CURVATURE IN HILBERT GEOMETRIES

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Abstract. We provide more transparent proofs for the facts that the curvature of a Hilbert geometry in the sense of Busemann can not be non-negative and a point of non-positive curvature is a projective center of the Hilbert geometry. Then we prove that the Hilbert geometry has non-positive curvature at its projective centers, and that a Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry if and only if it has non-positive curvature at every point of its intersection with a hyperplane. Moreover a 2-dimensional Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry if and only if it has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature.

1. Introduction

A *Hilbert geometry* is a pair $(\mathcal{I}, d_{\mathcal{I}})$ of an open, strictly convex domain $\mathcal{I} \subset \mathbb{R}^n$, and the *Hilbert metric* [2, page 297] $d_{\mathcal{I}} \colon \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ given by

$$d_{\mathcal{I}}(X,Y) = \begin{cases} 0, & \text{if } X = Y, \\ \frac{1}{2}\big|\ln(A,B;X,Y)\big|, & \text{if } X \neq Y, \text{ where } \overline{AB} = \mathcal{I} \cap XY. \end{cases} \qquad (1.1) \quad \langle 4, 5\rangle$$

Every geodesic $\tilde{\ell}$ of a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ is the intersection $\mathcal{I} \cap \ell$ of $\mathcal{I}$ with a straight line ℓ.

Busemann posed the problem [3, 34th on p. 406] if a Hilbert geometry that has non-positive curvature at every point is a Cayley–Klein model of Bolyai's hyperbolic geometry. This was affirmatively answered in [4, Theorem, p. 119], where Kelly and Strauss showed that if a point in a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature then it is a projective center of $\mathcal{I}$. They finished [4] by a conjecture that a Hilbert geometry can contain no points of non-negative curvature. This was proved in [6], where Kelly and Strauss closed the paper by discussing the problem if

a projective center has non-positive curvature. (1.2) $\langle 1\rangle$

In this paper we provide a bit more transparent proofs for the above mentioned results of Kelly and Strauss, and then we prove (1.2) in Theorem 4.2. Finally we obtain sharper affirmative answers for Busemann's problem [3, 34th on p. 406] in Section 5 as easy consequences.

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2. Notations and preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \dots$. The open segment with endpoints A and B is denoted by $\overline{AB}$, and AB denotes the line through A and B.

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *affine cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [2, page 243].

In this article $\mathcal{I}$ is an open, strictly convex domain in $\mathbb{R}^n$, where $n \geq 2$. We shall use without further notice the well-known fact [8, Theorem 25.3], that a convex function has both one-sided derivative at every point, and its derivative is strictly monotone, hence it is differentiable everywhere except at most a countable set. Moreover, a convex function has a second-order quadratic expansion at almost every point of its domain by Alexandrov's theorem [1] (see [9, Theorem 2.1]). These are called *Alexandrov points*, and in the expansions the usual big-O notation is used.

Given a point $P \in \mathcal{I}$, the *polar* P^* of P is defined as the locus of every point X that is the harmonic conjugate of P with respect to A and B, where $\overline{AB} = \mathcal{I} \cap PX$. It is easy to see [7, p. 64] that the polar P^* of a point $P \in \mathcal{I} \subset \mathbb{R}^n$ is a hyperplane outside $\mathcal{I}$ if and only if P is a *projective center* of $\mathcal{I}$, i.e. there is a projectivity ϖ such that $\varpi(P)$ is the affine center of $\varpi(\mathcal{I})$.

It is well known that a Hilbert geometry is the Cayley–Klein model of Bolyai's hyperbolic geometry if and only if it is given by an ellipsoid [2, 29.3].

A Hilbert geometry at a point O has *positive*, *non-negative*, *non-positive* and *negative curvature in the sense of Busemann* if there exists a neighborhood $\mathcal{U}$ of O such that for every pair of points $P, Q \in \mathcal{U}$ we have

$$2d_{\mathcal{I}}(\hat{P}, \hat{Q}) > d_{\mathcal{I}}(P, Q), \qquad 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \geq d_{\mathcal{I}}(P, Q),$$
$$2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \leq d_{\mathcal{I}}(P, Q), \qquad 2d_{\mathcal{I}}(\hat{P}, \hat{Q}) < d_{\mathcal{I}}(P, Q),$$

respectively, where $\hat{P}, \hat{Q}$ are the respective $d_{\mathcal{I}}$-midpoints of the geodesic segments $\overline{OP}$ and $\overline{OQ}$ [3, (36.1) on p. 237]. If neither of the cases is satisfied in any neighborhood of O, then we say that the curvature is *indeterminate* [4, Definition 1][1]. A projectivity ϖ is clearly a bijective isometry of $(\mathcal{I}, d_{\mathcal{I}})$ to $(\varpi(\mathcal{I}), d_{\varpi(\mathcal{I})})$, hence

$$\textit{Busemann's curvature is a projective invariant.} \tag{2.1}$$ ⟨6, 7, 9⟩

3. Preparations

We consider a Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ and a point O in $\mathcal{I}$.

