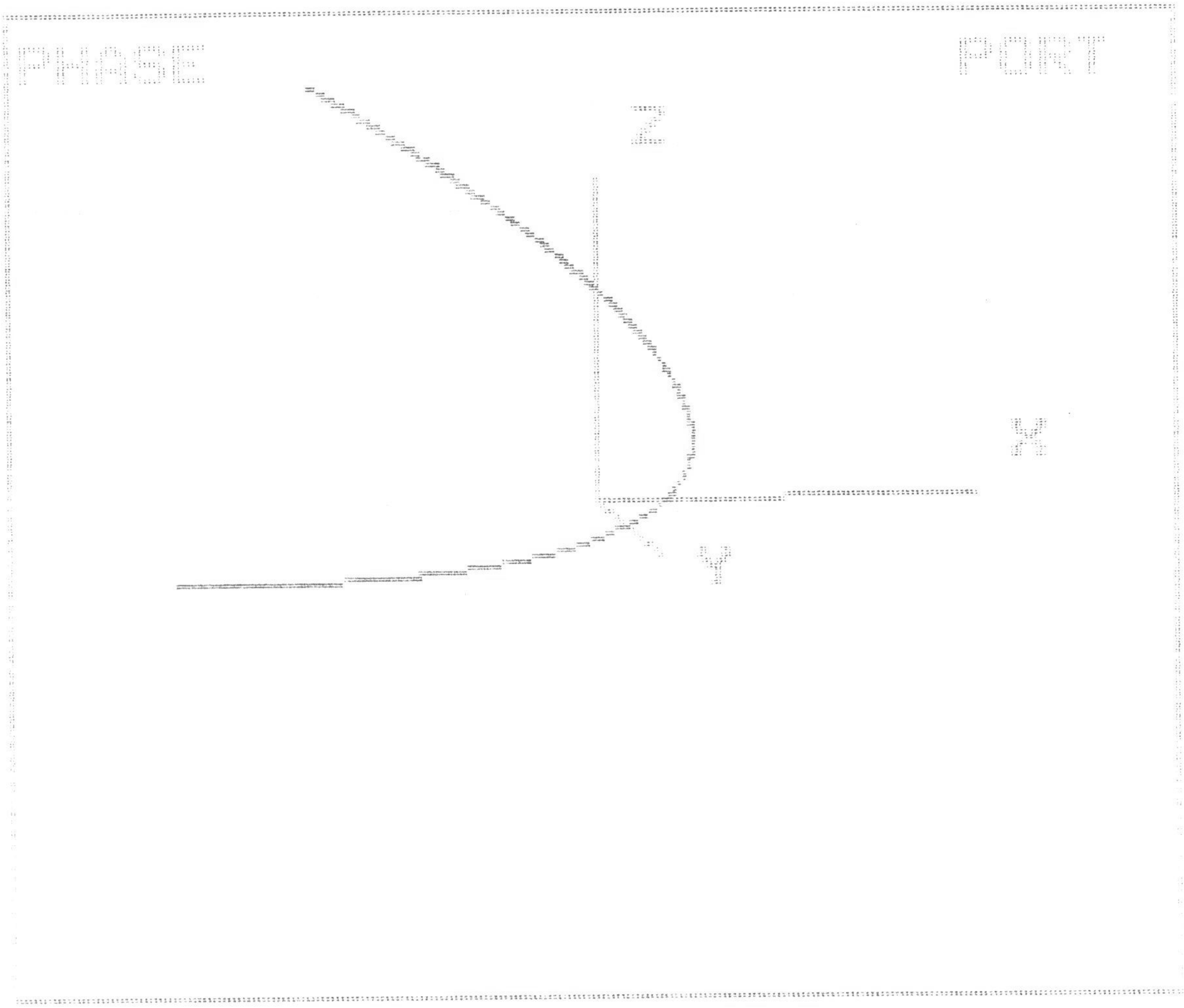

FIGURE 3.c : *The projection of geodesic is parabola.*

We can see on this figure the image of the geodesic in the three dimension space. As it can be easily seen the third coordinate a grows strictrly monotonuously as it was determined in the Theorem 11.

