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CONICS IN MINKOWSKI GEOMETRIES

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Abstract. Euclidean geometry is the only Minkowski geometry in which either there is a centrally symmetric, or a quadratic conic, or there is a conical ellipsoid or hyperboloid.

1. Introduction

Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^n$, (centrally) symmetric to the origin. Then function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\lambda > 0 : (\boldsymbol{y} - \boldsymbol{x})/\lambda \in \mathcal{I}\}$$

is a metric on $\mathbb{R}^n$ [1, IV.24], and is called *Minkowski metric on* $\mathbb{R}^n$. It satisfies the strict triangle inequality, i.e. $d(A,B) + d(B,C) = d(A,C)$ is valid if and only if $B \in \overline{AC}$. A pair $(\mathbb{R}^n, d)$, where d is a Minkowski metric, is called *Minkowski geometry*, and $\mathcal{I}$ is called the *indicatrix* of it. In a Minkowski geometry $(\mathbb{R}^n, d)$ a set

(D_1) $\mathcal{C}^{\varrho}_{d;F,\mathcal{H}} := \{X \in \mathbb{R}^n : \varrho d(X,\mathcal{H}) = d(F,\mathcal{H})d(F,X)\}$ is called a *conic*,

where $\mathcal{H}$ is a hyperplane, the *leading hyperplane*, $F \notin \mathcal{H}$ is a point, the *focus*, and $\varrho > 0$ is a number, the *radius*. A conic is said to be *elliptic*, *parabolic* and *hyperbolic*, if $\varrho < d(F,\mathcal{H})$, $\varrho = d(F,\mathcal{H})$ and $\varrho > d(F,\mathcal{H})$, respectively[1].

We prove in Theorem 4.2 and Theorem 4.3 that if *one* of the conics is centrally symmetric, then the Minkowski plane is Euclidean.

Further, we prove in Theorem 5.1 that if *one* of the conics is quadratical, then the Minkowski plane is Euclidean. This can be regarded as a generalization of the theorem in [1, 25.4] which states that a Minkowski geometry is Euclidean if and only if its indicatrix is an ellipsoid.

Finally, we prove in Theorem 6.1 that the Euclidean space is the only Minkowski geometry in which *one* of the ellipses or hyperbolas is a conic. For elliptic conics this strengthens [6, Theorem 2].

2. Notations and preliminaries

Points of $\mathbb{R}^n$ are denoted as $A, B, \dots$, vectors are $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots$, but we use these latter notations also for points if the origin is fixed. The open segment with endpoints A and B is denoted by $\overline{AB} = (A,B)$, $\overline{A}B$ is the open ray starting from A passing through B and the line through A and B is denoted by AB.

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[1]With a slightly more general interpretation of (D_1), we may allow $\mathcal{H}$ to be the ideal hyperplane and $\varrho = d(F,\mathcal{H})$ as infinity. Then we get the *spherical* conic, i.e. the sphere.

We denote the *affine ratio* of the collinear points A, B and C by $(A, B; C)$ that satisfies $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$.

Notations $\boldsymbol{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used. It is worth to note that by these we have $\frac{\mathrm{d}}{\mathrm{d}\varphi}\boldsymbol{u}_\varphi = \boldsymbol{u}_\varphi^\perp$.

A curve in the plane is called *quadratical*, if it is part of a *quadric* which has the equation of the form

$$\mathcal{Q}_{\mathfrak{s}}^\sigma := \left\{(x, y) : \begin{cases} 1 = x^2 + \sigma y^2, & \text{if } \sigma \in \{-1, 1\}, \\ x = y^2, & \text{if } \sigma = 0, \end{cases}\right\} \qquad (D_q) \quad \langle 11\rangle$$

in a suitable affine coordinate system $\mathfrak{s}$. A quadric is called *ellipse* (*affine circle*), *parabola* and *hyperbola*, if $\sigma = 1$, $\sigma = 0$ and $\sigma = -1$, respectively.

For given fixed points F_1, F_2, the *focuses*, and number $a \neq d(F_1, F_2)/2$, the *radius*, we can define

(D_2) the *ellipsoid* (*ellipse* in dimension 2) as the set
$\mathcal{E}^a_{d;F_1,F_2} := \{E : 2a = d(F_1, E) + d(E, F_2)\}$, and
(D_3) the *hyperboloid* (*hyperbola* in dimension 2) as the set
$\mathcal{H}^a_{d;F_1,F_2} := \{X : 2a = |d(F_1, X) - d(X, F_2)|\}$,

according to $a > d(F_1, F_2)/2$ or $a < d(F_1, F_2)/2$, respectively. Value $2f := d(F_1, F_2)$ is the *eccentricity*, and if the eccentricity vanishes, then the ellipsoid (ellipse) is called *sphere (circle)*. The metric midpoint of segment $\overline{F_1F_2}$ is called the *center*, and it is obviously the affine center of the ellipse or the hyperbola.

It is easy to observe that ellipsoids and hyperboloids intersect line F_1F_2, the *main axis*, in exactly two points A and B and these satisfy $2a = d(A, B)$. The *numerical eccentricity* is defined by $\varepsilon = f/a$.

We usually *polar parameterize* the boundary $\partial\mathcal{D}$ of a compact domain $\mathcal{D}$ in $\mathbb{R}^2$ starlike with respect to a point $P \in \mathcal{D}$ so that $\boldsymbol{r}\colon [-\pi, \pi) \to \mathbb{R}^2$ is defined by $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, where r is the *radial function* of $\mathcal{D}$ with *base point* P.