Lemma 3.1 ([4, Lemma 1 and Corollary]). *There exist two (maybe ideal) points X and Y in O^* such that line XY does not intersect $\mathcal{I}$, and $\partial\mathcal{I}$ is differentiable at the points in $\partial\mathcal{I} \cap (OX \cup OY)$.*

[1] Notice that positivity or negativity of the curvature in [4, Definition 1] corresponds to non-negativity, respectively non-positivity in our terms.

Proof. There is at least one chord $\overline{AB}$ of $\mathcal{I}$ which is bisected by O. Then the harmonic conjugate $\bar{X}$ of O with respect to A and B is on the line at infinity.

If $\bar{X}$ is the only point of O^* at infinity, then O^* cannot completely lie within the strip formed by the two supporting lines of $\mathcal{I}$ which are parallel to AB, because otherwise, as O^* is a connected curve, it would intersect $\mathcal{I}$. Thus, a further point $\bar{Y}$ of O^* outside this strip exists.

If $\bar{X}$ is not the only point of O^* at infinity, then let that point be $\bar{Y}$.

Then line $\bar{X}\bar{Y}$ does not intersect $\mathcal{I}$, but intersects O^* in the points $\bar{X}$ and $\bar{Y}$.

Since all but a denumerable set of points of $\partial\mathcal{I}$ are points of differentiability, we may choose points $X \in O^*$ and $Y \in O^*$ near $\bar{X}$ and $\bar{Y}$, respectively, so that $\partial\mathcal{I}$ is differentiable at the points in $\partial\mathcal{I} \cap (OX \cup OY)$, and XY does not intersect $\mathcal{I}$. $\square$

Let ℓ_1 and ℓ_2 be straight lines through O, and let $l_\pm$ be straight lines through O such that

$$-(\ell_1, \ell_2; l_-, l_+) \geq 1. \tag{3.1}$$ ⟨6, 9⟩

Denote by $Y_\pm$ the points where l_+ intersects $\partial\mathcal{I}$ so that

$$(Y_-, Y_+; O)^2 \leq 1. \tag{3.2}$$ ⟨6⟩

Let $t_\pm$ be the tangent lines of $\partial\mathcal{I}$ at $Y_\pm$.

Fix a coordinate system so that $O = (0,0)$, l_- is the x-axis, l_+ is the y-axis, and Y_+ is in the upper half-plane. For x in a small neighborhood of 0, let $y_\pm$ be the continuous functions of x such that $(x, y_\pm(x))$ are the two points of $\partial\mathcal{I}$ with abscissa x, and $Y_\pm = (0, y_\pm(0))$ so $\pm y_\pm(x) > 0$.

Fix an Euclidean metric d such that the two axes and the lines ℓ_1 and ℓ_2 are perpendicular to each other, respectively. Let $s > 0$ be the slope of ℓ_1, hence the slope of ℓ_2 is $-1/s$. Let $m_\pm$ be the slope of $t_\pm$, and if the intersection of $t_\pm$ and the x-axis exists, then denote it by $T_\pm$. So Figure 3.1 shows what we have.

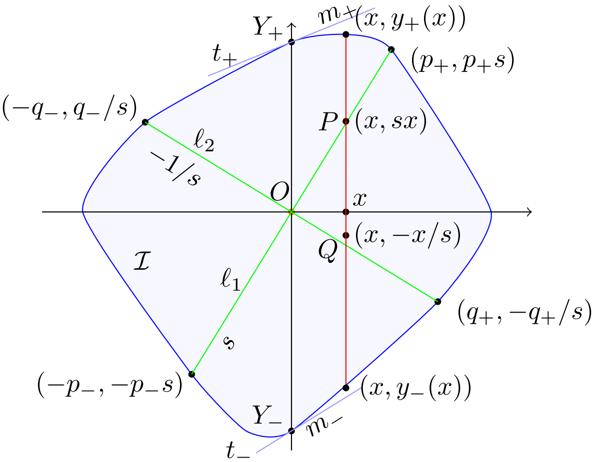

FIGURE 3.1. The configuration in euclidean plane.

