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EULER'S RATIO-SUM THEOREM REVISITED

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Abstract. We shortly cover the history of Euler's ratio-sum theorem, present a short proof for it, prove how it can be reversed, and convert Euler's ratio-sum formula into an interesting inequality.

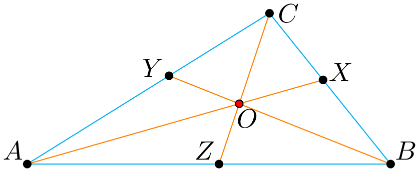
1. Introduction

Leonhard Euler (1707–1783), the greatest mathematician of his times, enriched geometry with numerous theorems which have come illustrious since then. In this survey, we deal with his less well known ratio-sum theorem. The literature of the ratio-sum theorem is not too extensive, however, articles appear from time to time (see [3, 4, 5, 7, 8]), and treat the ratio-sum formula in different settings. Interestingly enough, the ratio-sum formula has gotten into the field of vision of the present authors also in a different setting as a way to characterize projective-metric spaces [6].

Theorem 1.1 (Euler's ratio sum theorem [1]). *For every inner point O of a triangle $ABC\triangle$ in the Euclidean plane,*

$$\frac{d(A,O)}{d(O,X)}+\frac{d(B,O)}{d(O,Y)}+\frac{d(C,O)}{d(O,Z)}+2=\frac{d(A,O)}{d(O,X)}\cdot\frac{d(B,O)}{d(O,Y)}\cdot\frac{d(C,O)}{d(O,Z)}, \tag{1.1}$$

where $X = AO \cap BC$, $Y = BO \cap CA$, $Z = CO \cap AB$.



Although Euler submitted his paper [1] containing the ratio-sum theorem, it was published only 22 years after his death in 1873. Many of his works suffered a similar fate, due mainly to his peerless productivity in scientific article writing (in the course of his life, he wrote more than 800 articles, besides 28 extensive works), hence the considerable backlog of his unpublished works at his preferred journals of Academies of Saint Petersburg and Berlin.

Mathematicians of Euler's times, however, knew Euler's ratio-sum theorem, what can be seen from the publication [2] of Anders Johan Lexell (1740–1783)

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that appeared in 1873 and contains a survey on the spherical version of the ratio-sum theorem. Lexell was a good family friend of Euler, and a member of the same Russian Academy of Science in Saint Petersburg until his death.

Even less well known, less than the ratio-sum theorem itself, albeit it was included in Euler's original paper [1], that triangle $ABC\triangle$ can be constructed knowing the lengths in the ratio-sum formula (1.1).

Theorem 1.2 (Constructibility [1])**.** *If the length of segments $\overline{AO}$, $\overline{BO}$, and $\overline{CO}$, and further the length of segments $\overline{OX}$, $\overline{OY}$, and $\overline{OZ}$ are given, where O is a common point O of an unknown triangle $ABC\triangle$ such that $X = AO \cap BC$, $Y = BO \cap CA$, $Z = CO \cap AB$, then triangle $ABC\triangle$ is constructible.*

In this paper we prove not only the above theorems, but give the exact conditions under which three segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ with a given common point O can be rotated in a position such that $X = AO \cap BC$, $Y = BO \cap CA$, $Z = CO \cap AB$ [1].

Finally, although we present the possibility of an immediate generalization, we prefer to give and prove an inequality born by the proof of (1.1). This inequality happens to turn into an equality if and only if segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ pass through one point. This form of Euler's ratio-sum theorem, as phrased in this Theorem 3.1, is markedly reminiscent of Ceva's theorem.

2. Proofs and the converse of the ratio-sum theorem

Our notations mainly are the usual ones: points are denoted by capital letters; $d(A, B)$ denotes the distance of points A and B, AB respectively $\overline{AB}$ denotes the line respectively the segment with endpoints A and B. The triangle determined by points A, B and C is $ABC\triangle$, while its angle at vertex A is $\angle(BAC)$. The area function is $t(\cdot)$, so the area of triangle $ABC\triangle$ is given by $t(ABC) = t(ABC\triangle)$.