A point $F \in \ell$ is an *ℓ-foot of* $P \notin \ell$ with respect to d, if $d(P, Q) \geq d(P, F)$ for every $Q \in \ell$. A line ℓ' intersecting the line ℓ in a point F is said to be *perpendicular to* ℓ with respect to d, if F is an ℓ-foot of P for every $P \in \ell' \setminus \{F\}$. We denote this relation by $\ell' \perp_d \ell$. These give that a line ℓ' intersecting the line ℓ in a point F is perpendicular ($\ell' \perp_d \ell$) to ℓ if there is a point $P \in \ell'$ such that $\mathcal{I}_P$ is tangent to ℓ at F.

Ratio $\varepsilon := \varrho/d(F, \ell)$ is called the *numerical eccentricity* of the conic given in (D_1), so, a conic is *elliptic*, *parabolic* and *hyperbolic*, if $\varepsilon < 1$, $\varepsilon = 1$ and $\varepsilon > 1$, respectively.

In Minkowski plane elliptic conics are convex curves that are strictly convex if and only if the indicatrix is strictly convex [4] (see [5, Theorem 1]), hyperbolic conics are the union of two simple curves and if one of these contains a segment, then the indicatrix also contains a segment [5, Theorem 4], and parabolic conics are convex curves [4, Lemma 3] that contain segment if and only if the indicatrix contains a segment [5, Theorem 5].

3. Utilities

For the results we are proving in the next sections, it is enough to work in the affine plane, because, by [2, (16.12), p. 91],

a convex body in $\mathbb{R}^n$ $(n \geq 3)$ is an ellipsoid if and only if for any fixed $k \in \{2, \dots, n-1\}$ every k-plane through the origin intersects it in a k-dimensional ellipsoid. (3.1) ⟨7, 9, 10⟩

Lemma 3.1. *Let $\mathcal{C}^{\varrho}_{d;F,\ell}$ be a conic in the Minkowski plane $(\mathbb{R}^2, d)$, and let $F^{\perp}$ be the ℓ-foot of F. Then*

(1) *segment $\overline{FF^{\perp}}$ intersects $\mathcal{C}^{\varrho}_{d;F,\ell}$ in a unique point A,*
(2) *the only nearest point of $\mathcal{C}^{\varrho}_{d;F,\ell}$ to ℓ is A, and*
(3) *the only nearest point of $\mathcal{C}^{\varrho}_{d;F,\ell}$ to F is A.*

If $\varrho = d(F, \ell)$, then

(4) *A is the only common point of line AF and $\mathcal{C}^{\varrho}_{d;F,\ell}$.*

If $\varrho < d(F, \ell)$, then

(5) *open ray $\overline{A}F$ intersects $\mathcal{C}^{\varrho}_{d;F,\ell}$ in a unique point B,*
(6) *the only farthest point of $\mathcal{C}^{\varrho}_{d;F,\ell}$ to ℓ is B, and*
(7) *the only farthest point of $\mathcal{C}^{\varrho}_{d;F,\ell}$ to F is B.*

If $\varrho > d(F, \ell)$, then

(8) *open ray $\overline{F}A$ intersects $\mathcal{C}^{\varrho}_{d;F,\ell}$ in a unique point B,*
(9) *the only nearest point of $\mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^-$ to ℓ is B, and*
(10) *the only nearest point of $\mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^-$ to F is B.*

Proof. Let $\mathcal{L}^+$ be the half plane of ℓ that contains F, and let $\mathcal{L}^-$ be the other half plane. Further, let F_∞ and $F^{\perp}_\infty$ be the ideal points of rays $\overline{F^{\perp}F}$ and $\overline{F}F^{\perp}$, respectively.

Observe that ratio $r(X) = d(F, X)/d(F^{\perp}, X)$ is continuous along the geodesic line $FF^{\perp}$ except at $F^{\perp}$, because d is continuous. Moreover, it is strictly monotone on segment $\overline{F^{\perp}F}$ and rays $\overline{F}F_\infty$ $\overline{F^{\perp}}F^{\perp}_\infty$, because the positive additivity of d. As $r(F) = 0$, $\lim_{X \to F^{\perp}} r(X) = \infty$, $\lim_{X \to F_\infty} r(X) = 1$, and $\lim_{X \to F^{\perp}_\infty} r(X) = 1$, (1), (4), (5), and (8) follow.

For any point $P \in \mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^+$ we have $d(F, A) = \varepsilon d(A, F^{\perp})$ and $d(F, P) = \varepsilon d(P, P^{\perp})$, and the triangle inequality implies

$$\begin{aligned}(1+\varepsilon)d(P^{\perp}, P) &= d(P^{\perp}, P) + d(P, F) \\ &\geq d(F, P^{\perp}) \geq d(F, F^{\perp}) = d(F^{\perp}, A) + d(A, F) \\ &= (1+\varepsilon)d(F^{\perp}, A) = (1+\varepsilon)d(A^{\perp}, A).\end{aligned}$$

This with the strictness of the triangle inequality proves (3) in $\mathcal{L}^+$, and by $d(F,A) = \varepsilon d(A,F^\perp)$ and $d(F,P) = \varepsilon d(P,P^\perp)$, also (2) is proved in $\mathcal{L}^+$.

