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Support curves of invertible Radon transforms

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Abstract. Let S and the origin be different points of the closed curve $\mathcal{S}$ in the plane. For any point P there is exactly one orientation preserving similarity $\mathcal{A}_P$ which fixes the origin and takes S to P. The function transformation

$$\mathrm{R}_{\mathcal{S}}\, f(P) = \int_{\mathcal{A}_P \mathcal{S}} f(X)\, dX$$

is said to be the Radon transform with respect to the *support curve* $\mathcal{S}$, where dX is the arclength measure on $\mathcal{A}_P\mathcal{S}$. The invertibility of $\mathrm{R}_{\mathcal{S}}$ is proved on a subspace of the C^2 functions if $\mathcal{S}$ has strictly convex distance function. The support theorem is shown on a subspace of the L^2 functions for curves having exactly two cross points with any of the circles centered to the origin. Counterexample shows the necessity of this condition. Finally a generalization to higher dimensions and a continuity result are given.

1. Introduction

Radon's problem to recover a function from its integrals along straight lines in the plane has been generalized in many ways [1-7]. With respect to our following investigations the most interesting one is found in Cormack's paper [2]. In his paper Cormack gave an inversion formula for Radon's problem when the line integrals are evaluated along the so called α-curves given, for fixed polar coordinates (p, φ), by $r^\alpha \cos(\alpha(\psi - \varphi)) = p^\alpha$, where α is real and nonzero. This family of curves contains the most familiar curves like parabolas, straight lines, circles etc. The aim of this paper is to prove the invertibility and the support theorem for more general curves.

Our considerations constitute also a part of a more general question appears in the literature [1-3,6,8]. Let us take a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$, the *support hypersurface*

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in our terminology, and a transformation group G of $\mathbb{R}^n$. Then the corresponding Radon transform for continuous functions f of compact support is defined by

$$\mathrm{R}_{\mathcal{S}}\, f(g) = \int_{g\mathcal{S}} f(X)\, dX, \qquad (g \in G)$$

where dX is the natural surface measure on $g\mathcal{S}$. For the original Radon transform, $\mathcal{S}$ is a hyperplane, not intersecting the origin, and G contains the rotations around the origin and the translation with the multiples of a nonzero vector. Other way is to take G as the group of the rotations around the origin and the dilations [4]. This is the case in [2,3] too, where the support curves are α-curves and sphere. In [1] $\mathcal{S}$ is the unit sphere and G is the group of the dilations and the translations by multiples of $(n-1)$ independent vectors of $\mathbb{R}^n$. The group G is the same in [7], but the support hypersurface $\mathcal{S}$ is a general $\mathrm{C}^{(n+5)/2}$ hypersurface. Other example can be found in the first section of [7], where $\mathcal{S}$ is an ellipsoid with the origin as one of its focuses and G is the group of the rotations around the y-axis, the dilations and the dilations along the y-axis.

In our case, G will be the group of the rotations around the origin and the dilations as in [2,3]. We prove the invertibility on a subspace of the C^2 functions for C^2 support curves having strictly convex distance function. Then the support theorem is shown on a subspace of the square integrable functions for C^4 support curves having exactly two cross points with any of the circle centered to the origin. At the end of the second section counterexample shows the necessity of this condition. In the third section the analogous statements for higher dimensions and a continuity result are proved.

The paper closes with a discussion of the results and their generalizations. It turns out, that the invertibility of the Radon transform requests the support curve resp. hypersurface to satisfy our conditions only near its point farthest from the origin.

2. Support curves

Let S and the origin be different points of the closed curve $\mathcal{S}$ in the plane. For any point P there is exactly one orientation preserving similarity $\mathcal{A}_P$ which fixes the origin and takes S to P. The function transformation

$$\mathrm{R}_{\mathcal{S}}\, f(P) = \int_{\mathcal{A}_P\mathcal{S}} f(X)\, dX$$

is said to be the Radon transform with respect to the *support curve* $\mathcal{S}$, where dX is the arclength measure on $\mathcal{A}_P\mathcal{S}$.