Proof of Theorem 1.1. To reduce clutter, introduce notations $a = \frac{d(A,O)}{d(O,X)}$, $b = \frac{d(B,O)}{d(O,Y)}$ and $c = \frac{d(C,O)}{d(O,Z)}$. Then Euler's ratio-sum formula (1.1) takes the form

$$a + b + c + 2 = abc. \tag{2.1}$$

Adding expression $1 + a + b + c + ab + bc + ca$ to both sides, the right-hand side can be written in a form of product, while the left-hand side splits to a sum of products:

$$(1 + b)(1 + c) + (1 + a)(1 + c) + (1 + a)(1 + b) = (1 + a)(1 + b)(1 + c).$$

[1] In his article [7] Shephard gave a solution for the similar problem in affine plane, where the length of the segments is not taken into account, but only the ratios of the division by O counts. In this case condition (1.1) alone guaranties the existence of a triangle in which the ratios in equation (1.1) occur. Moreover, as Shephard notices too, every affine image of that triangle is appropriate.

Value of a, b and c is clearly different form -1, so dividing by the right-hand side leads to the equivalent equality

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1. \tag{2.2}$$

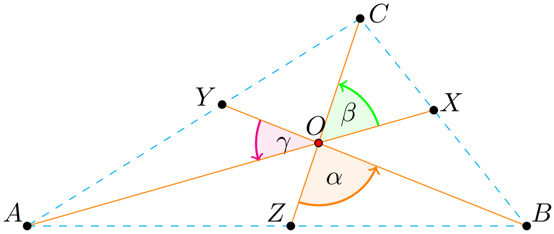
So, (1.1) is equivalent to equality

$$\frac{d(O,X)}{d(A,X)}+\frac{d(O,Y)}{d(B,Y)}+\frac{d(O,Z)}{d(C,Z)}=1. \tag{2.3}$$

This one, however, immediately follows from equalities

$$\frac{d(O,X)}{d(A,X)}=\frac{t(OBC)}{t(ABC)},\quad \frac{d(O,Y)}{d(B,Y)}=\frac{t(OCA)}{t(ABC)},\quad \frac{d(O,Z)}{d(C,Z)}=\frac{t(OAB)}{t(ABC)}. \qquad \square$$

Proof of Theorem 1.2. In order to avoid rather confusing complicated formulae, apply notations $a=\frac{d(A,O)}{d(O,X)}$, $b=\frac{d(B,O)}{d(O,Y)}$, and $c=\frac{d(C,O)}{d(O,Z)}$ this case as well, which, by our condition, fulfill (2.1), or what is the same, relation (2.2). Furthermore, introduce angles $\alpha=\angle(ZOB)$, $\beta=\angle(XOC)$, and $\gamma=\angle(YOA)$, for which relation $\alpha+\beta+\gamma=\pi$ is clearly true.



Our task therefore is to find values α, β, and $\gamma=\pi-\alpha-\beta$.

Point X happens to fall on segment $\overline{BC}$ if and only if

$$d(B,O)d(O,C)\sin\alpha=t(BOC)=d(B,O)d(O,X)\sin\gamma+d(C,O)d(O,X)\sin\beta,$$

that is,

$$\frac{d(A,O)}{d(O,X)}\frac{\sin\alpha}{d(O,A)}=\frac{\sin\alpha}{d(O,X)}=\frac{\sin\gamma}{d(O,C)}+\frac{\sin\beta}{d(B,O)}.$$