For any point $P \in \mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^-$, by the minimizing property of the foot we have $d(P,F) \geq d(X,F) \geq d(F,F^\perp) \geq d(A,F)$, $X = \ell \cap FP$. This proves (3) in $\mathcal{L}^-$. Again the minimizing property of the foot implies

$$\begin{aligned}(1+\varepsilon)d(P^\perp,P) &= d(P^\perp,P) + d(P,F) \geq d(P,F)\\ &\geq d(X,F) \geq d(F,F^\perp) = d(F^\perp,A) + d(A,F) = (1+\varepsilon)d(A^\perp,A),\end{aligned}$$

where equality never holds. This proves (2) in $\mathcal{L}^-$.

Assume $\varepsilon < 1$ from now on.

If $P \in \mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^-$ and $X = \ell \cap FP$, then $d(P,P^\perp) \geq \varepsilon d(P,P^\perp) = d(P,F) > d(P,X) \geq d(P,P^\perp)$, a contradiction, hence $\mathcal{C}^{\varrho}_{d;F,\ell} \subset \mathcal{L}^+$.

For any point $P \in \mathcal{C}^{\varrho}_{d;F,\ell}$, $d(F,B) = \varepsilon d(B,F^\perp)$, $d(F,P) = \varepsilon d(P,P^\perp)$, and the triangle inequality implies

$$\begin{aligned}(1-\varepsilon)d(P,P^\perp) &= d(P,P^\perp) - d(P,F) \leq d(P,F^\perp) - d(P,F)\\ &\leq d(F,F^\perp) = (1-\varepsilon)d(B^\perp,B).\end{aligned}$$

By the strictness of the triangle inequality, equality holds only if $P \in \overline{F}B$ which implies $F^\perp = P^\perp$, hence $B = P$. This proves (6). However, as $d(F,B) = \varepsilon d(B,F^\perp)$ and $d(F,P) = \varepsilon d(P,P^\perp)$, this also proves (7).

Assume $\varepsilon > 1$ from now on.

If $P \in \mathcal{C}^{\varrho}_{d;F,\ell} \cap \mathcal{L}^-$ and $X = \ell \cap FP$, then

$$(\varepsilon-1)d(P,F) = \varepsilon(d(P,X) + d(X,F)) - \varepsilon d(P,P^\perp) \geq \varepsilon d(X,F) \geq \varepsilon d(F,F^\perp),$$

where equality holds if and only if $P^\perp = X$ which implies $F^\perp = P^\perp$, hence $B = P$. This proves (10). However, as $d(F,B) = \varepsilon d(B,F^\perp)$ and $d(F,P) = \varepsilon d(P,P^\perp)$, this also proves (9).

The proof of the theorem is completed. □

Take a conic $\mathcal{C}^{\varrho}_{d;O,\ell}$ in the Minkowski plane $(\mathbb{R}^2,d)$, and let t_I, t_J be the two tangent lines of $\mathcal{I}_O$ that are parallel to ℓ, and denote the points, where t_I and t_J touch $\mathcal{I}_O$, by I and J, respectively. Then $O \in \overline{IJ}$ by the symmetry of $\mathcal{I}$, hence $IJ \perp_d \ell$, and therefore the ℓ-foot $O^\perp$ of O is in $\ell \cap IJ$. We have the freedom to assume that $d(\ell,I) < d(\ell,J)$.

Let $A = \mathcal{C}^{\varrho}_{d;O,\ell} \cap \overline{OO^\perp}$, and let ℓ_A denote the tangent line of $\mathcal{C}^{\varrho}_{d;O,\ell}$ at A. Observe that ℓ_A is parallel with ℓ by (2) of Lemma 3.1.

Fix a coordinate system with origin $O = (0,0)$, $I = (-1,0)$, $\ell = \{(-l,y) : y \in \mathbb{R}\}$ for some $l > 0$, and $\{(0,1),(0,-1)\} = \mathcal{I}_O \cap \{(0,y) : y \in \mathbb{R}\}$. Fix the Euclidean metric such that $\{(1,0),(0,1)\}$ is an orthonormal bases. In this Euclidean metric, let $\partial\mathcal{I}_O$ be polar parameterized by $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, so that $\boldsymbol{r}(\pi) = (-1,0)$, and $\mathcal{C}^{\varrho}_{d;O,\ell}$ be polar parameterized as $\boldsymbol{c}(\varphi) = c(\varphi)\boldsymbol{u}_\varphi$. (See Figure 3.1.)

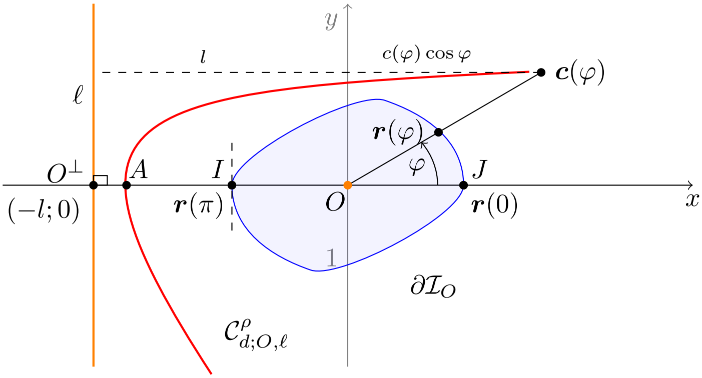

FIGURE 3.1. Conic $\mathcal{C}^{\rho}_{d;O,\ell}$ in a Minkowski plane of indicatrix $\mathcal{I}$.

Lemma 3.2. *The focal polar-equation of a conic in a Minkowski plane is*

$$c(\varphi) = \frac{l\varrho r(\varphi)}{l\sigma - \varrho r(\varphi)\cos\varphi}, \tag{3.2}$$ ⟨*5, 6, 10, 1*

where $\sigma \in \{-1, 1\}$.

