Visual distinguishability of segments

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Abstract. By considering the equioptics of segments in a plane we find some answers to the question that "if the measures of the visual angles of two segments are equal at some points in the plane do they coincide?".

1. Introduction

The *shadow picture* [4] of a compact convex set $\mathcal{B}$ is defined at each point $P \in \mathbb{R}^2 \setminus \mathcal{B}$ as the angle measure $\nu_{\mathcal{B}}(P)$ of the so called *visual angle* that $\mathcal{B}$ subtends at P. The point P and the set $\mathcal{B}$ are usually called the *source* and the *object* of shadow picture $\nu_{\mathcal{B}}(P)$. The shadow picture is often called *visual angle* (sometimes this makes some confusion) or *point projection* [2]. The function $\nu_{\mathcal{B}}$ is usually called *visual angle function*.

The central problem in this subject is to show such set $\mathcal{S}$ of sources and set $\mathcal{O}$ of objects that $\mathcal{O} \ni \mathcal{B} \mapsto \nu_{\mathcal{B}}|_{\mathcal{S}}$ is injective. There are a number of such results in the literature [3–7, 10] etc.

In this article we consider the distinguishability of segments by investigating first their equioptics in detail.

2. Equioptics of segments

The *equioptic* of two compact convex sets is the set of those points, where their shadow pictures are equal.[1] The *compoptic* of two compact convex sets is the set of those points, where the sum of their shadow pictures is π (these shadow pictures are said to be *supplementary*).

AMS Subject Classification (2012): 0052, 0054.
Key words and phrases: visual angle, shadow picture, masking function.
*Supported by the TÁMOP-4.2.2.A-11/1/KONV-2012-0073 project.

[1] As the shadow picture of a segment is not well defined at the endpoints of the segment, we think of it as equal to any angle in $[0, \pi]$.

Lemma 2.1. *The equioptic of two different segments is a union of subarcs of two cubic algebraic curves. The remaining subarcs of these two cubic algebraic curves constitute the compoptic of those segments.*

Proof. Denote the different segments by $\overline{AB}$ and $\overline{CD}$. Let $\mathbf{m}$ be a unit normal vector of the plane. If the segment $\overline{AB}$ subtends the same angle φ at X as $\overline{CD}$ does, then

$$(\overrightarrow{XA} \times \overrightarrow{XB})\,\langle \overrightarrow{XC}, \overrightarrow{XD}\rangle = \varepsilon(X)\,(\overrightarrow{XC} \times \overrightarrow{XD})\,\langle \overrightarrow{XA}, \overrightarrow{XB}\rangle,$$

where

$$\varepsilon(X) = \begin{cases} +1, & \text{if } |\overrightarrow{XC} \times \overrightarrow{XD}|\overrightarrow{XA} \times \overrightarrow{XB} = |\overrightarrow{XA} \times \overrightarrow{XB}|\overrightarrow{XC} \times \overrightarrow{XD}, \\ -1, & \text{otherwise.} \end{cases}$$

Obviously, $\varepsilon(X)$ is constant ± 1 in every quadrants of the straight lines AB and CD, and it changes sign if and only if X moves over one of the straight lines AB and CD.

Fix an origin O and let $\mathbf{x} = \overrightarrow{OX}$, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $\mathbf{d} = \overrightarrow{OD}$. Then

$$\begin{aligned} \overrightarrow{XA} \times \overrightarrow{XB} &= (\mathbf{a} - \mathbf{x}) \times (\mathbf{b} - \mathbf{x}) = \mathbf{x} \times (\mathbf{a} - \mathbf{b}) + \mathbf{a} \times \mathbf{b}, \\ \langle \overrightarrow{XA}, \overrightarrow{XB}\rangle &= \langle \mathbf{a} - \mathbf{x}, \mathbf{b} - \mathbf{x}\rangle = |\mathbf{x}|^2 - \langle \mathbf{x}, \mathbf{a} + \mathbf{b}\rangle + \langle \mathbf{a}, \mathbf{b}\rangle, \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{XC} \times \overrightarrow{XD} &= (\mathbf{c} - \mathbf{x}) \times (\mathbf{d} - \mathbf{x}) = \mathbf{x} \times (\mathbf{c} - \mathbf{d}) + \mathbf{c} \times \mathbf{d}, \\ \langle \overrightarrow{XC}, \overrightarrow{XD}\rangle &= \langle \mathbf{c} - \mathbf{x}, \mathbf{d} - \mathbf{x}\rangle = |\mathbf{x}|^2 - \langle \mathbf{x}, \mathbf{c} + \mathbf{d}\rangle + \langle \mathbf{c}, \mathbf{d}\rangle, \end{aligned}$$

therefore

$$\begin{aligned} &|\mathbf{x}|^2\langle \mathbf{x}, (\mathbf{a} - \mathbf{b}) \times \mathbf{m}\rangle + \langle \mathbf{a} \times \mathbf{b}, \mathbf{m}\rangle|\mathbf{x}|^2 - \langle \mathbf{x}, (\mathbf{a} - \mathbf{b}) \times \mathbf{m}\rangle\langle \mathbf{x}, \mathbf{c} + \mathbf{d}\rangle + \\ &\qquad + \langle \mathbf{x}, \langle \mathbf{c}, \mathbf{d}\rangle((\mathbf{a} - \mathbf{b}) \times \mathbf{m}) - \langle \mathbf{a} \times \mathbf{b}, \mathbf{m}\rangle(\mathbf{c} + \mathbf{d})\rangle + \langle \mathbf{a} \times \mathbf{b}, \mathbf{m}\rangle\langle \mathbf{c}, \mathbf{d}\rangle \\ &= \bar\varepsilon(\mathbf{x})(|\mathbf{x}|^2\langle \mathbf{x}, (\mathbf{c} - \mathbf{d}) \times \mathbf{m}\rangle + \langle \mathbf{c} \times \mathbf{d}, \mathbf{m}\rangle|\mathbf{x}|^2 - \langle \mathbf{x}, (\mathbf{c} - \mathbf{d}) \times \mathbf{m}\rangle\langle \mathbf{x}, \mathbf{a} + \mathbf{b}\rangle + \\ &\qquad + \langle \mathbf{x}, \langle \mathbf{a}, \mathbf{b}\rangle((\mathbf{c} - \mathbf{d}) \times \mathbf{m}) - \langle \mathbf{c} \times \mathbf{d}, \mathbf{m}\rangle(\mathbf{a} + \mathbf{b})\rangle + \langle \mathbf{c} \times \mathbf{d}, \mathbf{m}\rangle\langle \mathbf{a}, \mathbf{b}\rangle), \end{aligned} \tag{2.1}$$