A cyclic permutation of the vertexes gives the same way that $Y\in\overline{CA}$ and $Z\in\overline{AB}$ if and only if

$$\frac{d(B,O)}{d(O,Y)}\frac{\sin\beta}{d(O,B)}=\frac{\sin\alpha}{d(O,A)}+\frac{\sin\gamma}{d(C,O)},$$
$$\frac{d(C,O)}{d(O,Z)}\frac{\sin\gamma}{d(O,C)}=\frac{\sin\beta}{d(O,B)}+\frac{\sin\alpha}{d(A,O)},$$

respectively. Introducing $x = \frac{\sin\alpha}{d(A,O)}$, $y = \frac{\sin\beta}{d(O,B)}$, and $z = \frac{\sin\gamma}{d(O,C)}$ turns up that these fulfill the homogeneous linear system of equations

$$\begin{aligned} ax - y - z &= 0, \\ -x + by - z &= 0, \\ -x - y + cz &= 0. \end{aligned} \tag{2.4}$$

From the difference of equations of (2.4), one gets right away for the solutions that $(1+a)x = (1+b)y = (1+z)c$, hence all solutions are of the form $\left(\frac{\lambda}{1+a}, \frac{\lambda}{1+b}, \frac{\lambda}{1+c}\right)$, where $\lambda \in \mathbb{R}$. Accordingly,

$$\lambda\frac{d(A,O)}{1+a} = \sin\alpha, \quad \lambda\frac{d(B,O)}{1+b} = \sin\beta, \quad \text{and } \lambda\frac{d(C,O)}{1+c} = \sin\gamma. \tag{2.5}$$

This results, as well, in

$$\frac{d(O,B)}{1+b}\sin\alpha = \frac{d(O,A)}{1+a}\sin\beta,$$

and

$$\begin{aligned} \frac{d(O,C)}{1+c} = \frac{\sin\gamma}{\lambda} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\lambda} \quad &= \frac{\sin\alpha}{\lambda}\cos\beta + \cos\alpha\frac{\sin\beta}{\lambda} \\ = \frac{d(O,A)}{1+a}\cos\beta + \frac{d(O,B)}{1+b}\cos\alpha. \end{aligned}$$

Subtracting expression $\frac{d(O,B)}{1+b}\cos\alpha$ from both sides of the latter equation, then squaring the result and summing up to the first equation, we get

$$\frac{d^2(O,B)}{(1+b)^2}\sin^2\alpha + \left(\frac{d(O,C)}{1+c} - \frac{d(O,B)}{1+b}\cos\alpha\right)^2 = \frac{d^2(O,A)}{(1+a)^2}.$$

Performing the squaring of the difference on the left-hand side, we obtain

$$\frac{d^2(O,B)}{(1+b)^2} - 2\frac{d(O,C)}{1+c}\frac{d(O,B)}{1+b}\cos\alpha + \frac{d^2(O,C)}{(1+c)^2} = \frac{d^2(O,A)}{(1+a)^2}. \tag{2.6}$$

Pursuant to cosine theorem, this means that there exists a triangle $PQR\triangle$ such that for the length of its sides opposite to the vertexes, $p = \frac{d(O,A)}{1+a}$, $q = \frac{d(O,B)}{1+b}$, $r = \frac{d(O,C)}{1+c}$ holds true, respectively, and the magnitude of the angles at the vertexes are α, β, and $\gamma = \pi - \alpha - \beta$, respectively.

So thus, the origonal triangle can be constructed if length $p = \frac{d(A,O)d(O,X)}{d(A,X)}$, $q = \frac{d(B,O)d(O,Y)}{d(B,Y)}$, and $r = \frac{d(C,O)d(O,Z)}{d(C,Z)}$ are calculated. From these data triangle $PQR\triangle$ is constructible and its angles give α, β, and $\gamma = \pi - \alpha - \beta$, due to which segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ can be adjusted, properly to each other. $\square$

In virtue of the proofs of Theorem 1.1 and Theorem 1.2, it is clear that the conditions of the following theorem can not be lighten.

