Translation Invariant Radon Transforms

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Abstract. E. T. Quinto proved that for a generalized Radon transform R on $\mathbb{R}^n$ the translation invariance of the operator $R^t \circ R$ implies the invertibility of R. In this paper an other concept of the translation invariance is defined. We investigate the relation of these two concepts and determine the translation invariant Radon transforms to be a certain generalization of the Tretiak-Metz exponential Radon transform. Finally we give inversion formula and prove the support theorem for these transforms.

1. Introduction

Since in the first part of this century Radon created the classical Radon transform on points and lines in $\mathbb{R}^2$ several new Radon transforms was invented and two main ranges of the investigations of the Radon transforms have been developed. In the first one the Radon transforms are concretely considered [2] while in the other the classes of the Radon transforms are studied [6]. Our paper belongs to the second ranges. We investigate the class of translation invariant generalized Radon transforms on $\mathbb{R}^n$.

Our concept for translation invariance is based on the well known identity $Rf_a(\omega, p) = Rf(\omega, p + \langle a, \omega\rangle)$ for classical Radon transform [2]. There is another concept in [6], where Quinto defined the translation invariance by the translation invariance of $R^t \circ R$. However it is obvious that Quinto's definition covers a larger class in general then ours, we show out that the double fibration model to construct the generalized Radon transform introduced by Gelfand and used in [6] restricts his results for a smaller class.

One goal of this paper is to prove in Section 3. that the translation invariant Radon transforms, by our concept, are not other but a certain generalization of the Tretiak-Metz exponential Radon transforms.

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Our another goal in Section 4. is to give inversion formula, which proves the injectivity of our transforms, and the so-called support theorem.

These latter results are only known in dimension two [3,8] and may prove useful in the practice as A. Markoe [5] proved that the inversion of the variably attenuated X-ray, which occurs in single photon emission tomography, can be reduced to the inversion of the exponential transform. See [7] for application in diagnostic medicine. On the other hand the exponential Radon transform can be regarded as a first order approximation to a general attenuation.

2. Preliminaries and definitions

Let $\mu \in C^\infty(\mathbb{R}^n \times S^{n-1} \times \mathbb{R})$ be a strictly positive function such that $\mu(x,\omega,p) = \mu(x,-\omega,-p)$. Then the generalized Radon transform is

$$(1) \qquad R_\mu\colon D(\mathbb{R}^n) \to D(S^{n-1}\times\mathbb{R}), \quad R_\mu f(\omega,p) = \int_{H(\omega,p)} f(x)\mu(x,\omega,p)dx,$$

where $H(\omega,p) = \{x \in \mathbb{R}^n \colon \langle x,\omega\rangle = p\}$ is the hyperplane with normalvector $\omega \in S^{n-1}$ and distance p from the origin. $\mathrm{d}x$ is the surface element of $H(\omega,p)$ and $\langle .,.\rangle$ is the standard inner product on $\mathbb{R}^n$.

Let $\lambda \in C^\infty(\mathbb{R}^n \times S^{n-1} \times \mathbb{R})$ be a strictly positive function such that $\lambda(x,\omega,p) = \lambda(x,-\omega,-p)$. Then the generalized Radon transform is

$$(2) \ \ R^t_\lambda\colon D(S^{n-1}\times\mathbb{R}) \to C^\infty(\mathbb{R}^n), \quad R^t_\lambda f(x) = \int_{S^{n-1}} f(\omega,\langle\omega,x\rangle)\lambda(x,\omega,\langle\omega,x\rangle)d\omega,$$

where $d\omega$ is the surface measure on S^{n-1}. One can see here that our starting situation is more general then E. T. Quinto's at (20) and (21) in [6].

We call a generalized Radon transform exponential if

$$\mu(x,\omega,p) = \mu_1(\omega,p) \times \exp(\langle\mu_2(\omega),x\rangle),$$

where $\mu_1 \in C^\infty(S^{n-1}\times\mathbb{R})$ and $\mu_2\colon S^{n-1} \overset{C^\infty}{\to} \mathbb{R}^n$. Similarly R^t_λ is called exponential if

$$\lambda(x,\omega,p) = \lambda_1(\omega,p) \times \exp(\langle\lambda_2(\omega),x\rangle),$$

where $\lambda_1 \in C^\infty(S^{n-1}\times\mathbb{R})$ and $\lambda_2\colon S^{n-1} \overset{C^\infty}{\to} \mathbb{R}^n$.