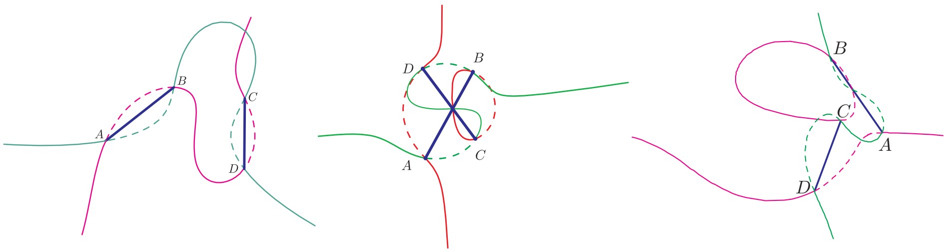
where $\bar\varepsilon(\mathbf{x}) = \bar\varepsilon(\overrightarrow{OX}) := \varepsilon(X)$. This proves that the equioptic is a union of arcs of two cubic algebraic curves. Since the equation for the compoptic is the same as (2.1), but with $-\bar\varepsilon$, the second statement of the theorem is also proven. ■

As $\mathbf{x} = \mathbf{0}$ is a solution of equation (2.1) if and only if $\langle \mathbf{a} \times \mathbf{b}, \mathbf{m}\rangle\langle \mathbf{c}, \mathbf{d}\rangle = \langle \mathbf{c} \times \mathbf{d}, \mathbf{m}\rangle\langle \mathbf{a}, \mathbf{b}\rangle$, we infer that the endpoints A, B, C and D, and, if they exist, the intersection points $M_0 := AB \cap CD$, $M_+ := AC \cap BD$ and $M_- := AD \cap BC$ are on the equioptic.

The cubic algebraic curves

$$
\begin{aligned}
&|\mathbf{x}|^2\langle \mathbf{x},(\mathbf{a}-\mathbf{b})\times\mathbf{m}\rangle+\langle \mathbf{a}\times\mathbf{b},\mathbf{m}\rangle|\mathbf{x}|^2-\langle \mathbf{x},(\mathbf{a}-\mathbf{b})\times\mathbf{m}\rangle\langle \mathbf{x},\mathbf{c}+\mathbf{d}\rangle+\\
&\qquad+\langle \mathbf{x},\langle \mathbf{c},\mathbf{d}\rangle((\mathbf{a}-\mathbf{b})\times\mathbf{m})-\langle \mathbf{a}\times\mathbf{b},\mathbf{m}\rangle(\mathbf{c}+\mathbf{d})\rangle+\langle \mathbf{a}\times\mathbf{b},\mathbf{m}\rangle\langle \mathbf{c},\mathbf{d}\rangle\\
&=\pm(|\mathbf{x}|^2\langle \mathbf{x},(\mathbf{c}-\mathbf{d})\times\mathbf{m}\rangle+\langle \mathbf{c}\times\mathbf{d},\mathbf{m}\rangle|\mathbf{x}|^2-\langle \mathbf{x},(\mathbf{c}-\mathbf{d})\times\mathbf{m}\rangle\langle \mathbf{x},\mathbf{a}+\mathbf{b}\rangle+\\
&\qquad+\langle \mathbf{x},\langle \mathbf{a},\mathbf{b}\rangle((\mathbf{c}-\mathbf{d})\times\mathbf{m})-\langle \mathbf{c}\times\mathbf{d},\mathbf{m}\rangle(\mathbf{a}+\mathbf{b})\rangle+\langle \mathbf{c}\times\mathbf{d},\mathbf{m}\rangle\langle \mathbf{a},\mathbf{b}\rangle),
\end{aligned}
\tag{2.2}
$$

are called *Apollonian curves* [11] and are denoted by $\mathcal{A}_\pm$ according to the sign of the right side of (2.2). The equioptic and the compoptic are assembled from the arcs of these Apollonian curves. Both Apollonian curves pass through the points A, B, C, D and M_0, but, in general, each one passes only the point $M_\pm$ with its index[2].



Observations 2.2. *The equioptic and compoptic subarcs of an Apollonian curve follow each other alternately and one takes over at those points, where the Apollonian curve intersects one of the straight lines of the segments. Each Apollonian curve has a straight line asymptotic to it at its both "ends".*

If an Apollonian curve intersects a straight line in more than three points, then by Bézout's theorem[3] it contains that straight line as a component, and therefore it is reducible.

Theorem 2.3. *If an Apollonian curve is reducible, then*

(a) *it is a nondegenerate circle and a straight line through its centre, or*

(b) *it is a degenerate circle and a straight line, or*

(c) *it is a nondegenerate equilateral hyperbola and the line at infinity, or*

(d) *it is two perpendicular straight lines and the line at infinity.*

[2]Only the coincidence of the indices of $\mathcal{A}_\pm$ and $M_\pm$ is invariant, because changing the order of the endpoints of one of the segment changes both indices.

[3]Roughly speaking Bézout's theorem [1] states that if two algebraic curves have more common points than the product of their degrees, then they have a common component.

Theorem 2.3 is clearly stated in [11, p. 358. l. -7] without proof, but it has a transparent proof in [12, Section 6] and a more detailed proof in [9, Section 2].

Theorem 2.4. *Let $\mathcal{A}$ be an Apollonian curve of the different nondegenerate segments $\overline{AB}$ and $\overline{CD}$. Then the followings hold.*

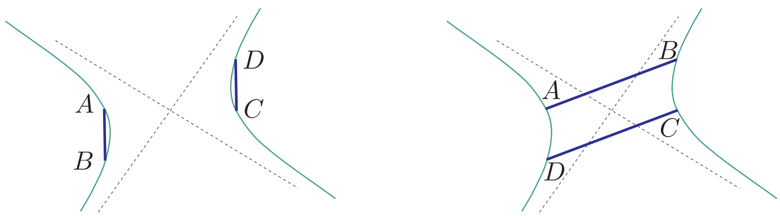
(1) *If $\mathcal{A}$ contains an open arc $\hat{\mathcal{H}}$ of a nondegenerate equilateral hyperbola $\mathcal{H}$, then the segments are opposite sides of a parallelogram and are separated by the branches of $\mathcal{H}$ if and only if they are the shorter edges in the parallelogram.*
(2) *If $\mathcal{A}$ contains an open arc $\hat{\mathcal{C}}$ of a circle $\mathcal{C}$, then it also contains a straight line ℓ passing through the centre of $\mathcal{C}$.*
(3) *If $\mathcal{A}$ contains an open segment $\hat{\ell}$ of a straight line ℓ, then the segments are*

(3.1) *the reflections of each other with respect to ℓ, or*
(3.2) *opposite sides of a kite symmetric with respect to ℓ, or*
(3.3) *adjoining sides of a kite symmetric with respect to ℓ, or*
(3.4) *placed on ℓ.*

Proof. In this proof we regard the common ideal point of two parallel straight lines as their intersection at the infinity.