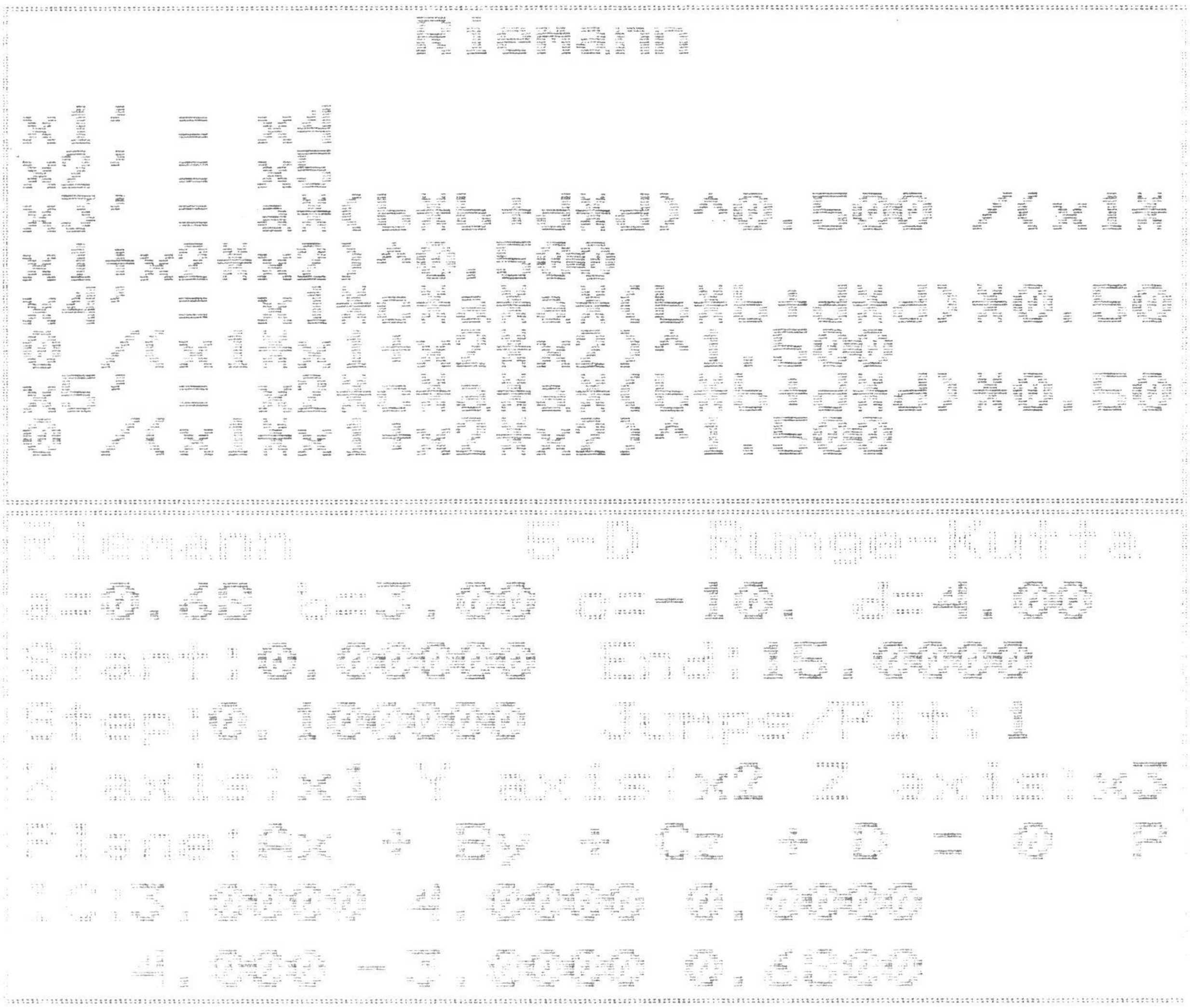

FIGURE 4.a : *The projection of geodesic is hyperbola.*

The datas printed on this figure are the measures in Corollary 10. The role of the scalar τ *is played here by* a . *As it can be easily seen, the conditions of Corollary 10 are satisfied by the parameters* a,b,c,d .