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Define the following function spaces for $k \in \mathbb{Z}$ and $m > 0$.

$$\mathrm{L}^2(\mathbb{R}^n, r^k) = \{f\colon \mathbb{R}^n \to \mathbb{R}\colon f(X)|X|^{k/2} \in \mathrm{L}^2(\mathbb{R}^n)\},$$
$$\mathrm{L}^2_m(\mathbb{R}^n, r^k) = \{f \in \mathrm{L}^2(\mathbb{R}^n, r^k)\colon |X| \le m \Rightarrow f(X) = 0\}$$

and $\mathrm{L}^2_*(\mathbb{R}^n, r^k) = \bigcup_{m>0} \mathrm{L}^2_m(\mathbb{R}^n, r^k)$.

Theorem 1. *If $\mathcal{S}$ is C^2 curve having strictly convex distance function then the Radon transform $\mathrm{R}_{\mathcal{S}}$ is invertible on the space C^2_c.*

Proof. Let $f \in \mathrm{L}^2_2(\mathbb{R}^2)$. We are tracing back this theorem to Mukhometov's Theorem 2 in [5] via the inversion of the plane to the unit circle. (See Figure 1; the idea to use inversion originates to [2]).

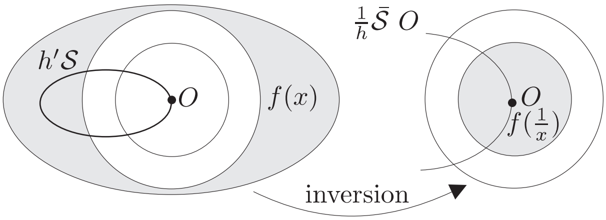

Figure 1

It is clear that the arclength measure transforms to the arclength measure multiplied by $|X|^{-2}$. Hence the Radon transform of the function f transforms to the Radon transform of the function $h(X) = f(X|X|^{-2})|X|^{-2}$. By our condition h is in $\mathrm{C}^2_c(\mathbb{R}^2)$. Thus, to use Mukhometov's theorem, we have to consider the family of the inversed support curves and, as we shall see below, the straight lines through the origin. Since the other two necessary conditions are obviously fulfilled in our situation we deal only with the first one. This condition says that any two points in the disc should be joined by exactly one curve. Translating this to our original support curve $\mathcal{S}$ we get that no two inscribed triangles with common vertex in the origin can be similar, but any triangle should have a similar inscribed triangle with one vertex in the origin.

To prove the uniqueness, let the curve $\mathcal{S}$ be parameterized in polar coordinate system as $(s(\zeta), \zeta)$, where ζ varies on $[0, \pi]$. The distance function s is convex and $s(0) = s(\pi) = 0$. If two inscribed triangles with common vertex in the origin are similar than there exist $\xi, \psi \in (0, \pi)$ such that

$$\frac{s(\xi + \alpha)}{s(\xi)} = \frac{s(\psi + \alpha)}{s(\psi)}.$$

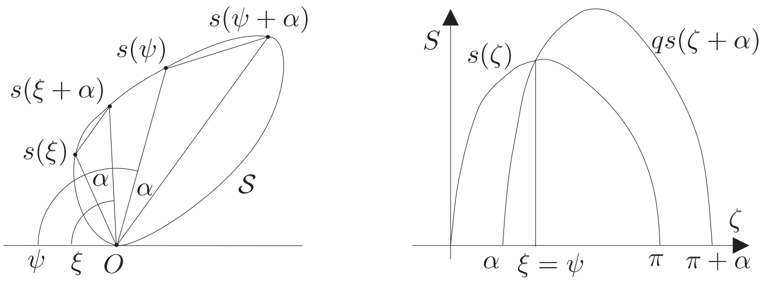

Figure 2.

Let $q = \frac{s(\xi)}{s(\xi+\alpha)}$. Then the equation $qs(\zeta+\alpha) = s(\zeta)$ has two solutions for ζ, namely ξ and ψ. But there must be exactly one solution as the right hand side of the Figure 2 shows.