(1) As $\mathcal{A}$ contains an arc $\hat{\mathcal{H}}$ of the nondegenerate equilateral hyperbola $\mathcal{H}$, by Bézout's theorem it also contains $\mathcal{H}$, hence it is reducible. Then by Theorem 2.3 the Apollonian cubic is the equilateral hyperbola $\mathcal{H}$ and the straight line ℓ_∞ at the infinity.

As the straight line ℓ_∞ is contained by $\mathcal{A}$, the straight lines AB and CD are the opposite sides of a parallelogram, say it is $ABCD$. Further, the pairs $\{A, B\}$ and $\{C, D\}$ are on the same branch of the hyperbola $\mathcal{H}$ if and only if the segments $\overline{AB}$ or $\overline{CD}$ subtend acute visual angle at the centre of the parallelogram $ABCD$.



(2) As $\mathcal{A}$ contains an arc $\hat{\mathcal{C}}$ of the circle $\mathcal{C}$, by Bézout's theorem it also contains $\mathcal{C}$, hence it is reducible. Then by Theorem 2.3 $\mathcal{A}$ is the circle $\mathcal{C}$ and a straight line ℓ through its centre.

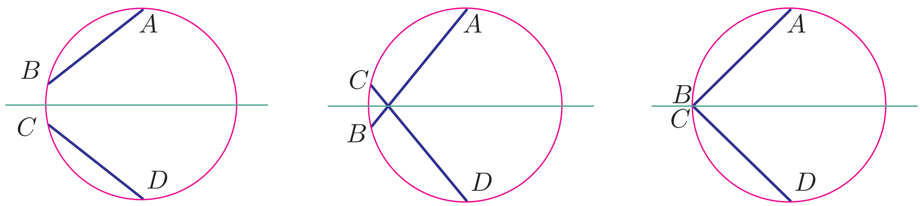
(3) As $\mathcal{A}$ contains the open segment $\hat{\ell}$ of the straight line ℓ, by Bézout's theorem it also contains ℓ, hence it is reducible. Then by Theorem 2.3 the Apollonian cubic can be of four types, that we are considering one-by-one.

Observe first that $\mathcal{A}$ contains the points A, B, C, D, M_0 and one of the points M_+ and M_-, and second that the segments subtend angles close to zero at the points of ℓ near infinity, therefore they subtend the same angle at those points, hence they have projections of equal length onto any straight line $\ell^\perp$ perpendicular to ℓ.

(a) In this case $\mathcal{A}$ is the union of the straight line ℓ and a nondegenerate circle $\mathcal{C}$ centred on ℓ. We consider subcases.

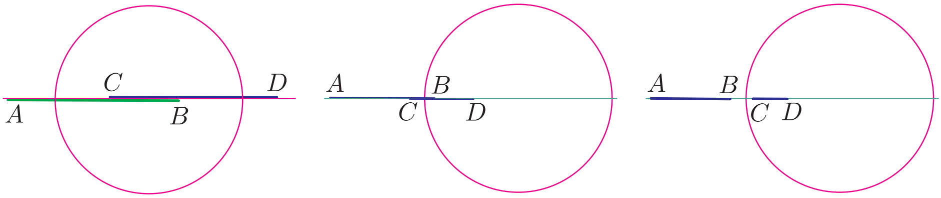
(a1) Assume that $A, B, C, D \in \mathcal{C}$ and all these points are different. Then $M_0 \in \ell$ and $M_+ \in \ell$ or $M_- \in \ell$. Suppose that M_0 is exterior to $\mathcal{C}$. As $ABCD$ is a cyclic quadrilateral in $\mathcal{C}$, $M_0AD\triangle$ is similar to $M_0CB\triangle$. Also the segments $\overline{AB}$ and $\overline{CD}$ are equal length chords of $\mathcal{C}$, hence by letting $x := d(A, B)$, $b := d(M_0, B)$ and $c := d(M_0, C)$ we get $(b + x)b = (c + x)c$, that is $(b - c)(b + c + x) = 0$. Thus $b = c$. If M_0 is interior to $\mathcal{C}$, then both $M_\pm$ are exterior to $\mathcal{C}$. Assume $M_+ \in \ell$. As $ABCD$ is a cyclic quadrilateral in $\mathcal{C}$, $M_+CD\triangle$ is similar to $M_+BA\triangle$. Also the segments $\overline{AC}$ and $\overline{BD}$ are equal length chords of $\mathcal{C}$, hence by letting $y := d(A, C)$, $a := d(M_+, A)$ and $d := d(M_+, D)$ we get $(a + y)a = (d + y)d$, that is $(a - d)(a + d + y) = 0$. Thus $a = d$. ($\Rightarrow$(3.1))

(a2) Assume that $A, B, C, D \in \mathcal{C}$ and $B = C$. The segments $\overline{AB}$ and $\overline{CD}$ are equal length chords of $\mathcal{C}$, and they also have equal length projection on $\ell^\perp$, hence they have also equal length projection on ℓ, and therefore BA and CD close the same (undirected) angle to ℓ. Thus, the point $B = C$ is on the bisector of $\overline{AD}$, which passes through the centre O of $\mathcal{C}$, hence $OBA\angle = OCD\angle$. This implies $OB = \ell$, hence the chord $\overline{AD}$ of $\mathcal{C}$ is perpendicular to ℓ, thus $B = C \in \ell$ and A is the reflection of D onto ℓ. ($\Rightarrow$(3.3))