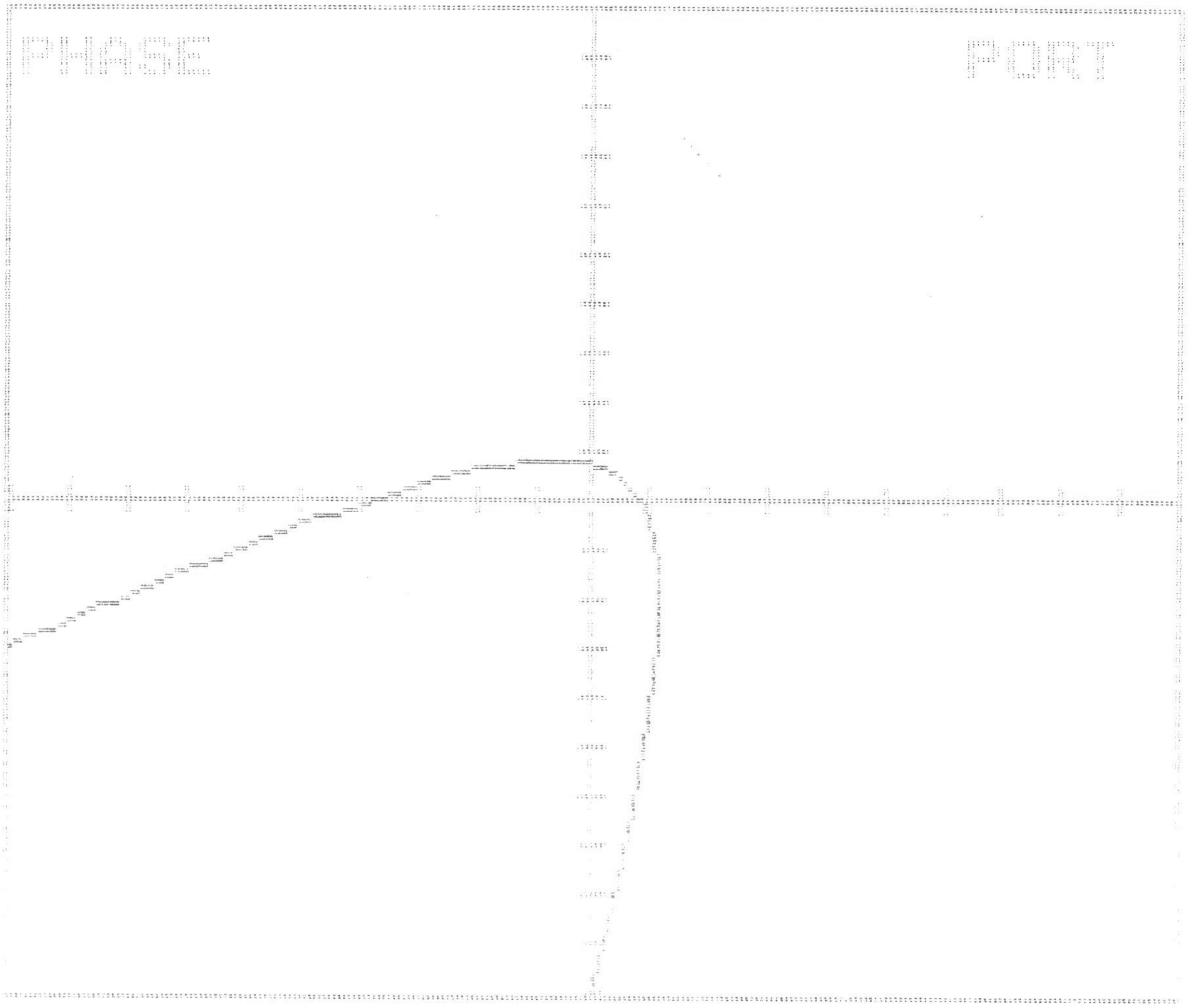

FIGURE 4.b : *The projection of geodesic is hyperbola.*

On this figure one can see the projection of the geodesic in the case determined in Corollary 10. As it can be easily computed from the datas of figure 4.a, the focal point of this hyperbola is the origin as it was determined in the Corollary 10.