Proof. A point $\boldsymbol{c}(\varphi)$ of $\mathcal{C}^{\varrho}_{d;O,\ell}$ satisfies $l\frac{|c(\varphi)|}{r(\varphi)} = ld(\boldsymbol{c}, O) = \varrho d(\boldsymbol{c}, \ell) = \varrho|l + c(\varphi)\cos\varphi|$, that can be reordered into the stated form, where $\sigma \in \{-1, 1\}$. $\square$

According to (3.2), we can also express the central polar-coordinates of $\partial\mathcal{I}_O$ by

$$r(\varphi) = \frac{l\sigma c(\varphi)}{\varrho(l + c(\varphi)\cos\varphi)}.$$

4. Symmetric conics

Let a conic $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ be given in the Minkowski plane $(\mathbb{R}^2, d)$. Let $F_1^{\perp}$ be the ℓ_--foot of F_1 on ℓ_-, $A = \mathcal{C}^{\varrho}_{d;F_1,\ell_-} \cap \overline{F_1F_1^{\perp}}$, and let ℓ_A be the tangent line of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ at A.

Assume that $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ is metrically symmetric in point O. Then O is the affine center of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$, and we introduce $F_2 = \bar{\rho}_O F_1$, $\ell_+ = \bar{\rho}_O \ell_-$, $B = \bar{\rho}_O A$, $\ell_B = \bar{\rho}_O \ell_A$, where $\bar{\rho}_O$ denotes the affine point reflection in point O, and define $l = d(\ell_-, F_1)$.

Let t_I, t_J be the two tangent lines of $\mathcal{I}_O$ that are parallel to ℓ_-, and denote the points, where t_I and t_J touch $\mathcal{I}_O$, by I and J, respectively. By the symmetry of $\mathcal{I}$ we have $O \in \overline{IJ}$, hence $IJ \perp_d \ell_-$. We have the freedom to assume that $d(\ell_-, I) < d(\ell_-, J)$,

As central symmetry maps every straight line onto a parallel straight line, we have $\ell_+ \parallel \ell_-$ and $\ell_A \parallel \ell_B$. Moreover, as $IJ \perp_d \ell$ and $F_1F_1^{\perp} \perp_d \ell$ imply $IJ \parallel F_1F_1^{\perp}$, we also have $\ell_A \parallel \ell_-$ and $\ell_B \parallel \ell_+$.

Lemma 4.1. *We have*

(1) $\varrho \neq l$,
(2) $F_1F_2 \equiv IJ$,
(3) $O \in \overline{F_1F_2} \cap \overline{AB}$.

Proof. (1) It follows from (3.2) that for $\varrho = d(\ell_-, F_1)$ there is only one ray starting from F_1 which does not intersect $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$. This contradicts the symmetry of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$, hence (1).

As we have either $\varrho < l$ or $\varrho > l$, by (8) and (5) of Lemma 3.1, there exists a unique common point A' of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ and $F_1F_1^{\perp}$ other than A, that, by (6) and (9), is an extremal point of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$. By the symmetry, with the same reasoning, there is a unique extremal common point B' of $\mathcal{C}^{\varrho}_{d;F_2,\ell_+} = \mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ and $F_2F_2^{\perp}$ too.

Thus, by Lemma 3.1, Figure 4.1 shows what we have.

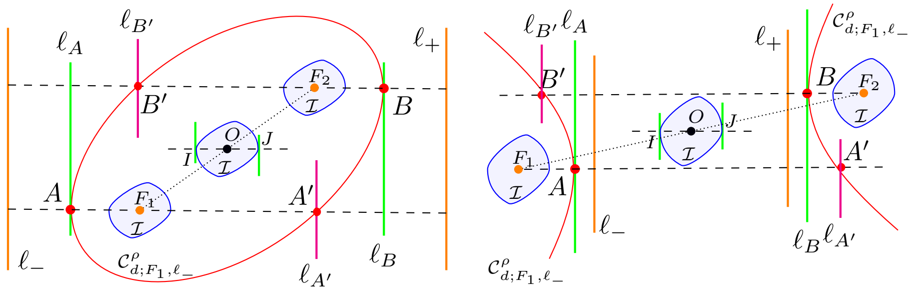

FIGURE 4.1. Center of symmetry of conics in a Minkowski plane

(2) If $\varrho < l$, then, by (6) of Lemma 3.1, the only farthest point of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ to ℓ_- is A', hence A' is the only nearest point of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ to ℓ_+, that, by (2) of Lemma 3.1, is B. This implies $A' \equiv B$ and $A \equiv B'$, hence $O \in \overline{F_1F_2} \subset \overline{AB}$.

If $\varrho > l$, then, by (9) of Lemma 3.1, the only nearest point of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ to ℓ_- is A', hence A' is the only nearest point of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ to ℓ_+, that, by (2) of Lemma 3.1, is B. This implies $A' \equiv B$ and $A \equiv B'$, hence $O \in \overline{AB} \subset \overline{F_1F_2}$. Thus, (2) is proved.

Now the lemma is completely proved. $\square$

Denote the straight line through O parallel to ℓ_- by ℓ_0, and let ℓ_i be the straight line through F_i parallel to ℓ_- for $i = 1,2$. We also let $\{H_0^-, H_0^+\} = \partial\mathcal{I} \cap \ell_0$ and $\{P_i^-, P_i^+\} = \mathcal{C}^{\varrho}_{d;F_1,\ell_-} \cap \ell_i$ for $i = 1,2$.