Theorem 2.1 (Converse of the ratio-sum theorem)**.** *If for a common point O of segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$, Euler's ratio-sum formula (1.1) holds true, and each of the numbers $p = \frac{d(A,O)d(O,X)}{d(A,X)}$, $q = \frac{d(B,O)d(O,Y)}{d(B,Y)}$, and $r = \frac{d(C,O)d(O,Z)}{d(C,Z)}$ is smaller than the sum of the other two, then segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ can be turned around O so that points X, Y, Z fall onto the sides of triangle $ABC\triangle$, respectively.*

Proof. For the sake of simplicity let $a = \frac{d(A,O)}{d(O,X)}$, $b = \frac{d(B,O)}{d(O,Y)}$, and $c = \frac{d(C,O)}{d(O,Z)}$. By the assumption and (2.2), we have

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1. \tag{2.7}$$

Now construct a triangle $PQR\triangle$ such that length of its sides opposite to the vertexes, are p, q, r, respectively. Let the angles at vertexes be α, β, and $\gamma = \pi - \alpha - \beta$, respectively. Rotate segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ so that $\angle(ZOB)$ be α, $\angle(XOC)$ be β, and $\angle(YOA)$ be γ. Hence

$$\begin{aligned} q^2 - 2rq\cos\alpha + r^2 &= p^2, \\ p^2 - 2rp\cos\beta + r^2 &= q^2, \\ p^2 - 2qp\cos\gamma + q^2 &= r^2. \end{aligned} \tag{2.8}$$

Now we should prove that resulted triangle $ABC\triangle$ is such that its sides opposite to the vertexes contain points X, Y, and Z, respectively.

Let $\hat{X} = AX \cap BC$, $\hat{Y} = BY \cap CA$, and $\hat{Z} = CZ \cap AB$, furthermore, $\hat{a} = \frac{d(A,O)}{d(O,\hat{X})}$, $\hat{b} = \frac{d(B,O)}{d(O,\hat{Y})}$, and $\hat{c} = \frac{d(C,O)}{d(O,\hat{Z})}$. Relation (1.1) is true for triangle $ABC\triangle$, so (2.2) gives

$$\frac{1}{1+\hat{a}} + \frac{1}{1+\hat{b}} + \frac{1}{1+\hat{c}} = 1. \tag{2.9}$$

Introducing notations $\hat{p} = \frac{d(A,O)d(O,\hat{X})}{d(A,\hat{X})}$, $\hat{q} = \frac{d(B,O)d(O,\hat{Y})}{d(B,\hat{Y})}$, and $\hat{r} = \frac{d(C,O)d(O,\hat{Z})}{d(C,\hat{Z})}$, the construction procedure and (2.6) imply

$$\begin{aligned} \hat{q}^2 - 2\hat{r}\hat{q}\cos\alpha + \hat{r}^2 &= \hat{p}^2, \\ \hat{p}^2 - 2\hat{r}\hat{p}\cos\beta + \hat{r}^2 &= \hat{q}^2, \\ \hat{p}^2 - 2\hat{q}\hat{p}\cos\gamma + \hat{q}^2 &= \hat{r}^2. \end{aligned} \tag{2.10}$$

Comparing equations (2.8) with equations (2.10), as both triple of equations applies to triangles with the same corresponding angle, i.e. to similar triangles, it follows that $\hat{p} = \lambda p$, $\hat{q} = \lambda q$, and $\hat{r} = \lambda r$ for some number $\lambda > 0$. From the definition of p, q, r and $\hat{p}, \hat{q}, \hat{r}$, equalities $\frac{1}{1+\hat{a}} = \frac{\lambda}{1+a}$, $\frac{1}{1+\hat{b}} = \frac{\lambda}{1+b}$, and $\frac{1}{1+\hat{c}} = \frac{\lambda}{1+c}$ follow. Substituting these into (2.9) and comparing the result to (2.9), $\lambda = 1$ presents itself. Hence $\hat{a} = a$, $\hat{b} = b$, and $\hat{c} = c$, that is, $\hat{X} = X$, $\hat{Y} = Y$, and $\hat{Z} = Z$.