Now we present our concept for translation invariance of generalized Radon transforms. Our idea is based on two simple observations on the classical Radon transform R and R^t (see [2] and [4]) :

$$Rf_a(\omega,p) = Rf(\omega,p+\langle\omega,a\rangle) \ \text{ and } (R^tf)_a = R^t(f(\omega,p+\langle\omega,a\rangle)),$$

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where f_a denotes the translation of the function f with $a \in \mathbb{R}^n$. The following definitions generalize these properties. We call R_μ respectively R^t_λ translation invariant, if there is a ν respectively η in $C^\infty(\mathbb{R}^n \times S^{n-1} \times \mathbb{R})$ for which

$$R_\mu f_a(\omega, p) = \nu(a, \omega, p) R_\mu f(\omega, p + \langle \omega, a\rangle)$$

respectively

$$(R^t_\lambda f)_a = R^t_\lambda(f(\omega, p + \langle \omega, a\rangle)\eta(a, \omega, p)).$$

In the classical case $\nu \equiv \eta \equiv 1$. At the same time it is obvious that if $\nu \equiv \eta$ then the operator $R^t \circ R$ is translation invariant.

3. Determination of the translation invariant transforms

Theorem 3.1. *The generalized Radon transformation R_μ on $C^\infty_c(\mathbb{R}^n)$ is translation invariant if and only if it is exponential.*

Proof. Because of the simplicity of the sufficiency we only prove the necessity. It is immediate from our definition of translation invariance that

$$\mu(x - a, \omega, p) = \nu(a, \omega, p)\mu(x, \omega, p + \langle \omega, a\rangle), \tag{3}$$

where $x \in H(\omega, p + \langle \omega, a\rangle)$. Let $a = \alpha\omega + \beta\bar{\omega}_a$ and $x = p\omega + a + \kappa\bar{\omega}_x$, where $\bar{\omega}_a$ and $\bar{\omega}_x$ are perpendicular to $\omega \in S^{n-1}$. Taking $\kappa = 0$ one can reject $\nu(a, \omega, p)$ from (3). For $\alpha = 0$ and $\bar{\omega}_a = \bar{\omega}_x$ the result is

$$\frac{\mu(p\omega + (\beta + \kappa)\bar{\omega}_a, \omega, p)}{\mu(p\omega, \omega, p)} = \frac{\mu(p\omega + \beta\bar{\omega}_a, \omega, p)}{\mu(p\omega, \omega, p)} \times \frac{\mu(p\omega + \kappa\bar{\omega}_a, \omega, p)}{\mu(p\omega, \omega, p)}.$$

Thus the map

$$\beta \to \mu(p\omega + \beta\bar{\omega}_a, \omega, p)/\mu(p\omega, \omega, p)$$

is a continuous homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_+, \bullet)$ and so we have

$$\mu(p\omega + \beta\bar{\omega}_a, \omega, p) = \bar{\mu}_1(\omega, p)\exp(\beta\bar{\mu}(\bar{\omega}_a, \omega, p)),$$

where $\bar{\mu}_1(\omega, p) = \mu(p\omega, \omega, p)$. Writing back this formula into the previous equation and letting $\bar{\omega}_a \neq \bar{\omega}_x$ we get the homogenity of $\bar{\mu}$ in its first parameter *i.e.* $\beta\bar{\mu}(\bar{\omega}_a, \omega, p) = \langle \beta\bar{\omega}_a, \bar{\mu}_2(\omega, p)\rangle$, where $\bar{\mu}_2(\omega, p) \perp \omega$. Again let $a = \alpha\omega$ and write back μ in its present form into (3). We obtain

$$\bar{\mu}_1(\omega, p)\exp(\langle \kappa\bar{\omega}_x, \bar{\mu}_2(\omega, p)\rangle) = \nu(\alpha\omega, \omega, p)\bar{\mu}_1(\omega, p + \alpha)\exp(\langle \kappa\bar{\omega}_x, \bar{\mu}_2(\omega, p + \alpha)\rangle).$$