Further, we fix the affine coordinate system for which $O = (0,0)$, $I = (-1,0)$, $J = (1,0)$ and $H_0^{\pm} = (0,\pm 1)$, and choose the Euclidean metric d_e such that $\{(0,1),(1,0)\}$ is an orthonormal basis. Equipped with these, we clearly have $F_1 = (-f,0)$ and $F_2 = (f,0)$, $A = (-a,0)$ and $B = (a,0)$, where $a = \varepsilon l$, and $\ell_{\pm} = \{(\pm l, y) : y \in \mathbb{R}\}$.

Let the border $\partial\mathcal{I}$ of the indicatrix $\mathcal{I}$ be polar parameterized by $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, where $\boldsymbol{r}(0) = (1,0)$. Further, let $\mathcal{C}^\varrho_{d;F_1,\ell_-}$ be parameterized by $\boldsymbol{c}(\varphi) = c(\varphi)\boldsymbol{u}_\varphi$, where $\boldsymbol{c}(0) = (a,0)$. Let the point $C(\varphi)$ of $\mathcal{C}^\varrho_{d;F_1,\ell_-} = \mathcal{C}^\varrho_{d;F_2,\ell_+}$ be defined by $\overrightarrow{OC(\varphi)} = \boldsymbol{c}(\varphi)$ (hence $C(\pi) = (-a,0)$), and let $C^\perp_\pm$ be the $\ell_\pm$-foot of $C(\varphi)$, which is clearly the same for both metrics d and d_e. Further, we introduce the Euclidean angles α and β, such that $C(\varphi) = F_1 + c_1(\alpha(\varphi))\boldsymbol{u}_{\alpha(\varphi)}$ and $C(\varphi) = F_2 + c_2(\beta(\varphi))\boldsymbol{u}_{\beta(\varphi)}$, where c_1 and c_2 are positive.

4.1. Elliptic symmetric conics. In this subsection we consider elliptic conics $\mathcal{C}^\varrho_{d;F_1,\ell_-}$, where $\varrho < l$, i.e. $\varepsilon \in [0,1)$. Figure 4.2 shows our configuration

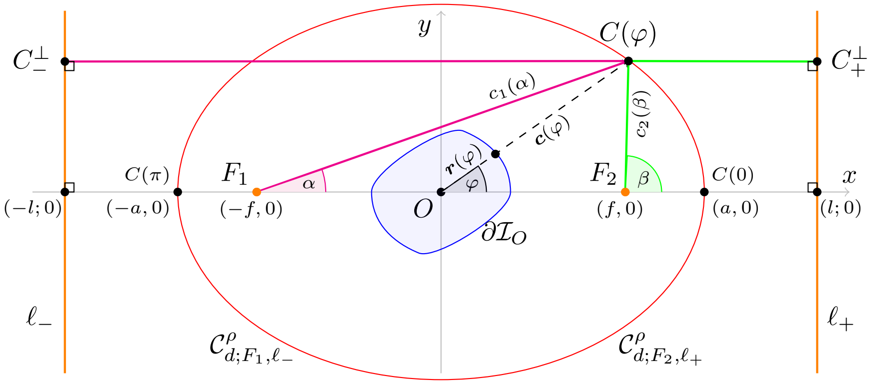

Figure 4.2. Notations: an elliptic conic symmetric to O.

By the definitions of $\mathcal{C}^\varrho_{d;F_1,\ell_-}$ and $\mathcal{C}^\varrho_{d;F_2,\ell_+}$ we have

$$\begin{aligned}\frac{c_1(\alpha)}{r(\alpha)} &= \varepsilon d(C^\perp_-, C) = \varepsilon\frac{l-f+c_1(\alpha)\cos\alpha}{r(0)} = \varepsilon(l-f+c_1(\alpha)\cos\alpha),\\ \frac{c_2(\beta)}{r(\beta)} &= \varepsilon d(C, C^\perp_+) = \varepsilon\frac{l-f-c_2(\beta)\cos\beta}{r(0)} = \varepsilon(l-f-c_2(\beta)\cos\beta),\end{aligned} \tag{4.1}$$ ⟨7⟩

where $\varepsilon = \varrho/(l-f) < 1$.

Theorem 4.2. *Euclidean is the only Minkowski geometry that has centrally symmetric elliptic conic.*

Proof. By (3.1) we need to work only in the plane. Formulas (4.1) give

$$c_1(\alpha) = \frac{\varepsilon(l-f)r(\alpha)}{1-\varepsilon r(\alpha)\cos\alpha}, \quad\text{and}\quad c_2(\beta) = \frac{\varepsilon(l-f)r(\beta)}{1+\varepsilon r(\beta)\cos\beta}.$$

Substituting these into $c_1(\alpha)\cos\alpha - c_2(\beta)\cos\beta = 2f$ and into $c_1(\alpha)\sin\alpha = c_2(\beta)\sin\beta$ results in

$$\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha} - \frac{\varepsilon r(\beta)\cos\beta}{1+\varepsilon r(\beta)\cos\beta} = \frac{2f}{l-f}, \quad\text{and}$$

$$\frac{r(\alpha)\sin\alpha}{1-\varepsilon r(\alpha)\cos\alpha}-\frac{r(\beta)\sin\beta}{1+\varepsilon r(\beta)\cos\beta}=0. \tag{4.2}$$ ⟨8⟩