Herewith the theorem is proven. □

3. Instead of generalization – inequality

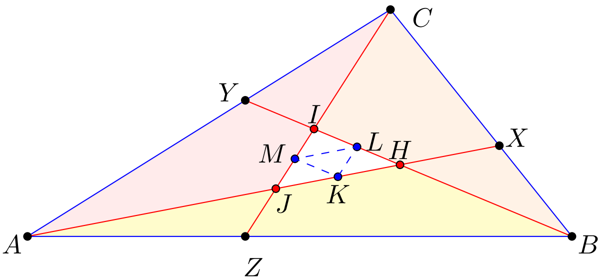
It can be shown that Euler's ratio-sum theorem remains true with appropriate interpretation if the common point of the lines through the vertexes of the triangle does not fall on the lines of the sides of the triangle. Moreover, the reverse statement for this more general case holds true, as well, if one requires for points X, Y, Z only to fall onto the lines of the sides of the triangle. We do not give proof for these generalizations here, but show an inequality instead.

Let us consider the ratio-sum formula in its equivalent form (2.3). This formula is valid when the three segments pass through one common point. When the segments meet each other in the pairwise different points H, I, J, a ratio still can be defined on each segment, taking the midpoint of the two points of intersection as a new dividing point on the respective segments.

Theorem 3.1. *Let X, Y, Z be points on the sides of triangle $ABC\triangle$ opposite to vertexes A, B, C, respectively. Furthermore, let $H = AX \cap BY$, $I = BY \cap CZ$, and $J = CZ \cap AX$ be the points of intersection of segments joining these points with the opposite vertexes, respectively. Finally, let the midpoints of the triangle $HIJ\triangle$, determined by the latter three points, be $K = (J + H)/2$, $L = (H + I)/2$, and $M = (I + J)/2$. Then,*

$$\frac{d(K,X)}{d(A,X)} + \frac{d(L,Y)}{d(B,Y)} + \frac{d(M,Z)}{d(C,Z)} \geq 1, \tag{3.1}$$

where equality stands if and only if $K = L = M$.



Proof. We can calculate the area ratio of triangles with same base the following way: $\frac{t(HBC)}{t(ABC)} = \frac{d(H,X)}{d(A,X)}$, $\frac{t(JBC)}{t(ABC)} = \frac{d(J,X)}{d(A,X)}$, $\frac{t(ICA)}{t(ABC)} = \frac{d(I,Y)}{d(B,X)}$, $\frac{t(HCA)}{t(ABC)} = \frac{d(H,Y)}{d(B,X)}$, $\frac{t(JAB)}{t(ABC)} = \frac{d(J,Z)}{d(C,X)}$, $\frac{t(IAB)}{t(ABC)} = \frac{d(I,Z)}{d(C,X)}$. Substituting these into the doubled left-hand side of (3.1) gives

$$\frac{d(H,X)+d(J,X)}{d(A,X)} + \frac{d(I,Y)+d(H,Y)}{d(B,Y)} + \frac{d(J,Z)+d(I,Z)}{d(C,Z)}$$
$$= \frac{t(HBC)}{t(ABC)} + \frac{t(JBC)}{t(ABC)} + \frac{t(ICA)}{t(ABC)} + \frac{t(HCA)}{t(ABC)} + \frac{t(JAB)}{t(ABC)} + \frac{t(IAB)}{t(ABC)}$$

$$= \frac{t(HBC)}{t(ABC)} + \frac{t(HCA)}{t(ABC)} + \frac{t(ICA)}{t(ABC)} + \frac{t(IAB)}{t(ABC)} + \frac{t(JBC)}{t(ABC)} + \frac{t(JAB)}{t(ABC)}$$
$$= 1 - \frac{t(HAB)}{t(ABC)} + 1 - \frac{t(IBC)}{t(ABC)} + 1 - \frac{t(JAC)}{t(ABC)} = 3 - \frac{t(ABC) - t(HIJ)}{t(ABC)}$$
$$= 2 + \frac{t(HIJ)}{t(ABC)}.$$

This completes the proof. □

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