Let $p_\pm, q_\pm > 0$ be such that $(\pm p_\pm, \pm p_\pm s)$ are the points of $\ell_1 \cap \partial\mathcal{I}$, and $(\pm q_\pm, \mp q_\pm/s)$ are the points of $\ell_2 \cap \partial\mathcal{I}$.

Lemma 3.2. *If O is the affine midpoint of the chords $\ell_1 \cap \mathcal{I}$ and $\ell_2 \cap \mathcal{I}$, and the points $Y_\pm$ are Alexandrov points of $\partial\mathcal{I}$, then in a small neighborhood of O we have*

$$d_{\mathcal{I}}(P,Q) - 2d_{\mathcal{I}}(\hat{P},\hat{Q}) \geq \frac{x^2}{2s}\Big(\frac{s^2 m_-}{y_-^2(0)} - \frac{m_+}{y_+^2(0)}\Big) + x^3 O(1), \tag{3.3}$$ ⟨4, 7, 8, 9⟩

where $\hat{P}, \hat{Q}$ are the $d_{\mathcal{I}}$-midpoints of the geodesic segments $\overline{OP}$ and $\overline{OQ}$, respectively.

Proof. Since O is the midpoint of the chords $\tilde{\ell}_1 = \ell_1 \cap \mathcal{I}$ and $\tilde{\ell}_2 = \ell_2 \cap \mathcal{I}$, we have $p := p_+ = p_-$ and $q := q_+ = q_-$.

We have to show that there is an $\varepsilon > 0$ such that the points $P = (x, sx) \in \tilde{\ell}_1$, $Q = (x, -x/s) \in \tilde{\ell}_2$ $(x \in (0,\varepsilon))$, and the respective $d_{\mathcal{I}}$-midpoints $\hat{P}, \hat{Q}$ of the geodesic segments $\overline{OP}$ and $\overline{OQ}$ satisfy (3.3).

The strict triangle inequality $d_{\mathcal{I}}(\hat{P},\hat{Q}) < d_{\mathcal{I}}(\hat{P},\bar{P}) + d_{\mathcal{I}}(\bar{P},\bar{Q}) + d_{\mathcal{I}}(\bar{Q},\hat{Q})$, where $\bar{P} = (\frac{x}{2}, \frac{sx}{2})$ and $\bar{Q} = (\frac{x}{2}, \frac{-x}{2s})$, gives

$$d_{\mathcal{I}}(P,Q) - 2d_{\mathcal{I}}(\hat{P},\hat{Q}) \geq (d_{\mathcal{I}}(P,Q) - 2d_{\mathcal{I}}(\bar{P},\bar{Q})) - 2(d_{\mathcal{I}}(\hat{P},\bar{P}) + d_{\mathcal{I}}(\bar{Q},\hat{Q})), \tag{3.4}$$ ⟨5⟩

so it is enough to estimate the right-hand side of this inequality from below.

By (1.1) and the Taylor series expansion of the logarithm, we have

$$d_{\mathcal{I}}(O,P) = \frac{1}{2}\ln\frac{p+x}{p-x} = \frac{1}{2}\Big(\ln\Big(1+\frac{x}{p}\Big) - \ln\Big(1-\frac{x}{p}\Big)\Big) = \frac{1}{2}\sum_{i=1}^{\infty}\frac{1-(-1)^i}{i}\Big(\frac{x}{p}\Big)^i,$$

hence

$$d_{\mathcal{I}}(O,P) = \sum_{j=0}^{\infty}\frac{1}{2j+1}\Big(\frac{x}{p}\Big)^{2j+1}, \quad \text{and} \quad d_{\mathcal{I}}(O,\bar{P}) = \sum_{j=0}^{\infty}\frac{2^{-1-2j}}{2j+1}\Big(\frac{x}{p}\Big)^{2j+1}.$$

The same calculation for Q and $\bar{Q}$ leads to

$$d_{\mathcal{I}}(O,Q) = \sum_{j=0}^{\infty}\frac{1}{2j+1}\Big(\frac{x}{q}\Big)^{2j+1}, \quad \text{and} \quad d_{\mathcal{I}}(O,\bar{Q}) = \sum_{j=0}^{\infty}\frac{2^{-1-2j}}{2j+1}\Big(\frac{x}{q}\Big)^{2j+1}.$$