For the existence part of the condition, the above argument gives a solution to $q = \frac{s(\xi)}{s(\xi+\alpha)}$ for any $0 < \alpha < \pi$ and $0 < q$. Hence the only problem is the case $\alpha = \pi$, that means the segment of the two points contains the origin. Therefore we need the integrals of h along the straight lines going through the origin. For any $\varepsilon > 0$, we know the integral $I(\varepsilon)$ of h along the inversed curve $c(\varepsilon)$ through the points $p_{-1}(\varepsilon) = (\cos(\pi-\varepsilon), \sin(\pi-\varepsilon))$ and $p_1(\varepsilon) = (\cos\varepsilon, \sin\varepsilon)$. The curve $c(\varepsilon)$ is convex and the angles to the x-axis $\alpha_{-1}(\varepsilon)$ and $\alpha_1(\varepsilon)$ of its tangents at $p_{-1}(\varepsilon)$ and $p_1(\varepsilon)$ go to zero as $\varepsilon \to 0$, because $\mathcal{S}$ is C^2 near S, the farthest point of $\mathcal{S}$ from the origin. Furthermore $\alpha_{-1}(\varepsilon) < 0 < \alpha_1(\varepsilon)$, hence the arclength measure on $c(\varepsilon)$ approaches uniformly to the arclength measure of the segment $p_{-1}(0)p_1(0)$. This implies $\lim_{\varepsilon\to 0} I(\varepsilon) = I(0)$, where $I(0)$ is the integral of h along the segment $p_{-1}(0)p_1(0)$, that completes the proof.

■

Now we present a theorem for an other family of support curves on a bigger function space. The invertibility will be a consequence of the support theorem here.

Let S be the point of the support curve $\mathcal{S}$ farthest from the origin. Suppose that $|OS| = 1$ and $\mathcal{S}$ has two parts parameterized in polar coordinate system by $(r, \varphi(r))$ and $(r, \psi(r))$ on OS, where S is the point $(1, 0)$.

Theorem 2. *Let the symmetric curve $\mathcal{S}$ have curvature $\kappa > 1$ at S and* $\frac{\varphi(r)}{\sqrt{1-r}} \in \mathrm{C}^2([0,1])$.

i) *If $f \in \mathrm{L}^2_*(\mathbb{R}^2)$ and $\mathrm{R}_{\mathcal{S}} f(P) = 0$ for $|P| \le m$, then $f \in \mathrm{L}^2_m(\mathbb{R}^2)$.*

ii) *$\mathrm{R}_{\mathcal{S}}$ is one-to-one on $\mathrm{L}^2_*(\mathbb{R}^2)$.*

Proof. Since (ii) is clearly implied by (i), we deal only with the first statement that is in fact the support theorem.

There must be $n>0$ that $f\in \mathrm{L}^2_n(\mathbb{R}^2)$ and so the claim is not obvious only in the case when $m>n$. If we regard f and $\mathrm{R}_{\mathcal{S}}\, f$ in polar coordinates, our Radon transform takes the form

(1)
$$\mathrm{R}_{\mathcal{S}}\, f(r,\alpha)=\int_0^1\Big(f(rt,\alpha+\varphi(t))\sqrt{1+t^2\dot\varphi^2(t)}+f(rt,\alpha+\psi(t))\sqrt{1+t^2\dot\psi^2(t)}\Big)r\,dt.$$

The expansions of f and $\mathrm{R}_{\mathcal{S}}\, f$ into Fourier series are

$$f(r,\alpha)=\sum_{k=-\infty}^{\infty}f_k(r)e^{ik\alpha}\quad\text{and}\quad \mathrm{R}_{\mathcal{S}}\, f(r,\alpha)=\sum_{k=-\infty}^{\infty}(\mathrm{R}_{\mathcal{S}}\, f)_k(r)e^{ik\alpha}.$$

Using the symmetry of $\mathcal{S}$ (i.e. $\varphi=-\psi$) a quick calculation shows

(2)
$$(\mathrm{R}_{\mathcal{S}}\, f)_k(r)=2\int_n^r f_k(t)\cos(k\varphi(t/r))\sqrt{1+\frac{t^2}{r^2}\dot\varphi^2\left(\frac{t}{r}\right)}\,dt=2\int_n^r f_k(t)\bar K(r,t)\,dt.$$