(a3) Assume $A, B \in \ell$. Then also $C, D \in \ell$, and an easy calculation with (2.1) shows that $\mathcal{C}$ is the Apollonian circle of such different length segments $\overline{AB}$ and $\overline{CD}$

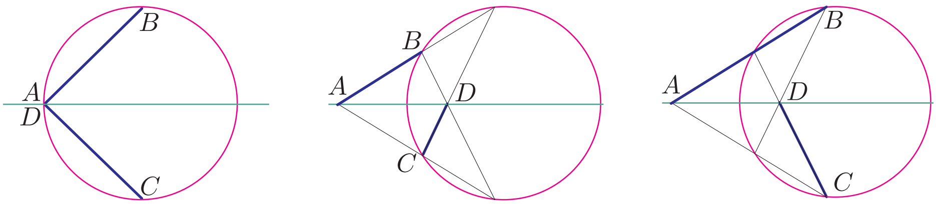
that neither one contains the other. (⇒(3.4))



(a4) Assume $A, D \in \ell$, $A = D$ and $B \in \mathcal{C} \setminus \ell$. Then also $C \in \mathcal{C} \setminus \ell$. The segments $\overline{AB}$ and $\overline{CD}$ subtend equal or supplementary angles at the projection $B^{\perp}$ of B to ℓ, which is $\pi/2$, so BC is perpendicular to ℓ. Since the projections of the segments are also of equal length, ℓ is the bisector of $\overline{BC}$. (⇒(3.3))

(a5) Assume $A, D \in \ell$, $A \neq D$ and $B \in \mathcal{C} \setminus \ell$. Then also $C \in \mathcal{C} \setminus \ell$. The segments $\overline{AB}$ and $\overline{CD}$ have equal length projection on any straight line $\ell^{\perp}$ perpendicular to ℓ, therefore B and C are in the same distance from ℓ. If $B = C$, an easy calculation with (2.1) shows that the other component of the Apollonian that contains the straight line is a degenerate circle with centre in $B = C$, therefore $B \neq C$, and the same reasoning as in (a4) implies that ℓ is the bisector of $\overline{BC}$. (⇒(3.2))

(a6) Assume $A, C \in \ell$ and $B \in \mathcal{C} \setminus \ell$. Then also $D \in \mathcal{C} \setminus \ell$, and a simple exchanging of the name of C and D leads us to (a4) or (a5) according to whether $A = C$ or $A \neq C$.



(b) In this case $\mathcal{A}$ is the union of the straight line ℓ and a degenerate circle $\mathcal{C} = \{P\} \not\subset \ell$.

(b1) If no points of A, B, C, D are in $\mathcal{C}$, then an easy calculation with (2.1) shows that the other component of the Apollonian that contains ℓ is nondegenerate circle or a straight line, therefore this case can not happen.

(b2) If $A, D \in \ell$ and $B \in \mathcal{C}$ then also $C \in \mathcal{C}$, and therefore $A \neq D$. (⇒(3.3))

(b3) If $A, C \in \ell$ and $B \in \mathcal{C}$ then also $D \in \mathcal{C}$, therefore this is just a renaming of (b2). (⇒(3.3))

(c) This case can not occur, because the straight line ℓ is not at the infinity.

(d) In this case $\mathcal{A}$ is the union of the straight line ℓ and straight line $\ell^{\perp}$ that is perpendicular to ℓ.

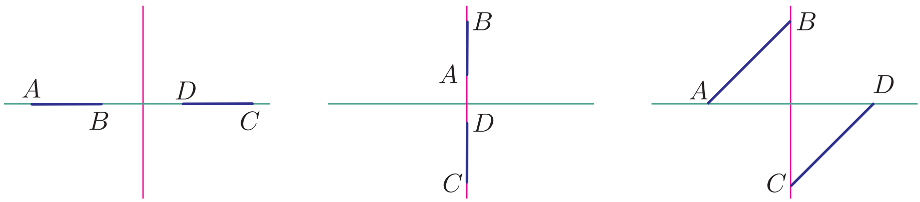
Observe that the segments subtend angles close to zero at the points of ℓ and $\ell^{\perp}$ near infinity, therefore the segments subtend the same angle at those points, hence they have projections of equal length onto any of the straight lines ℓ or $\ell^{\perp}$, and hence they have the same length.

(d1) Assume $\overline{AB} \subset \ell$. Then also $\overline{CD} \subset \ell$, and considering the angles the segments subtend at points of $\ell^{\perp}$ near $\ell \cap \ell^{\perp}$ shows that the segments are the reflection of each other with respect to $\ell^{\perp}$. ($\Rightarrow$(3.4))

(d2) Assume $\overline{AB} \subset \ell^{\perp}$. Then also $\overline{CD} \subset \ell^{\perp}$, and by following the reasoning in (d1), we get that the segments are the reflection of each other with respect to ℓ. ($\Rightarrow$(3.1))

(d3) Assume $A, D \in \ell$, $A = D$ and $B \in \ell^{\perp} \setminus \ell$. Then also $C \in \ell^{\perp} \setminus \ell$, and since $B \neq C$ the point C is the reflection of B with respect to ℓ. Then, as we saw earlier, $\mathcal{A}$ is $\ell^{\perp}$ and the circle through the points A, B, C that contradicts the assumption, therefore this case can not happen.

(d4) Assume $A, D \in \ell$, $A \neq D$ and $B \in \ell^{\perp} \setminus \ell$. Then also $C \in \ell^{\perp} \setminus \ell$, and as in (d3) we conclude that $B \neq C$. Since the segments have equal length corresponding projections, B and C can not be on the same side of ℓ, thus B and C are the reflections of each other with respect to ℓ, and in the same way we deduce that A and D are the reflections of each other with respect to $\ell^{\perp}$. This proves that $ABDC$ is a square. ($\Rightarrow$(3.2))



The proof is now completed. ∎

3. Distinguishability of segments

A set $\mathcal{P}$ of points in the plane is called *set of injectivity* if the coincidence of any two segments follows from the equality of the measures of their respective visual angles at every point of $\mathcal{P}$. Obviously, if a set is covered by an Apollonian curve, then it is not a set of injectivity. In this section we show sets of injectivity.