The previous one implies

$$\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha}+\frac{1}{1+\varepsilon r(\beta)\cos\beta}=\frac{2f}{l-f}+1=\frac{l+f}{l-f} \tag{4.3}$$ ⟨8⟩

which gives

$$\varepsilon r(\beta)\cos\beta=\Big(\frac{l+f}{l-f}-\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big)^{-1}-1=\frac{(l+f)\varepsilon r(\alpha)\cos\alpha-2f}{l+f-2l\varepsilon r(\alpha)\cos\alpha}.$$

Putting (4.3) into (4.2) leads to

$$r(\beta)\sin\beta=\frac{r(\alpha)\sin\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big(\frac{l+f}{l-f}-\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big)^{-1}=\frac{(l-f)r(\alpha)\sin\alpha}{l+f-2l\varepsilon r(\alpha)\cos\alpha}.$$

As $f=l\varepsilon^2$, we have the map $\Phi\colon r(\alpha)\boldsymbol{u}_\alpha\to r(\beta)\boldsymbol{u}_\beta$, as

$$\Phi\colon (x,y)\mapsto\frac{1}{(1+\varepsilon^2)-2\varepsilon x}((1+\varepsilon^2)x-2\varepsilon,(1-\varepsilon^2)y). \tag{4.4}$$ ⟨8, 10⟩

It is clear that $\partial\mathcal{I}_O$ is an invariant curve of Φ. Looking for an invariant curve as a function $\phi(x)=y>0$, we get $\phi(\Phi_1(x))=\Phi_2(\phi(x))$ from (4.4), that is

$$\frac{(1+\varepsilon^2)-2\varepsilon x}{1-\varepsilon^2}\phi\Big(\frac{(1+\varepsilon^2)x-2\varepsilon}{(1+\varepsilon^2)-2\varepsilon x}\Big)=\phi(x). \tag{4.5}$$ ⟨8⟩

This is an equation of type [3, (4.20)], where $f(z)=\frac{(1+\varepsilon^2)z-2\varepsilon}{(1+\varepsilon^2)-2\varepsilon z}$, $g(z)=\frac{(1+\varepsilon^2)-2\varepsilon x}{1-\varepsilon^2}$, and $h(z)\equiv 0$ are all analytic. Then, [3, Theorem 4.6] proves that (4.5) has one and only one solution in a neighborhood of $(1,0)$ which is, in addition, analytic. However, the circle $x^2+y^2=1$ is an invariant curve of Φ, because $x^2+y^2=1$ implies

$$((1-\varepsilon)^2-2\varepsilon x)^2=((1+\varepsilon^2)x-2\varepsilon)^2+(1-\varepsilon^2)^2y^2, \tag{4.6}$$ ⟨8⟩

so, in a neighborhood of $(1,0)$, $\partial\mathcal{I}_O$ coincides with the circle $x^2+y^2=1$.

Let $\alpha_0=\sup\{\alpha_0\in(0,\pi):r(\alpha)=1 \text{ for every } \alpha\in(0,\alpha_0)\}$. Then, for every $\alpha\in(0,\alpha_0)$, we have $\boldsymbol{u}_\beta=\Phi(\boldsymbol{u}_\alpha)$, where $\beta\in(0,\pi)$ by (4.6). Further, we also have $\beta>\alpha$, because

$$\cot\beta=\frac{(1+\varepsilon^2)x-2\varepsilon}{(1-\varepsilon^2)y}<\frac{x}{y}=\cot\alpha$$

by (4.4). Applying this inequality to α_0 leads to contradiction unless $\alpha_0=\pi$, hence the circle $x^2+y^2=1$ coincides with $\partial\mathcal{I}_O$ everywhere. □

4.2. Hyperbolic symmetric conics. In this subsection we consider hyperbolic conics $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$, where $\varrho > l$, i.e. $\varepsilon > 1$. Figure 4.3 shows what we have.

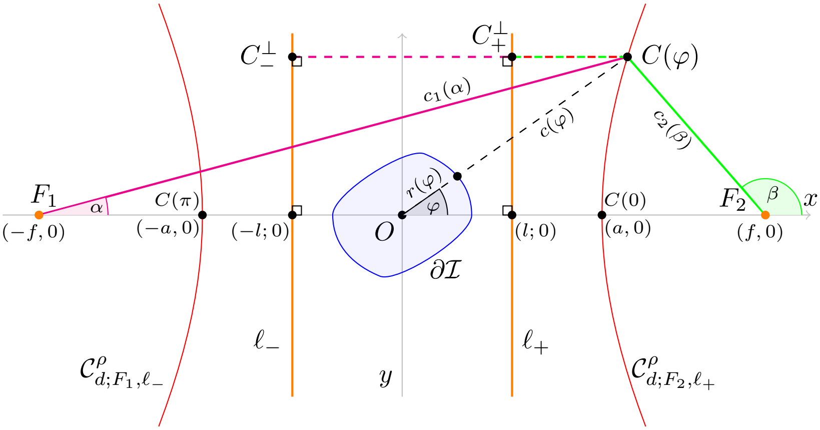

Figure 4.3. Notations: a hyperbolic conic symmetric to O.

By the definitions of $\mathcal{C}^{\varrho}_{d;F_1,\ell_-}$ and $\mathcal{C}^{\varrho}_{d;F_2,\ell_+}$ that

$$\begin{aligned}\frac{c_1(\alpha)}{r(\alpha)} &= \varepsilon d(C_-^{\perp}, C) = \varepsilon\frac{c_1(\alpha)\cos\alpha - (f-l)}{r(0)} = \varepsilon(l - f + c_1(\alpha)\cos\alpha),\\ \frac{c_2(\beta)}{r(\beta)} &= \varepsilon d(C, C_+^{\perp}) = \varepsilon\frac{(f-l) + c_2(\beta)\cos\beta}{r(0)} = \varepsilon(f - l + c_2(\beta)\cos\beta),\end{aligned} \tag{4.7}$$ ⟨9⟩

where $\varepsilon = \varrho/(f-l) > 1$.