Further, as $d_{\mathcal{I}}(O,\hat{P}) = d_{\mathcal{I}}(O,P)/2$, and $d_{\mathcal{I}}(O,\hat{Q}) = d_{\mathcal{I}}(O,Q)/2$, the above formulas also imply

$$d_{\mathcal{I}}(\bar{P},\hat{P}) = |d_{\mathcal{I}}(O,\hat{P}) - d_{\mathcal{I}}(O,\bar{P})| = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1-2^{-2j}}{2j+1}\Big(\frac{x}{p}\Big)^{2j+1}, \tag{3.5}$$ ⟨5⟩

$$d_{\mathcal{I}}(\bar{Q},\hat{Q}) = |d_{\mathcal{I}}(O,\hat{Q}) - d_{\mathcal{I}}(O,\bar{Q})| = \frac{1}{2}\sum_{j=1}^{\infty}\frac{1-2^{-2j}}{2j+1}\Big(\frac{x}{q}\Big)^{2j+1}. \tag{3.6}$$ ⟨5⟩

Since $d_{\mathcal{I}}(P,Q)=\frac{1}{2}\Big|\ln\Big(\frac{sx-y_-(x)}{y_+(x)-sx}:\frac{-x/s-y_-(x)}{y_+(x)+x/s}\Big)\Big|$ by (1.1), the Taylor series expansion of the logarithm gives

$$
\begin{aligned}
&d_{\mathcal{I}}(P,Q)\\
&=\frac{1}{2}\Big(\ln\Big(1+\frac{sx}{-y_-(x)}\Big)-\ln\Big(1-\frac{sx}{y_+(x)}\Big)+\ln\Big(1+\frac{x/s}{y_+(x)}\Big)-\ln\Big(1-\frac{x/s}{-y_-(x)}\Big)\Big)\\
&=\frac{1}{2}\Big(\sum_{i=1}^{\infty}\frac{1-(-1)^i}{i}\Big(\frac{sx}{-y_-(x)}\Big)^i+\sum_{i=1}^{\infty}\frac{1-(-1)^i}{i}\Big(\frac{x/s}{y_+(x)}\Big)^i\Big)\\
&=\sum_{j=0}^{\infty}\frac{-s^{2j+1}}{2j+1}\Big(\frac{x}{y_-(x)}\Big)^{2j+1}+\sum_{j=0}^{\infty}\frac{s^{-2j-1}}{2j+1}\Big(\frac{x}{y_+(x)}\Big)^{2j+1}. \qquad (3.7)\ \langle 5\rangle
\end{aligned}
$$

In the same way $d_{\mathcal{I}}(\bar{P},\bar{Q})=\frac{1}{2}\Big|\ln\Big(\frac{sx/2-y_-(x/2)}{y_+(x/2)-sx/2}:\frac{-x/2/s-y_-(x/2)}{y_+(x/2)+x/2/s}\Big)\Big|$ implies

$$
d_{\mathcal{I}}(\bar{P},\bar{Q})=\sum_{j=0}^{\infty}\frac{-s^{2j+1}}{2j+1}\Big(\frac{x/2}{y_-(x/2)}\Big)^{2j+1}+\sum_{j=0}^{\infty}\frac{s^{-2j-1}}{2j+1}\Big(\frac{x/2}{y_+(x/2)}\Big)^{2j+1}. \qquad (3.8)\ \langle 5\rangle
$$

Since the points $Y_\pm$ are Alexandrov points of $\partial\mathcal{I}$, we have the Taylor series expansions $\bar{y}_\pm(t)=\bar{y}_\pm(0)+t\bar{y}'_\pm(0)+t^2O(1)$ of the functions $\bar{y}_\pm:=1/y_\pm$. For easy handling of this we define $\bar{y}_\pm^{\langle i\rangle}(0)$ $(i=0,1,2)$ so that $\bar{y}_\pm(t)=\sum_{i=0}^{2}t^i\bar{y}_\pm^{\langle i\rangle}(0)/i!$.

Substituting (3.5), (3.6), (3.7), (3.8), and the above Taylor expansion of $\bar{y}_\pm(x)$ into the right-hand side of (3.4), we obtain