Denote the cubic polynomials on the left- and right-hand side of (2.2) by f and g, respectively, and observe that $f^2 = g^2$ satisfied exactly by the points of the two Apollonian curve. Since the degree of the polynomial equation $f^2 = g^2$ is 6, it has at most $\binom{6+2}{2} = 28$ independent coefficients, hence there can be chosen 29 points so that no polynomial equation of the form $f^2 = g^2$ can have them all as solutions. Thus, there are sets of injectivity that contain only 29 points, and the problem arises to

(3.1) *determine the minimal cardinality of a set of injectivity.*

The following theorem proves that the minimal cardinality of a set of injectivity can not be more than 16.

Theorem 3.1. *Let I be the intersection point of the straight lines ℓ_1 and ℓ_2 that are not orthogonal to each other. Let $P_1, P_2, P_3, P_4, P_5, P_6 \in \ell_1$ and $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \in \ell_2$ be different points outside I and finally let R_1, R_2, R_3 collinear points outside of $\ell_1 \cup \ell_2 \cup \ell_1^\perp \cup \ell_2^\perp$, where the straight lines $\ell_1^\perp, \ell_2^\perp$ are perpendicular to ℓ_1 and ℓ_2, respectively, and pass through I. If the nondegenerate segments $\overline{AB}$ and $\overline{CD}$ have equal or supplementary shadow pictures at every points of $\{I, P_1, \dots, P_6, Q_1, \dots, Q_6, R_1, R_2, R_3\}$, then they coincide.*

Proof. Assume that $\overline{AB}$ and $\overline{CD}$ are different.

As ℓ_1 has 7 common points with the Apollonian curves of $\overline{AB}$ and $\overline{CD}$, one of the Apollonian curve, say $\mathcal{A}_-$, has at least 4 points common with ℓ_1, hence by Bézout's theorem ℓ_1 is a component of $\mathcal{A}_+$.

By Theorem 2.3 we see, that outside of ℓ_1 the Apollonian curve $\mathcal{A}_-$ may intersect ℓ_2 in at most 2 points, hence $\mathcal{A}_+$ must intersect ℓ_2 in at least 4 points. Thus by Bézout's theorem ℓ_2 is a component of $\mathcal{A}_+$.

Thus both Apollonian curves $\mathcal{A}_\pm$ are of type (3) in our Theorem 2.4, and therefore we have to investigate the pairs of the cases (3.1)–(3.4).

If $\mathcal{A}_-$ is of (3.1) and $\mathcal{A}_+$ is of (3.1), then the rotation given by the reflections with respect to ℓ_1 and ℓ_2 is by angle 2π around I, hence $\ell_1 = \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.1) and $\mathcal{A}_+$ is of (3.2), then the kite $\mathcal{K}$ mentioned in (3.2) is also symmetric to ℓ_1, hence ℓ_1 is the middle line of $\mathcal{K}$, hence $\mathcal{K}$ is a square, and therefore $\mathcal{A}_-$ is ℓ_1 and the circumscribed circle, $\mathcal{A}_+$ is ℓ_2 and the straight line of the other diagonal of $\mathcal{K}$.

If $\mathcal{A}_-$ is of (3.1) and $\mathcal{A}_+$ is of (3.3), then the kite $\mathcal{K}$ mentioned in (3.3) is also symmetric to ℓ_1, hence ℓ_1 and ℓ_2 are the diagonals of $\mathcal{K}$, that implies $\ell_1 \perp \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.1) and $\mathcal{A}_+$ is of (3.4), then $\ell_1 \perp \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.2) and $\mathcal{A}_+$ is of (3.2), then ℓ_1 and ℓ_2 are the diagonals of the kite $\mathcal{K}$ mentioned in (3.2), that implies $\ell_1 \perp \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.2) and $\mathcal{A}_+$ is of (3.3), we have a contradiction, as the segments can not be opposite and adjoining sides of a kite at once.

If $\mathcal{A}_-$ is of (3.2) and $\mathcal{A}_+$ is of (3.4), then ℓ_1 intersects ℓ_2 in two points, hence $\ell_1 = \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.3) and $\mathcal{A}_+$ is of (3.3), then the kite $\mathcal{K}$ mentioned in (3.3) is symmetric to ℓ_1 and ℓ_2, hence ℓ_1 and ℓ_2 are the diagonals of $\mathcal{K}$, that implies $\ell_1 \perp \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.3) and $\mathcal{A}_+$ is of (3.4), then ℓ_1 intersects ℓ_2 in two points, hence $\ell_1 = \ell_2$, which is a contradiction.

If $\mathcal{A}_-$ is of (3.4) and $\mathcal{A}_+$ is of (3.4), then $\ell_1 = \ell_2$, which is a contradiction.

In sum, the only case with no contradiction to the conditions is the case that $\mathcal{A}_-$ is of (3.1) and $\mathcal{A}_+$ is of (3.2), in which case $\mathcal{A}_-$ is ℓ_1 and a circle $\mathcal{C}$, and $\mathcal{A}_+$ is ℓ_2 and a straight line $\ell_2^\perp$ orthogonal to ℓ_2 passing through the centre of $\mathcal{C}$.

But the points R_1, R_2, R_3 are collinear and they are outside of $\ell_1 \cup \ell_2 \cup \ell_1^\perp \cup \ell_2^\perp$, hence they can not be covered by ℓ_1, $\mathcal{C}$,ℓ_2 and $\ell_2^\perp$. This proves the theorem. ∎

To describe the sets of injectivity is a very different task. A first step to have such decsription is the following improvement of [4, Lemma 2.1, Lemma 3.2].

Theorem 3.2. *Assume that two closed segments subtend equal angles at the points of the border $\partial\mathcal{D}$ of an open domain $\mathcal{D}$.*

(1) *If the straight lines of the segments do not intersect $\mathcal{D}$, they coincide.*
(2) *If the segments do not end on $\partial\mathcal{D}$, then one of the followings is valid:*
   (2a) *the segments are collinear with two-two endpoints inside and outside of $\mathcal{D}$, and $\partial\mathcal{D}$ is a circle or a straight line;*
   (2b) *$\mathcal{D}$ is a half plane and the segments are the reflections of each other with respect to $\partial\mathcal{D}$;*
   (2c) *$\mathcal{D}$ is an equilateral quadrant and the segments are the reflections of each other with respect to the straight lines of the border of $\partial\mathcal{D}$.*

Proof. Denote the segments by $\overline{AB}$ and $\overline{CD}$. Let ν_{AB} and ν_{CD} be the visual angle functions of them, respectively, and let $\nu = \nu_{AB} - \nu_{CD}$. By the conditions ν vanishes on the border $\partial\mathcal{D}$ of $\mathcal{D}$.