Because of the dependence on κ this implies the orthogonality of $\bar{\omega}_x$ and $\bar{\mu}_2(\omega, p+\alpha) - \bar{\mu}_2(\omega, p)$. Since $\bar{\mu}_2 \perp \omega$ and $\bar{\omega}_x$ is arbitrary in $\omega^\perp$, the orthogonal complement of ω, this yields to the independence of $\bar{\mu}_2$ from its second argument. Let $\tilde{\mu}_2(\omega) = \bar{\mu}_2(\omega, 0)$, $\hat{\mu}_2(\omega) \in C(S^{n-1})$, $\mu_2(\omega) = \tilde{\mu}_2(\omega) + \omega\hat{\mu}_2(\omega)$ and $\mu_1(\omega, p) = \bar{\mu}_1(\omega, p)\exp(-p\tilde{\mu}_2(\omega))$. Then we conclude

$$\mu(x, \omega, p) = \mu_1(\omega, p)\exp(\langle \mu_2(\omega), x\rangle),$$

which was to be proved.

■

Theorem 3.2. *The generalized dual Radon transformation R^t_λ on $C^\infty_c(S^{n-1} \times \mathbb{R})$ is translation invariant if and only if it is exponential.*

This theorem can be proven by imitation of the previous proof so we leave it to the reader. The following theorem shows how to choose R^t_λ so that the operator $R^t_\lambda \circ R_\mu$ is translation invariant.

Theorem 3.3. *If R_μ and R^t_λ are exponential, the operator $R^t_\lambda \circ R_\mu$ is translation invariant if and only if $\mu_1(\omega, p)\lambda_1(\omega, p)$ does not depend on p and $\mu_2(\omega) + \lambda_2(\omega) \equiv 0$.*

Proof. It is clear that $R^t_\lambda \circ R_\mu$ is translation invariant if and only if $\nu \equiv \eta$ *i.e.*

$$\frac{\mu_1(\omega, p)\exp(\langle -\mu_2(\omega), a\rangle)}{\mu_1(\omega, p + \langle a, \omega\rangle)} = \frac{\lambda_1(\omega, p + \langle a, \omega\rangle)\exp(\langle \lambda_2(\omega), a\rangle)}{\lambda_1(\omega, a)}.$$

Let $a = \alpha\omega + \beta\bar{\omega}$, where $\omega \perp \bar{\omega}$. The dependence on β gives our second statement from which the other statement of the theorem easily follows.

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We note here that Quinto has got by his Proposition 4.1. of [6] the following formulas in (20) and (21) for the translation invariant Radon transforms

$$Rf(\omega, p) = \int_{H(\omega,p)} f(x)\mu_1(\omega, p)\exp(\langle z, x\rangle)dx, \tag{20*}$$

$$R^t f(x) = \exp(\langle -z, x\rangle)\int_{S^{n-1}} f(\omega, \langle \omega, x\rangle)\frac{a(\omega)}{\mu_1(\omega, \langle \omega, x\rangle)}d\omega, \tag{21*}$$

where we used Quinto's notations and $\mu_1(\omega, p) = \sqrt{m(0)a(\omega)/n(\langle \omega, p\rangle)}$. Since z constant this is clearly less general than our formulas. In the proof of the following theorem we will show the point where the double fibration model, used by Quinto, simplifies his result.

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Theorem 3.4. *The operator $R^t_\lambda \circ R_\mu$ is translation invariant if and only if for all $y \in \mathbb{R}^n$ and $\omega \in y^\perp \cap S^{n-1}$ the function*

$$\lambda(x, \omega, \langle\omega, x\rangle)(\mu(x + y, \omega, \langle\omega, x\rangle) + \mu(x - y, \omega, \langle\omega, x\rangle)),$$

does not depend on x for $x \perp y$.