$$
\begin{aligned}
&(d_{\mathcal{I}}(P,Q)-2d_{\mathcal{I}}(\bar{P},\bar{Q}))-2(d_{\mathcal{I}}(\hat{P},\bar{P})+d_{\mathcal{I}}(\bar{Q},\hat{Q}))\\
&=\sum_{j=0}^{\infty}\frac{-s^{2j+1}}{2j+1}\Big(\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_-^{\langle i\rangle}(0)}{i!}\Big)^{2j+1}+\sum_{j=0}^{\infty}\frac{s^{-2j-1}}{2j+1}\Big(\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_+^{\langle i\rangle}(0)}{i!}\Big)^{2j+1}-\\
&\quad-2\sum_{j=0}^{\infty}\frac{-s^{2j+1}}{2j+1}\Big(\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_-^{\langle i\rangle}(0)}{i!2^{i+1}}\Big)^{2j+1}-2\sum_{j=0}^{\infty}\frac{s^{-2j-1}}{2j+1}\Big(\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_+^{\langle i\rangle}(0)}{i!2^{i+1}}\Big)^{2j+1}-\\
&\quad-\sum_{j=1}^{\infty}\frac{1-2^{-2j}}{2j+1}\Big(\frac{x}{p}\Big)^{2j+1}-\sum_{j=1}^{\infty}\frac{1-2^{-2j}}{2j+1}\Big(\frac{x}{q}\Big)^{2j+1}.
\end{aligned}
$$

Separating the summands with index $j=0$ from the sums with running variable j, and moving them to the beginning result in

$$
\begin{aligned}
&(d_{\mathcal{I}}(P,Q)-2d_{\mathcal{I}}(\bar{P},\bar{Q}))-2(d_{\mathcal{I}}(\hat{P},\bar{P})+d_{\mathcal{I}}(\bar{Q},\hat{Q}))\\
&\qquad=-s\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_-^{\langle i\rangle}(0)}{i!}+\frac{1}{s}\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_+^{\langle i\rangle}(0)}{i!}+s\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_-^{\langle i\rangle}(0)}{i!2^i}-\\
&\qquad\quad-\frac{1}{s}\sum_{i=0}^{2}x^{i+1}\frac{\bar{y}_+^{\langle i\rangle}(0)}{i!2^i}+x^3O(1).
\end{aligned}
$$

The summands with index $i=0$ just cancel each other, the summands with index $i=2$ has multiplier x^3, so we obtain

$$(d_{\mathcal{I}}(P,Q)-2d_{\mathcal{I}}(\bar{P},\bar{Q}))-2(d_{\mathcal{I}}(\hat{P},\bar{P})+d_{\mathcal{I}}(\bar{Q},\hat{Q}))\\ =x^2\Big(\frac{1}{2s}\bar{y}'_+(0)-\frac{s}{2}\bar{y}'_-(0)\Big)+x^3O(1).$$

Since $y_\pm := 1/\bar{y}_\pm$, one gets

$$e:=\frac{1}{2s}\bar{y}'_+(0)-\frac{s}{2}\bar{y}'_-(0)=\frac{-1}{2s}\frac{y'_+(0)}{y_+^2(0)}+\frac{s}{2}\frac{y'_-(0)}{y_-^2(0)}=\frac{1}{2s}\Big(\frac{s^2m_-}{y_-^2(0)}-\frac{m_+}{y_+^2(0)}\Big)$$

that proves the lemma. □

4. Curvature in Hilbert geometry

Firstly we reprove the result of [6] using our preparatory Lemma 3.2.

Theorem 4.1. *A Hilbert geometry can not have positive or non-negative curvature at any point.*

Proof. It is enough to prove that

through every point O of a Hilbert geometry $(\mathcal{I},d_{\mathcal{I}})$ there are two geodesics $\tilde{\ell}_1$ and $\tilde{\ell}_2$ such that in any suitable small open neighborhood $\mathcal{U}$ of O inequality $2d_{\mathcal{I}}(\hat{P},\hat{Q})<d_{\mathcal{I}}(P,Q)$ is fulfilled for some points $P\in\tilde{\ell}_1\cap\mathcal{U}$ and $Q\in\tilde{\ell}_2\cap\mathcal{U}$, where $\hat{P},\hat{Q}\in\mathcal{U}$ are the $d_{\mathcal{I}}$-midpoints of the geodesic segments $\overline{OP}$ and $\overline{OQ}$, respectively. (4.1) ⟨6, 7⟩

As two geodesics lie always in a common plane, it is enough to prove (4.1) in the plane. Let O be an arbitrary point in $\mathcal{I}\subset\mathbb{R}^2$.

By Lemma 3.1, there is a projectivity ϖ such that $\varpi(O)$ is the affine center of at least two geodesics $\varpi(\tilde{\ell}_1)$ and $\varpi(\tilde{\ell}_2)$. So taking (2.1) into account, we assume from now on that O *is the affine center of the segments* $\ell_1\cap\mathcal{I}$ *and* $\ell_2\cap\mathcal{I}$.