(1) Since the straight lines of the segments do not intersect $\partial\mathcal{D}$, ν_{AB} and ν_{CD} are harmonic functions on $\mathcal{D}$ (see Lemma A.1 in the Appendix), hence ν is also harmonic on $\mathcal{D}$. As ν vanishes on $\partial\mathcal{D}$, the maximum principle of harmonic

functions implies that ν vanishes on the whole domain $\mathcal{D}$. Now Theorem 3.1 implies the coincidence of the segments by choosing suitable points in $\mathcal{D}$.

(2) We have $\{A,B,C,D\} \cap \partial\mathcal{D} = \emptyset$.

If $AB = CD$, then (2a) follows from Theorem 2.4, therefore we assume $AB \not\equiv CD$ from now on.

Knowing (1) we conclude that one of the straight lines AB and CD, say AB, intersects $\partial\mathcal{D}$ in at least one point P. Then the straight line CD also passes through P, because $\nu_{CD}(P) = \nu_{AB}(P) = 0$. As $AB \not\equiv CD$ we infer $P = M_0$.

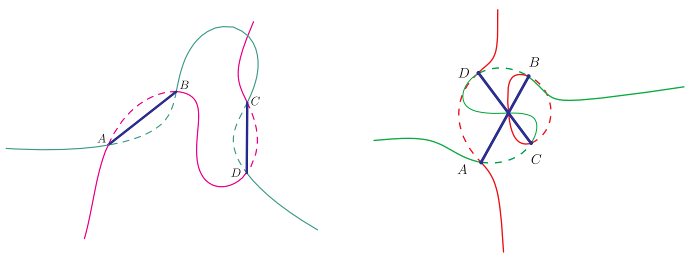
Since $\overline{AB} \not\equiv \overline{CD}$, the equioptic $\mathcal{E}$ is the union of subarcs of the Apollonian curves $\mathcal{A}_+$ and $\mathcal{A}_-$ of the segments $\overline{AB}$ and $\overline{CD}$, and therefore

$$(AB\cup CD)\cap\partial\mathcal{D} \subseteq (AB\cup CD)\cap\mathcal{E} \subseteq (AB\cup CD)\cap(\mathcal{A}_+\cup\mathcal{A}_-) \subseteq \{A,B,C,D,M_0\}.$$

As also $\partial\mathcal{D}\cap\{A,B,C,D\} = \emptyset$, we obtain $(AB\cup CD)\cap\partial\mathcal{D} = \{M_0\}$.

The straight lines AB and CD divides the plane into four open quadrants $\mathcal{Q}_{-2}$, $\mathcal{Q}_{-1}$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$, where the quadrants are indexed so that $\partial\mathcal{Q}_i \cap \partial\mathcal{Q}_{-i} = \{M_0\}$ $(i = 1,2)$. It clearly follows then that $\partial\mathcal{D} \subset \{M_0\}\cup\mathcal{Q}$, where $\mathcal{Q}$ is the union of two quadrants. These two quadrants are adjacent, if there is a neighbourhood of M_0 in which $\partial\mathcal{D}$ is covered by only one Apollonian curve, and they are opposite, if in any neighbourhood of M_0 both Apollonian curves are necessary to cover $\partial\mathcal{D}$.

The point M_0 is simultaneously inner or external for both segments $\overline{AB}$ and $\overline{CD}$, because otherwise the arcs of $\mathcal{A}_+$ and $\mathcal{A}_-$ near $M_0 \in \mathcal{E}$ would be on the compoptic and therefore $\partial\mathcal{D}$ could not pass M_0.



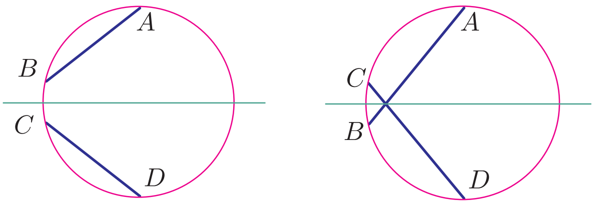
According to Observation 2.2 the intersection of an Apollonian curve wit a quadrant $\mathcal{Q}_i$ $(i = \pm1,\pm2)$ is the intersection of the quadrant $\mathcal{Q}_i$ with either the equioptic or the compoptic, hence the curves $\partial\mathcal{D}\cap\mathcal{Q}_i$ are covered by exactly one of the Apollonian curves for each $i = \pm1,\pm2$.

One of the quadrants that covers "half" of $\partial\mathcal{D}$, say $\mathcal{Q}_1$, has in its border (two rays ℓ_+ and ℓ_- of the straight lines CD and AB, respectively, both starting at M_0) one of the points A,B,C,D, say it is D.

In the quadrant $\mathcal{Q}_1$ the Apollonian curve $\mathcal{A}$ (one of $\mathcal{A}_\pm$) that covers $\partial\mathcal{D}$ passes through the points M_0 and D and it has an asymptotic ray ℓ_+. As $\partial\mathcal{D}$ is the border of the open domain $\mathcal{D}$, the curve $\partial\mathcal{D}\cap\mathcal{Q}_1$ is connected. Denoting the point of $\mathcal{A}_1 := \mathcal{A}\cap\mathcal{Q}_1$ at the infinity by A_∞ we conclude that M_0 and A_∞ is in one component of $\bar{\mathcal{A}}_1 := \mathcal{A}\cap(\mathcal{Q}_1\cup\partial\mathcal{Q}_1)$ and also $C\in\bar{\mathcal{A}}_1$.

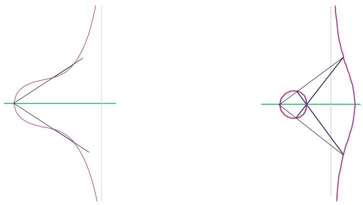
We proceed by considering different cases.

(2.1) *Assume $\mathcal{A}$ is reducible.* Then by Theorem 2.3 the Apollonian curve $\mathcal{A}$ is the straight line $\ell := M_0A_\infty$ and a circle $\mathcal{C}$ centred at a point on ℓ. As $M_0 \notin \{A,B,C,D\}$ Theorem 2.4 proves that $\overline{AB}$ is the reflection of $\overline{CD}$ with respect to ℓ (we may assume that ℓ is the bisector of $\overline{AD}$ and $\overline{BC}$).