To find out the singularity of the kernel $\bar K(r,t)$ we prove that

$$(*)\qquad \lim_{\varepsilon\to 0}\dot\varphi(1-\varepsilon)\sqrt{\varepsilon}=\frac{-1}{\sqrt{2\kappa-2}}.$$

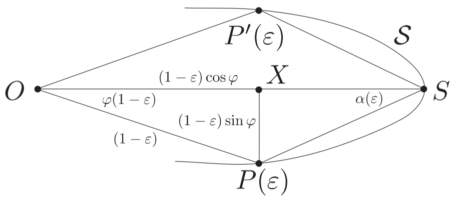

Figure 3.

See Figure 3. Let $P(\varepsilon)$ be the point $(1-\varepsilon,\varphi(1-\varepsilon))$, and $\alpha(\varepsilon)$ be the angle $P(\varepsilon)SO\angle$. X is the orthogonal projection of $P(\varepsilon)$ to OS and $P'(\varepsilon)$ is the reflection of $P(\varepsilon)$ to OS. The basic theory of the curves states that the radius $\varrho(\varepsilon)$ of the circle determined by the points $P(\varepsilon)$, S and $P'(\varepsilon)$ tends to $1/\kappa$. Taking into account the symmetry of $\mathcal{S}$ we obtain

$$\varrho(\varepsilon)=\frac{|SP|/2}{\cos\alpha(\varepsilon)}=\frac{|SX|^2+|XP|^2}{2|SX|}\longrightarrow\frac{1}{\kappa}.$$

Since SX approaches the zero as ε goes to zero this implies

$$\kappa = \lim \frac{2(1-(1-\varepsilon)\cos\varphi(1-\varepsilon))}{(1-\varepsilon)^2\sin^2\varphi(1-\varepsilon)} = \lim \frac{2(\varepsilon + 2(1-\varepsilon)\sin^2\frac{\varphi(1-\varepsilon)}{2})}{4(1-\varepsilon)^2\sin^2\frac{\varphi(1-\varepsilon)}{2}\cos^2\frac{\varphi(1-\varepsilon)}{2}}$$
$$\overset{(**)}{=} 1 + \lim \frac{2\varepsilon}{\varphi^2(1-\varepsilon)} = 1 + \lim \frac{-1}{\varphi(1-\varepsilon)\dot\varphi(1-\varepsilon)},$$

where the last equation comes from $(**)$ using the L'Hospital rule and knowing the existence of the last limit by the differentiability condition. The last two equations imply $(*)$ and so we can write our integral equation (2) in the form

$$(\mathrm{R}_{\mathcal{S}} f)_k(r) = 2\int_n^r f_k(t)\frac{K(r,t)}{\sqrt{r-t}}\,dt, \tag{2'}$$

where $K(r,t) = \bar K(r,t)\sqrt{r-t}$, $K(t,t) = \sqrt{\frac{2r}{\kappa-1}} \neq 0$ for $t \in [n,m]$ and $K \in \mathrm{C}^1([n,m]^2)$ by the differentiability condition on φ. Therefore, Theorem B in [6] implies that $(2')$ has unique solution on $[n,m]$ that, of course, can only be the zero function. This was to be proved.

■

We note that $(**)$ shows $\kappa \geq 1$ as a consequence of the parameterization and that the differentiability of $\frac{\varphi(r)}{\sqrt{1-r}}$ can be proved assuming C^4 differentiability for the inverse function $r(\varphi)$ of φ near S and $\varphi \in \mathrm{C}^2((0,1))$.

Geometrically, our theorem above is based on two essential conditions:

(α) The curve $\mathcal{S}$ is symmetric.

(β) The half of the curve $\mathcal{S}$ has exactly one cross point with any of the circle centered to the origin.

We shall prove that the symmetry condition (α) is not necessary. We use the notations of Theorem 2.