Theorem 4.3. *Euclidean is the only Minkowski geometry that has centrally symmetric hyperbolic conic.*

Proof. By (3.1) we need to work only in the plane. Formulas (4.7) give

$$c_1(\alpha) = \frac{\varepsilon(l-f)r(\alpha)}{1-\varepsilon r(\alpha)\cos\alpha}, \quad \text{and} \quad c_2(\beta) = \frac{\varepsilon(f-l)r(\beta)}{1-\varepsilon r(\beta)\cos\beta}.$$

Substituting these into $c_1(\alpha)\cos\alpha - c_2(\beta)\cos\beta = 2f$ and into $c_1(\alpha)\sin\alpha = c_2(\beta)\sin\beta$ results in

$$\begin{aligned}\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha} + \frac{\varepsilon r(\beta)\cos\beta}{1-\varepsilon r(\beta)\cos\beta} &= \frac{2f}{l-f}\\ \frac{r(\alpha)\sin\alpha}{1-\varepsilon r(\alpha)\cos\alpha} + \frac{r(\beta)\sin\beta}{1-\varepsilon r(\beta)\cos\beta} &= 0.\end{aligned} \tag{4.8}$$ ⟨10⟩

The first one implies

$$\frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha} + \frac{1}{1-\varepsilon r(\beta)\cos\beta} = \frac{2f}{l-f} + 1 = \frac{l+f}{l-f} \tag{4.9}$$ ⟨10⟩

which gives

$$\varepsilon r(\beta)\cos\beta = 1 - \Big(\frac{l+f}{l-f} - \frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big)^{-1}\frac{2f-(l+f)\varepsilon r(\alpha)\cos\alpha}{l+f-2l\varepsilon r(\alpha)\cos\alpha}$$

Putting (4.9) into (4.8) gives

$$r(\beta)\sin\beta = \frac{-r(\alpha)\sin\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big(\frac{l+f}{l-f} - \frac{\varepsilon r(\alpha)\cos\alpha}{1-\varepsilon r(\alpha)\cos\alpha}\Big)^{-1} = \frac{(f-l)r(\alpha)\sin\alpha}{l+f-2l\varepsilon r(\alpha)\cos\alpha}.$$

As $f = l\varepsilon^2$, the map $\Psi\colon r(\alpha)\boldsymbol{u}_\alpha \to r(\beta)\boldsymbol{u}_\beta$ takes the form

$$\Psi\colon (x,y) \mapsto \frac{1}{(1+\varepsilon^2)-2\varepsilon x}\big(2\varepsilon-(1+\varepsilon^2)x, (\varepsilon^2-1)y\big),$$

that, compared to (4.4), shows that $\Psi(x,y) = -\Phi(x,y)$. Now the proof can be finished in the same way as the previous one. $\square$

5. Quadratical conics

We start with the setup given at Lemma 3.2, and consider (3.2).

If $\rho \le l$, then, as $r(\varphi)|\cos\varphi| \le 1$ by the convexity of $\mathcal{I}$, we have $1-\frac{\rho}{l}r(\varphi)\cos\varphi \ge 0$, hence $\sigma = 1$ and c is bounded unless $\rho = l$ and $\varphi = 0$. In the other hand, if $\rho > l$, then there exists at least one $\varphi \in (0,\pi)$ which satisfies $\frac{l\sigma}{\rho} = r(\varphi)\cos\varphi$. Based on these observations and by the continuity of $\mathcal{C}^{\varrho}_{d;O,\ell}$ we deduce that a quadratical conic $\mathcal{C}^{\varrho}_{d;O,\ell}$ is a quadric that is

- (T_1) elliptic if $\rho \in (0,l)$, because c is bounded,
- (T_2) parabolic if $\rho = l$, because c tends to infinity exclusively if $\varphi \to 0$, and
- (T_3) hyperbolic if $\rho > l$, because c tends to infinity for at least two different angles.

Theorem 5.1. *Euclidean is the only Minkowski plane in which quadratical conic exists.*

Proof. By (3.1) we need to work only in the plane. We continue to work in the setup given in the proof of Lemma 3.2.

$\boxed{\textit{First we assume that } \ell = \infty.}$

Then (3.2) gives $c(\varphi) = \rho r(\varphi)$, therefore $\partial\mathcal{I}$ is a homothetic image of $\mathcal{C}^{\varrho}_{d;O,\ell}$, which, by (T_1), is an ellipse. Consequently the indicatrix is an ellipse, hence the geometry is Euclidean [1, 25.4].

$\boxed{\textit{Now we assume that } \varrho \in (0,l).}$

As an elliptic quadric is symmetric, Theorem 4.2 implies that the Minkowski plane is Euclidean.

Now we assume that $\varrho = l$.

Then, by (T_2), $\mathcal{C}^{\varrho}_{d;O,\ell}$ is a parabolic Euclidean conic. This parabolic Euclidean conic is intersected by the x-axis perpendicularly, and the x-axis intersects it in exactly one point.