Then the asymptotic straight line $\ell^\perp$ of the other Apollonian curve $\mathcal{A}_\pm$ of the segments is perpendicular to ℓ, and it goes through M_0 if and only if $\overline{AB}$ is the reflection of $\overline{CD}$ with respect to $\ell^\perp$, and we get the configurations described in (2b) and (2c).

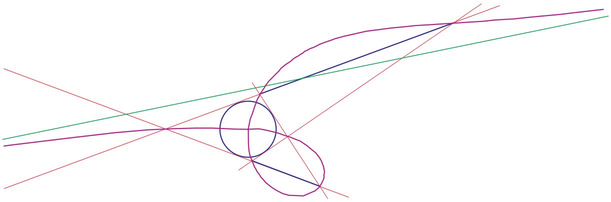
If $\ell^\perp$ does not pass M_0, then it can not intersect $\mathcal{A}_\pm$, because otherwise by symmetry it would have two intersections with $\mathcal{A}_\pm$ that counted together with its second order intersection B_∞ with $\mathcal{A}_\pm$ at infinity would imply $\mathcal{A}_\pm \equiv \ell^\perp$ by Bézout's theorem.



If $\ell^\perp$ does not intersect the segments, then the strip between AD and BC separates B_∞ from M_0 in $\mathcal{Q}_{\pm 2}$, because AD and BC intersect $\mathcal{A}_\pm$ in exactly three points A, D, B_∞ and B, C, B_∞, respectively. If $\ell^\perp$ intersects the segments, then it separates B_∞ from M_0. By this disconnectedness in both cases, the arc of $\mathcal{A}_\pm$ starting from B_∞ can not be in the border $\partial\mathcal{D}$ of $\mathcal{D}$, hence we get the configuration described in (2b).

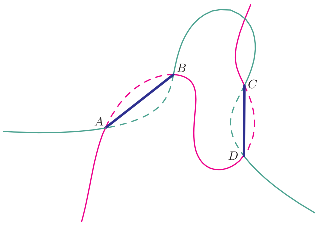
(2.2) *Assume $\mathcal{A}$ is irreducible and has a double point*, i.e. it is singular [12]. Then by [12, 5.7. Theorem] the straight lines AB, CD and either AC and BD

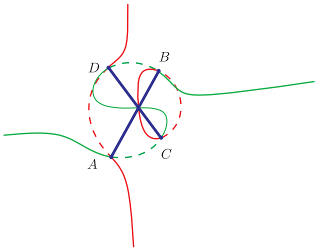
or AD and BC are tangent to the same circle $\mathcal{C}$ centred to the double point S. Without loss of generality we may assume that the straight lines AB, CD, AC and BD are tangent to the circle $\mathcal{C}$.



Let ℓ be the asymptotic straight line of $\mathcal{A}$ and let $\ell^{\|}$ be the straight line through S that is parallel to ℓ. The straight line $\ell^{\|}$ intersects $\mathcal{A}$ exactly in S, but the other straight lines through S intersect $\mathcal{A}$ in exactly one more point. This means that the connected part of $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{Q}_1$ that contains A_∞ is in the domain $\mathcal{F}$ of the angle $DS\ell^{\|}\angle$. This domain does not contains M_0 and $\mathcal{A}$ intersects its border exactly in the point D. This proves that M_0 and $\mathcal{A}$ cannot be in one component of $\mathcal{A}_1$, hence this case can not happen.

(2.3) *Assume that $\mathcal{A}$ is irreducible, has no double point and has only one component.* As $\mathcal{A}$ intersects the straight line CD in three points, it has a bounded and an unbounded (asymptotic) component (arc) on both sides of CD, and these components are separated, because $\mathcal{A}$ does not have double point.





Either M_0 separates D and C, or C separates M_0 and D, or D separates M_0 and C.

If M_0 separates the points C and D, then it is on the bounded arc of $\mathcal{A}$ between C and D, and A_∞ is on the unbounded components of $\mathcal{A}_1$, hence M_0 and A_∞ can not be in a common component of $\mathcal{A}_1$.

If C (or D) separates the points M_0 and D (C, respectively), then it is on the bounded arc of $\mathcal{A}$ between M_0 and D (C, respectively), and therefore M_0 and A_∞ is in a common component of $\mathcal{A}_1$ if the border of the quadrant $\mathcal{Q}_1$ contains neither C nor D. This contradicts the assumption that $D \in \partial\mathcal{Q}_1$, hence M_0 and A_∞ can not be in a common component of $\mathcal{A}_1$.

(2.4) *Assume that $\mathcal{A}$ is irreducible, has no double point and consists of two components.* As $\mathcal{A}$ intersects the straight line CD in three points, two of these points are on the bounded component, and the third one is on the unbounded. As M_0 and A_∞ are in the same component of $\mathcal{A}_1$, they are also in a common component of $\mathcal{A}$, hence M_0 is in the unbounded component of $\mathcal{A}$. Using the above reasoning, we get that the bounded component of $\mathcal{A}$ contains the points A, B, C, D which is clearly a contradiction[4]. This contradiction shows that M_0 and A_∞ can not be in a common component of $\mathcal{A}_1$.

The theorem is now completely proven. ∎

Note that Theorem 3.2 fails without the condition $\{A, B, C, D\} \cap \partial\mathcal{D} = \emptyset$ as shown[5] by the figures in Lemma 2.1. Nevertheless we obtain the following as a corollary.

Theorem 3.3. *If the centres of three circles of equal radius ϱ form a triangle so that every heights of that triangle is bigger than 2ϱ, then the union of the circles constitute a set of injectivity.*

For the proof we only have to observe, that a segment can intersect only two of the circles.

A. Appendix

Let $\mathcal{B}$ be a convex body with the boundary $\partial\mathcal{B}$, and let $g \colon [-1, 1] \to \mathbb{R}^2$ be a C^2 curve outside $\mathcal{B}$, that is parametrized by arc length and has no tangents meeting $\mathcal{B}$. Denote the tangent line, the tangent vector and the curvature of the curve g at $g(s)$, by $\ell(s)$, $\mathbf{t}(s)$ and $\kappa(s)$, respectively. Let $\mathcal{T}^a(s)$ and $\mathcal{T}^b(s)$ be the tangents of $\mathcal{B}$ through the point $g(s)$ so that their respective unit directional vectors $\mathbf{t}^a(s)$, $\mathbf{t}^b(s)$ pointing toward $\mathcal{B}$ are such that $\mathbf{t}^a(s)$ is a positive linear combination of $\mathbf{t}^b(s)$ and $\mathbf{t}(s)$. Let $\alpha(s)$ and $\beta(s)$ be the respective angles of $\mathbf{t}^a(s)$ and $\mathbf{t}^b(s)$ to $\mathbf{t}(s)$. Then $0 < \alpha(s) < \beta(s) < \pi$ and $\nu(s) = \beta(s) - \alpha(s)$ is the shadow picture of $\mathcal{B}$ at $g(s)$.