Proof. We follow Quinto's method and notations [6]. Let K be $SO(n)$, L be the isotropy subgroup of $e_x = x/|x|$ and M be the isotropy subgroup of $e \in S^{n-1} \cap e_x^\perp$. Furthermore let dk, dl and dm be the invariant measures on these groups with total measure one. Then

$$R^t_\lambda \circ R_\mu f(x) = \int_K^\infty \int_0^\infty \int_L^\infty (r^{n-2} f(x + rkle)\lambda(x, ke_x, \langle ke_x, x\rangle)\times \\ \times \mu(x + rkle, ke_x, \langle ke_x, x\rangle))dldrdk,$$

where $C = |S^{n-1}||S^{n-2}|$. Let $\bar{a}(x, y, \omega) = \lambda(x, \omega, \langle\omega, x\rangle)\mu(x + y, \omega, \langle\omega, x\rangle)$. Using $\bar{a}$, reversing the integrations with respect to dl and dk, the substitution kl^{-1} for *kgives*

$$R^t_\lambda \circ R_\mu f(x) = C \int_0^\infty \int_K (r^{n-2} f(x + rke)\bar{a}(x, rke, ke_x))dk\, dr$$

by the right invariance of dk. Let dk_m be the K invariant measure on K/M satisfying $dk = dm\, dk_m$. Then

$$R^t_\lambda \circ R_\mu f(x) = C \int_0^\infty \int_{K/M} \Big(r^{n-2} f(x + rke) \int_M \bar{a}(x, rke, ke_x)dm\Big)dk_m dr. \tag{4}$$

The inner integral over M multiplied by $|S^{n-2}|$ is clearly the integral of $\bar{a}(x, rke, ke_x)$ over the great sphere $S^{n-1} \cap H(ke, 0)$ in its standard measure. We denote it by $\hat{a}(x, rke, ke)$. Thus we have

$$R^t_\lambda \circ R_\mu f(x) = \int_{\mathbb{R}^n} f(x + y)\hat{a}\Big(x, y, \frac{y}{|y|}\Big)/|y|dy. \tag{5}$$

Using the translation invariance and evaluating $R^t_\lambda \circ R_\mu f$ on two specific distributions Quinto proved that

$$\hat{a}(x, 0, \omega) = \hat{a}(0, 0, \omega) \quad \text{and if} \quad x \perp \omega \quad \text{then} \quad \hat{a}(x, r\omega, \omega) = \hat{a}(0, r\omega, \omega), \tag{6}$$

where $\omega \in S^{n-1}$. Writing the second equation into (5) one can easily see that these equations are not only necessary but sufficient too. At the same time the second equation clearly implies the first one.

Since the transformation '$\hat{\ }$' is invertible on even functions of $\omega \in S^{n-1}$[2], the first equation of (6) is equivalent to $\bar{a}(x,0,\omega) = \bar{a}(0,0,\omega) = a(\omega)$. Using the speciality of the double fibration model Quinto could calculate the measure μ here, substantially from the square root of a. We can not do this in our situation.

Unfortunately the integrand in the second equation of (6) may be not even and so it is equivalent to the even part of the integrand being zero. This is just the condition in the theorem.

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One can easily find μ and λ for $R^t_\lambda \circ R_\mu$ being translation invariant, but no R_μ nor R^t_λ are exponential if μ and λ are allowed to be not strictly positiv. For example $\mu(x,\omega,p) = \cos(\langle x, \varphi(\omega)\rangle)$ and $\lambda(x,\omega,p) = 1/\cos(\langle x, \varphi(\omega)\rangle)$, where $\varphi(\omega) \perp \omega$. But if I insisted on the strictly positivity of μ and λ I could only find the exponential μ and λ while I was unable to prove the uniqueness of this type.

4. Inversion formula and support theorem

Let R_μ and R^t_λ be exponential i.e.

$$Rf(\omega,p) = \int_{H(\omega,p)} f(x)\mu_1(\omega,p)\exp(\langle \mu_2(\omega), x\rangle)dx,$$

$$R^t_\lambda f(x) = \exp(\langle -z, x\rangle)\int_{S^{n-1}} f(\omega,\langle\omega,x\rangle)\frac{\exp(\langle\mu_2(\omega),x\rangle)}{\mu_1(\omega,\langle\omega,x\rangle)}d\omega.$$