Choose the straight lines $l_\pm$ through O so that $Y_\pm$ are Alexander points of $\partial\mathcal{I}$, and $-(\ell_1,\ell_2;l_-,l_+)>1$. This is possible because if equality happened in (3.1), then rotating l_- a little bit helps. So by (3.2) we have

$$-(\ell_1,\ell_2;l_-,l_+)>(Y_-,Y_+;O)^2. \tag{4.2}$$ ⟨7⟩

If either one of the tangents $t_\pm$ is parallel to l_-, then slightly rotate l_- around O so that it keeps the properties required above and intersects the tangents $t_\pm$ in some points, say $T_\pm=t_\pm\cap l_-$. If $|(T_+,T_-;O)|<|(Y_+,Y_-;O)|$, then change the indexing from $\pm$ to $\mp$, so we have $|(T_+,T_-;O)|\geq|(Y_+,Y_-;O)|$.

Now we choose a coordinate system so that the positive half of the x-axis contains T_-. Figure 4.1 shows what we have if $O\in\overline{T_-T_+}$.

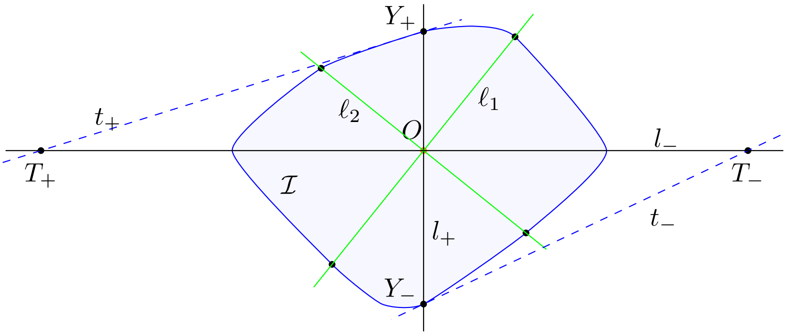

FIGURE 4.1. The affine configuration if $O \in \mathcal{I} \cap \overline{T_+T_-}$.

By Lemma 3.2 statement (4.1) fulfills if the main term $\frac{m_-}{2sy_+^2(0)}\big(s^2\frac{y_+^2(0)}{y_-^2(0)} - \frac{m_+}{m_-}\big)$ in (3.3) is positive, i.e. $s^2\frac{y_+^2(0)}{y_-^2(0)} > \frac{m_+}{m_-}$. Observe that (4.2) implies

$$s^2\frac{y_+^2(0)}{y_-^2(0)} = -\frac{-s}{1/s}\frac{|Y_+O|^2}{|OY_-|^2} = \frac{-(\ell_1,\ell_2;l_-)}{(Y_-,Y_+;O)^2} = \frac{-(\ell_1,\ell_2;l_-,l_+)}{(Y_-,Y_+;O)^2} > 1.$$

So we need to prove that $\frac{m_+}{m_-} \le 1$. If $0 < (T_+,T_-;O)$, then $m_+ < 0$ and therefore $\frac{m_+}{m_-} < 0$. If $(T_+,T_-;O) < 0$, then

$$\frac{m_+}{m_-} = \frac{|Y_+O|/|T_+O|}{|OY_-|/|OT_-|} = \frac{|Y_+O|}{|OY_-|}\frac{|OT_-|}{|T_+O|} = \frac{|(Y_+,Y_-;O)|}{|(T_+,T_-;O)|} \le 1,$$

so the proof is complete. □

We use again Lemma 3.2 to improve [4, the first statement of Theorem].

Theorem 4.2. *A point O in the Hilbert geometry $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature if and only if it is a projective center of $\mathcal{I}$.*

Proof. Firstly we prove the necessity part[2].

We assume that $(\mathcal{I}, d_{\mathcal{I}})$ has non-positive curvature at O, and have to prove that O^* is a hyperplane. For this it is enough to prove that every plane section of O^* is a straight line. So, from now on we assume that $\mathcal{I} \subset \mathbb{R}^2$, and need to prove that

there is a projectivity ϖ such that $\varpi(O)$ is the affine center of $\varpi(\mathcal{I})$.