Let $\mathcal{A}(s) = \mathcal{B} \cap \mathcal{T}^a(s)$ and $\mathcal{B}(s) = \mathcal{B} \cap \mathcal{T}^b(s)$ and define the points $A^\pm(s) := \lim_{x \searrow 0} \mathcal{A}(s \pm x)$, $B^\pm(s) := \lim_{x \searrow 0} \mathcal{B}(s \pm x)$. Their respective distances from $g(s)$ are $a^\pm(s) = |A^\pm(s) - g(s)|$ and $b^\pm(s) = |B^\pm(s) - g(s)|$.

[4] Using the main involution [12, 4.6.] and [12, 4.3. Theorem] one sees $\{A, B\}^* = \{CD\}$ (and $\{A, B\} = \{CD\}^*$), and by [12, 4.19.] this implies that the points A, B, C, D can not be in the same component.

[5] Look for the points near A and C on the first and second figures, or the points near the segment $\overline{AB}$ on the third figure.

If $A^+(s) = A^-(s)$ and $B^+(s) = B^-(s)$, then let $A(s) = A^-(s)$, $B(s) = B^-(s)$, $a(s) = a^-(s)$, $b(s) = b^-(s)$ and let $\kappa^a(s)$, $\kappa^b(s)$ be the (maybe infinite) curvatures of $\partial\mathcal{B}$ at $A(s)$ and $B(s)$, respectively.

If more curves occur, then they and their objects are indexed consequently. A function without argument is understood at the appropriate parameter $s \in [-1,1]$.

Lemma A.1. *Let $\mathcal{B}$ be a convex body and let $\Sigma\mathcal{B}$ denote the union of the straight lines of the segments in $\partial\mathcal{B}$.*

(1) *The visual angle function of $\mathcal{B}$ is subharmonic in $\mathbb{R}^2 \setminus \mathcal{B} \setminus \Sigma\mathcal{B}$.*
(2) *The visual angle function of $\mathcal{B}$ is harmonic at a point $P \in \mathbb{R}^2 \setminus \mathcal{B} \setminus \Sigma\mathcal{B}$ if and only if every tangent of $\mathcal{B}$ through P touches $\mathcal{B}$ in a singular point.*

Proof. Let $P \in \mathbb{R}^2 \setminus \mathcal{B} \setminus \Sigma\mathcal{B}$ and take two orthogonal straight lines g_1 and g_2 through P so that $g_1(0) = g_2(0) = P$. Then $A_i^+(0) = A_i^-(0)$ and $B_i^+(0) = B_i^-(0)$ $(i = 1, 2)$, and therefore the second equation in [6, Lemma 1] implies

$$\begin{aligned}\Delta\nu(P) &= \ddot{\nu}_1(0) + \ddot{\nu}_2(0)\\ &= \frac{\sin 2\beta_1}{b^2} - \frac{\sin 2\alpha_1}{a^2} + \frac{\sin^2\beta_1}{b^3\kappa^b} + \frac{\sin^2\alpha_1}{a^3\kappa^a} +\\ &\quad + \frac{\sin(2\beta_1+\pi)}{b^2} - \frac{\sin(2\alpha_1+\pi)}{a^2} + \frac{\sin^2(\beta_1+\pi/2)}{b^3\kappa^b} + \frac{\sin^2(\alpha_1+\pi/2)}{a^3\kappa^a}\\ &= \frac{1}{b^3\kappa^b} + \frac{1}{a^3\kappa^a}.\end{aligned}$$

This proves statement (1), because b, a, κ^b and κ^a are nonnegative functions. Statement (2) is justified because $\kappa^b = \kappa^a = +\infty$ if and only if the tangents goes through singular points of $\mathcal{B}$. ■

References

[1] E. Brieskorn and H. Knörrer, *Plane algebraic curves*, Birkhäuser Verlag, Basel, 1986.

[2] R. J. Gardner, *Geometric tomography*, Encyclopedia of Math. and its Appl. **58**, Cambridge University Press, Cambridge, 2006 (first edition in 1996).

[3] J. Kincses, The determination of a convex set from its angle function, *Discrete Comput. Geom.*, **30** (2003), 287–297.

[4] J. Kincses and Á. Kurusa, Can you recognize the shape of a figure from its shadows?, *Beiträge zur Alg. und Geom.*, **36** (1995), 25–34.

[5] J. Kincses, An example of a stable, even order Quadrangle which is determined by its angle function, *Discrete Geometry, in honor of W. Kuperberg's 60th birthday* (ed.: A. Bezdek), CRC Press (Marcel Dekker), New York – Basel, 2003, 367–372.

[6] Á. Kurusa, You can recognize the shape of a figure by its shadows!, *Geom. Dedicata*, **59** (1996), 103–112.

[7] Á. Kurusa, The shadow picture problem for nonintersecting curves, *Geom. Dedicata*, **59** (1996), 113–125.

[8] Á. Kurusa, Is a convex plane body determined by an isoptic?, *Beiträge Algebra Geom.*, **53** (2012), 281–294; DOI: 10.1007/s13366-011-0074-2.

[9] Á. Kurusa, Equioptics of segments: generalizing Apollonius' theorem, *Polygon*, **21** (2013), 43–57 (in hungarian: "*Szakaszok ekvioptikusai: Apollóniosz tételének általánosítása*").

[10] Á. Kurusa, Visual distinguishability of polygons, *Beiträge Algebra Geom.* (2013), DOI: 10.1007/s13366-012-0121-7.

[11] P. Pamfilos and A. Thoma, Apollonian cubics: An application of group theory to a problem in Euclidean geometry, *Mathematics Magazine*, **72** (1999), 356–366.

[12] P. Pamfilos, *Theory of Isoptic cubics*, Help file of Isoptikon program that is freely available at http://www.math.uoc.gr/∼pamfilos/#iso , 1998.

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