We know then that the operator $R^t_\lambda \circ R_\mu$ is translation invariant on $C^\infty_c(\mathbb{R}^n)$. Furthermore one can easily prove that $R^t_\lambda \circ R_\mu$ is continuous linear map of $C^\infty_c(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$. Thus there must exist a temperate distribution T [1] for which $R^t_\lambda \circ R_\mu f = T * f$, where $*$ is the convolution operator. This gives inversion formula through the Fourier transform F [1] as

$$f = R^t_\lambda \circ R_\mu f \ * F^{-1}(1/FT). \tag{7}$$

Theorem 4.1. *If R_μ is an exponential i.e. translation invariant Radon transform, then $R^t_\lambda \circ R_\mu$ is invertible on $C^\infty_c(\mathbb{R}^n)$ with the formula (7), where*

$$T(x) = \int_{S^{n-2}\perp x} \exp(\langle\mu_2(\omega),x\rangle)d\omega.$$

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The proof of this theorem is immediate from (5) and left to the reader.

To finish the paper now we prove the so called support theorem for translation invariant Radon transforms. Our proof is a generalization to higher dimensions of A. Hertle's proof in [3].

Theorem 4.2. *Let R_μ be exponential Radon transform and f be a Lipschitz continuous function of compact support on $\mathbb{R}^n$. If $r > 0$ and $R_\mu f$ is supported in $\{(\omega, p) \in S^{n-1} \times \mathbb{R}\colon |p| \le r\}$, then f is supported in $\{x \in \mathbb{R}^n\colon |x| \le r\}$.*

Proof. If U is a rotation, then $R_\mu f \circ U = R_{\bar\mu}(f \circ U)$, where $\bar\mu_1 = \mu_1 \circ U$ and $\bar\mu_2 = U^{-1} \circ \mu_2 \circ U$. Since $\bar\mu$ is also exponential it suffices to show that f is zero on every hyperplane perpendicular to the first coordinate axis with $x_1 = p > r$. Since μ_1 is strictly positive we can omit it. Thus it remains to prove

$$\int_{H(\omega,p)} x_i f(x) \exp(\langle \mu_2(\omega), x\rangle) dx = 0 \qquad 1 \le i \le n, \tag{8}$$

where $\omega = (1, 0, \ldots, 0)$ and $x = (x_1, \ldots, x_n)$. Then by induction the factor x_i can be replaced by any polynomial in $x_1, \ldots, x_n$ that implies $f \equiv 0$ on $H(\omega, p)$ by the Weierstrass theorem. To show (8) let U^i_φ be the rotation in the plane of the first and i-th coordinate axis by angle φ $i, e,$

$$U^i_\varphi(x_1, \ldots, x_n) = (x_1 \cos\varphi + x_i \sin\varphi, x_2, \ldots, x_{i-1}, -x_1 \sin\varphi + x_i \cos\varphi, x_{i+1}, \ldots, x_n).$$

Differentiating the equation $R_\mu f(U^i_\varphi \omega, p) = 0$ with respect to φ and putting $\varphi = 0$ we obtain for $i \ge 2$ that

$$\begin{aligned} 0 = \int [-x_i \partial_1 f(x) + p \partial_i f(x)] \exp(\langle \mu_2(\omega), x\rangle) dx + \\ + \int f(x) \exp(\langle \mu_2(\omega), x\rangle)[\langle \partial_i \mu_2(\omega), x\rangle + p\mu_2^i - x_i \mu_2^1] dx, \end{aligned}$$

where the integration is over $H(\omega, p)$ and μ_2^i is the i-th coordinate function of μ_2. Let the left hand side function in (8) be denoted by $f_i(p)$. Then this result gives the differential equation system

$$\frac{d}{dp} f_i(p) = \sum_{j=2}^{n} \partial_i \mu_2^j(\omega) f_j(p),$$

from which

$$(f_2(p), ..., f_n(p)) = (\text{vector}) \exp([\partial_i \mu_2^j(\omega)] p)$$

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follows for all $p \geq r$, where $[\partial_i \mu_2^j(\omega)]$ is $(n-1) \times (n-1)$ matrix. Since the left hand side has compact support so does the right hand side too , *i.e.* vector $= 0$ and the theorem is proved.

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