By Lemma 3.1, there is a projectivity ϖ such that $\varpi(O)$ is the affine center of at least two geodesics $\varpi(\tilde{\ell}_1)$ and $\varpi(\tilde{\ell}_2)$, so, according to (2.1), we may assume without loss of generality that *O is the affine center of the segments $\ell_1 \cap \mathcal{I}$ and $\ell_2 \cap \mathcal{I}$.*

[2]This is [4, first statement of Theorem]

This time we choose the straight lines $l_\pm$ through O so that

$$-(\ell_1, \ell_2; l_-, l_+) = 1, \tag{4.3}$$ ⟨8⟩

$Y_\pm$ are Alexander points of $\partial\mathcal{I}$, and l_- intersects both $t_\pm$. This can be achieved easily, because except the two directions, where l_- is parallel to one of the tangents $t_\pm$, and where a point $Y_\pm$ is not an Alexander point of $\partial\mathcal{I}$, the direction of l_- can be chosen freely, and l_+ is determined change accordingly by (4.3).

Choose the direction of the x-axes so that the abscissa of T_- be positive. Again Figure 4.1 shows what we have if $O \in \overline{T_+T_-}$.

Since the Busemann curvature is non-positive, i.e. $2d_{\mathcal{I}}(\hat{P}, \hat{Q}) \geq d_{\mathcal{I}}(P, Q)$, the main term in (3.3) of Lemma 3.2 should vanish, i.e. $\frac{s^2 m_-}{y_-^2(0)} = \frac{m_+}{y_+^2(0)}$. However $s^2 = -\frac{-s}{1/s} = -(\ell_1, \ell_2; l_-) = -(\ell_1, \ell_2; l_-, l_+) = 1$, so $-\frac{y_-'(0\pm)}{y_-^2(0)} = \frac{y_+'(0\pm)}{y_+^2(0)}$ follows, where the sign $\pm$ at $0\pm$ is determined by the direction of the x-axis. Rearrangement gives

$$y_+(0)\frac{y_+(0)}{y_+'(0\pm)} = (-y_-)(0)\frac{(-y_-)(0)}{(-y_-)'(0\pm)},$$

that, as $\pm y_+(0) = d(O, Y_\pm)$ and $\pm y_\pm(0)/(\pm y_\pm'(0)) = d(O, T_\pm)$, means that the triangles $\triangle OY_+T_+$ and $\triangle OY_-T_-$ have equal areas.

Change now to a Euclidean metric d_e such that ℓ_1 and ℓ_2 are orthogonal. Let the direction vector of l_+ be $(\cos\varphi, \sin\varphi)$, hence the direction vector of l_- is $(\cos\varphi, -\sin\varphi)$, and let r be the radial function of $\partial\mathcal{I}$ from the point O, hence $Y_+ = r(\varphi)(\cos\varphi, \sin\varphi)$ and $Y_- = r(\varphi+\pi)(\cos(\varphi+\pi), \sin(\varphi+\pi))$. See Figure 4.2.

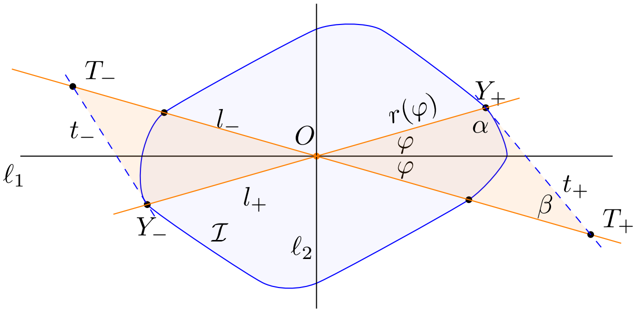

FIGURE 4.2. We have $\mathrm{area}(\triangle OY_+T_+) = \mathrm{area}(\triangle OY_-T_-)$ for every φ.

Define $\alpha := \angle(O, Y_+, T_+)$, $\beta := \pi - \alpha - 2\varphi$. Then $\cot\alpha = -\dot{r}(\varphi)/r(\varphi)$ and $a(\varphi) := 2\,\mathrm{area}(\triangle OY_+T_+) = r^2(\varphi)\frac{\sin(2\varphi)}{\sin\beta}\sin\alpha$, hence

$$\begin{aligned}\frac{\sin(2\varphi)}{a(\varphi)} &= r^{-2}(\varphi)\frac{\sin(2\varphi+\alpha)}{\sin\alpha} = r^{-2}(\varphi)\big(\sin(2\varphi)\cot\alpha + \cos(2\varphi)\big)\\ &= \sin(2\varphi)\frac{-\dot{r}(\varphi)}{r^3(\varphi)} + \cos(2\varphi)\frac{1}{r^2(\varphi)} = \frac{1}{2}\Big(\frac{\sin(2\varphi)}{r^2(\varphi)}\Big)'.\end{aligned}$$