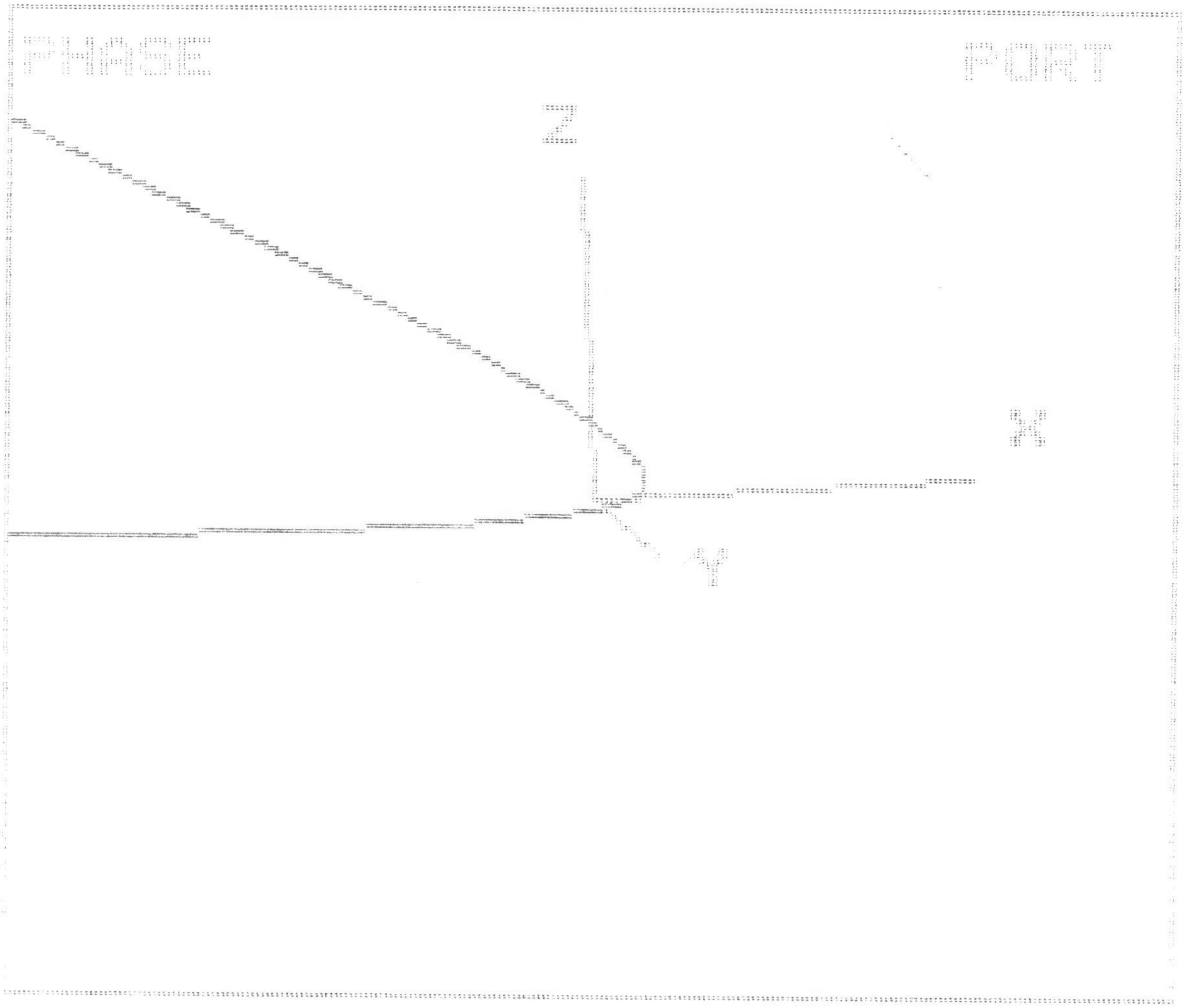

FIGURE 4.c : *The projection of geodesic is hyperbola.*

We can see on this figure the image of the geodesic in the three dimension space. As it can be easily seen the third coordinate a *grows strictrly monotonuously as it was determined in the Theorem 11.*

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