Therefore, by (D_q), the affine equation of $\mathcal{C}^{\varrho}_{d;O,\ell}$ is

$$a(x-p) = y^2 \tag{5.1}$$ ⟨11⟩

with parameters $a > 0$ and $p \in \mathbb{R}$. According to (3.2) point $\boldsymbol{c}(\varphi)$ has coordinates

$$x = \frac{lr(\varphi)\cos\varphi}{r(\pi) - r(\varphi)\cos\varphi}, \qquad y = \frac{lr(\varphi)\sin\varphi}{r(\pi) - r(\varphi)\cos\varphi}.$$

Substituting these into (5.1) gives

$$\frac{alr(\varphi)\cos\varphi}{r(\pi) - r(\varphi)\cos\varphi} - ap = \frac{l^2 r^2(\varphi)\sin^2\varphi}{(r(\pi) - r(\varphi)\cos\varphi)^2}.$$

i.e.

$$\begin{aligned} ar(\pi)lr(\varphi)\cos\varphi - alr^2(\varphi)\cos^2\varphi - l^2r^2(\varphi)\sin^2\varphi& \\ = ap(r^2(\pi) - 2r(\pi)r(\varphi)\cos\varphi + r^2(\varphi)\cos^2\varphi)&. \end{aligned} \tag{5.2}$$ ⟨11⟩

Taking this at $\varphi + \pi$, by $r(\varphi + \pi) = r(\varphi)$, we get

$$\begin{aligned} -ar(\pi)lr(\varphi)\cos\varphi - alr^2(\varphi)\cos^2\varphi - l^2r^2(\varphi)\sin^2\varphi& \\ = ap(r^2(\pi) + 2r(\pi)r(\varphi)\cos\varphi + r^2(\varphi)\cos^2\varphi)&. \end{aligned} \tag{5.3}$$ ⟨11⟩

Substraction of (5.3) from (5.2) yields $ar(\pi)r(\varphi)\cos\varphi(l + 2p) = 0$, i.e. $l + 2p = 0$. Summing up (5.2) and (5.3) gives

$$\begin{aligned} -alr^2(\varphi)\cos^2\varphi - l^2r^2(\varphi)\sin^2\varphi &= apr^2(\pi) + apr^2(\varphi)\cos^2\varphi \\ r^2(\varphi)\big(a(p+l)\cos^2\varphi + l^2\sin^2\varphi\big) &= -apr^2(\pi) \\ r^2(\varphi)\Big(\frac{a}{2}\cos^2\varphi + l\sin^2\varphi\Big) &= \frac{a}{2}r^2(\pi). \end{aligned}$$

Choosing $\varphi = \pi/2$ implies $a = \frac{2lr^2(\pi/2)}{r^2(\pi)}$, hence, we obtain

$$\frac{1}{r^2(\varphi)} = \frac{\cos^2\varphi}{r^2(\pi)} + \frac{\sin^2\varphi}{r^2(\pi/2)}, \tag{5.4}$$

which is the polar form of an Euclidean ellipse relative its center [8]. Thus $\partial\mathcal{I}$ is a Euclidean ellipse, and therefore the Minkowski plane is Euclidean.

Now we assume that $\varrho > l$.

As every hyperbolic quadric is symmetric, Theorem 4.3 implies that the Minkowski plane is Euclidean. □

6. Conical ellipses and hyperbolas

According to [6], A. Moór posed the problem to determine those Finsler manifolds in which

(1) the class of elliptic conics coincides with the class of ellipses, or
(2) the class of hyperbolic conics coincides with the class of hyperbolas.

Tamássy and Bélteky considered only case (1) and they found in [7, Theorem 2], that only the Euclidean space fulfills (1).

We call an ellipse or hyperbola *conical* if it is a conic.

As every ellipse and every hyperbola is symmetric in Minkowski geometry, every conical ellipse and every conical hyperbola is a symmetric conic, hence Theorem 4.2 and 4.3 imply the following.

Theorem 6.1. *Euclidean is the only Minkowski geometry in which either a conical ellipsoid or a conical hyperboloid exists.*

We raise the analogous problem to determine projective-metric spaces in which

(a) some or all ellipses are conical, or
(b) some or all hyperbolas are conical.

As Minkowski geometries are projective-metric spaces, our results give some support to conjecture that in both cases only the Euclidean space is the solution.

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References

[1] H. Busemann and P. J. Kelly, *Projective Geometries and Projective Metrics,* Academic Press, New York, 1953. ⟨1, 10⟩
[2] H. Busemann, *The Geometry of Geodesics,* Academic Press, New York, 1955. ⟨3⟩
[3] S. S. Cheng and W. Li, *Analytic Solutions of Functional Equations,* World Scientific, New Jersey, 2008. ⟨8⟩
[4] M. A. Ghandehari, Heron's problem in the Minkowski plane, *Technical Report of Math. Dept. of University of Texas at Arlington,* **306** (1997), http://hdl.handle.net/10106/2500. ⟨2, 3⟩
[5] Á. G. Horváth and H. Martini, Conics in normed planes, *Extracta Math.*, **26:1** (2011), 29–43; available also at arXiv: 1102.3008. ⟨2, 3⟩
[6] L. Tamássy and K. Bélteky, On the coincidence of two kinds of ellipses in Minkowskian and Finsler planes, *Publ. Math. Debrecen*, **31:3-4** (1984), 157–161. ⟨1, 12⟩
[7] L. Tamássy and K. Bélteky, On the coincidence of two kinds of ellipses in Riemannian and in Finsler spaces, *Coll. Math. Soc. J. Bolyai*, **46** (1988), 1193-1200. ⟨12⟩
[8] `https://en.wikipedia.org/wiki/Ellipse`. ⟨11⟩

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