Thus, we have

$$\Big(\frac{\sin(2\varphi)}{r^2(\varphi)}\Big)' = \frac{\sin(2\varphi)}{a(\varphi)} = \frac{\sin(2(\varphi+\pi))}{a(\varphi+\pi)} = \Big(\frac{\sin(2(\varphi+\pi))}{r^2(\varphi+\pi)}\Big)',$$

and also $\lim_{\varphi\to 0}\frac{\sin(2\varphi)}{r^2(\varphi)} = 0 = \lim_{\varphi\to 0}\frac{\sin(2(\varphi+\pi))}{r^2(2(\varphi+\pi))}$. Thus $r(\varphi)\equiv r(\varphi+\pi)$ follows, meaning that $\mathcal{I}$ is affine symmetric with respect to O.

Thus the necessity part of the theorem is proved.

Next we prove the sufficiency part[3].

We assume that O is a projective center of $\mathcal{I}$, and we have to prove that

there is a suitable small open neighborhood $\mathcal{U}$ of O that for every geodesics $\tilde{\ell}_1$ and $\tilde{\ell}_2$ through O inequality $2d_{\mathcal{I}}(\hat{P},\hat{Q}) \le d_{\mathcal{I}}(P,Q)$ is fulfilled for every points $P\in\tilde{\ell}_1\cap\mathcal{U}$ and $Q\in\tilde{\ell}_2\cap\mathcal{U}$, where $\hat{P},\hat{Q}\in\mathcal{U}$ are the $d_{\mathcal{I}}$-midpoints of the geodesic segments $\overline{OP}$ and $\overline{OQ}$, respectively. (4.4) ⟨9⟩

According to (2.1), we may assume without loss of generality that O *is the affine center of* $\mathcal{I}$. Since two geodesics lie in a common plane, it is enough to prove (4.4) in the plane, so we assume that O is the affine center of $\mathcal{I}\subset\mathbb{R}^2$.

Choose the straight lines $l_\pm$ so that $Y_\pm$ are Alexander points of $\partial\mathcal{I}$, and

$$-(\ell_1,\ell_2;l_-,l_+) > 1. \tag{4.5}$$ ⟨9⟩

This is possible because if equality happened in (3.1), then rotating l_- a little bit helps. Moreover, if t_+ is parallel to l_-, then one can slightly rotate l_- around O so that (4.5) remains valid and intersects t_+. Thus, we can assume that the point T_+ exists. Since O is the affine center of $\mathcal{I}$, we have $t_+\parallel t_-$, so also point T_- exists, and O is clearly the affine center of $\overline{T_-T_+}$.

Now we fix the coordinate system and euclidean metric given in Section 3 so that the positive half of the x-axes contains T_-. Again Figure 4.1 shows what we have.

By Lemma 3.2 statement (4.4) fulfills if the main term $\frac{m_-}{2sy_+^2(0)}\big(s^2\frac{y_+^2(0)}{y_-^2(0)}-\frac{m_+}{m_-}\big)$ in (3.3) is positive. This fulfills because $\frac{m_+}{m_-}=1$ by $t_+\parallel t_-$, $m_->0$, and $s^2\frac{y_+^2(0)}{y_-^2(0)}-1=-\frac{-s}{1/s}-1=-(\ell_1,\ell_2;l_-,l_+)-1>0$ by (4.5). □

5. Consequences

The following statements sharpen and extend the solution [4, second statement in Theorem] of Kelly and Strauss given to Busemann's [3, Problem 34, p. 406].

Theorem 5.1. *A Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry if and only if there is a hyperplane intersecting the Hilbert geometry so that every point of the intersection is of non-positive curvature.*

[3] The last paragraph of [6] argues that this "does not seem easy".

Proof. If the Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry, then it has non-positive curvature at every point.

If there is a hyperplane intersecting the Hilbert geometry so that the Hilbert geometry has non-positive curvature at every point in the intersection, then all these points are projective centers by Theorem 4.2, and therefore [7, Theorem 3.3(a)] implies that the domain is an ellipsoid, hence the Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic geometry. □

For dimension 2 we have an even sharper version.

Theorem 5.2. *A 2-dimensional Hilbert geometry is a Cayley–Klein model of the hyperbolic space if and only if it has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature.*

Proof. If the 2-dimensional Hilbert geometry is a Cayley–Klein model of Bolyai's hyperbolic plane, then it has non-positive curvature at every point.

If the 2-dimensional Hilbert geometry has two points of non-positive curvature and its boundary is twice differentiable where it is intersected by the line joining those points of non-positive curvature, then [5, Theorem 3] implies that the domain is an ellipse. □

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