Theorem 3. *The statements of Theorem 2 remain true if we have* $\frac{\varphi}{\sqrt{1-r}}, \frac{\psi}{\sqrt{1-r}} \in \mathrm{C}^2([0,1])$ *instead of the symmetry.*

Proof. Without the symmetry we have to modify the previous proof at the formula (2) first. The present situation gives

$$(\mathrm{R}_{\mathcal{S}} f)_k(r) = \int_n^r f_k(t)\left(e^{ik\varphi\left(\frac{t}{r}\right)}\sqrt{1+\frac{t^2}{r^2}\dot\varphi^2\left(\frac{t}{r}\right)} + e^{ik\psi\left(\frac{t}{r}\right)}\sqrt{1+\frac{t^2}{r^2}\dot\psi^2\left(\frac{t}{r}\right)}\right)dt. \tag{2''}$$

Let f^r (resp. f^i) denote the real (resp. the imaginary) part of the function f and let g and h denote the real and the imaginary part of the kernel of this integral equation ($2''$). Then

$$
\begin{aligned}
g(x) &= \cos(k\varphi(x))\sqrt{1+x^2\dot\varphi^2(x)} + \cos(k\psi(x))\sqrt{1+x^2\dot\psi^2(x)},\\
h(x) &= \sin(k\varphi(x))\sqrt{1+x^2\dot\varphi^2(x)} + \sin(k\psi(x))\sqrt{1+x^2\dot\psi^2(x)}
\end{aligned}
$$

and the system of integral equations

$$
\begin{aligned}
(\mathrm{R}_{\mathcal{S}}\, f)^r_k(r) &= \int_n^r f^r_k(t)g(t/r)\,dt - \int_n^r f^i_k(t)h(t/r)\,dt\\
(\mathrm{R}_{\mathcal{S}}\, f)^i_k(r) &= \int_n^r f^r_k(t)h(t/r)\,dt + \int_n^r f^i_k(t)g(t/r)\,dt,
\end{aligned}
$$

is equivalent to ($2''$). Changing the order of the integrations on the left hand sides below one gets

$$
\begin{aligned}
\int_n^q \left((\mathrm{R}_{\mathcal{S}}\, f)^i_k(r)\frac{g(r/q)}{r/q} - (\mathrm{R}_{\mathcal{S}}\, f)^r_k(r)\frac{h(r/q)}{r/q}\right) dr &= \int_n^q f^i_k(t)B(t,q)\,dt + I^r,\\
\int_n^q \left((\mathrm{R}_{\mathcal{S}}\, f)^i_k(r)\frac{h(r/q)}{r/q} + (\mathrm{R}_{\mathcal{S}}\, f)^r_k(r)\frac{g(r/q)}{r/q}\right) dr &= \int_n^q f^r_k(t)B(t,q)\,dt + I^i,
\end{aligned}
$$

where

$$
\begin{aligned}
B(t,q) &= \int_t^q \left(\frac{g(t/r)g(r/q)+h(t/r)h(r/q)}{r/q}\right) dr,\\
I^* &= \int_n^q f^*_k(t)\int_t^q \left(\frac{h(t/r)g(r/q)-g(t/r)h(r/q)}{r/q}\right) dr\,dt
\end{aligned}
$$

and '$*$' is either 'r' or 'i' accordingly. Introducing the new variable $s = qt/r$ in the inner integral we obtain $I^* = -I^*$ hence $I^* \equiv 0$.

Therefore we have two integral equations with the same kernel $B(t,q)$, where the left hand sides are zero for $q < m$. To finish the proof in the way of Theorem 2 we need to see that $B(t,t) \neq 0$ and $B \in \mathrm{C}^1([n,m]^2)$.

We know the functions $G(x) = g(x)\sqrt{1-x^2}$ and $H(x) = h(x)\sqrt{1-x^2}$ are in $\mathrm{C}^1((0,1])$ and $G(1) = \frac{2}{\sqrt{\kappa-1}}$, $H(1) = 0$, because reflecting φ or ψ to OS we could get a symmetric curve satisfying all the conditions of Theorem 2. Now we can write

$$
B(t,q) = q^2\int_t^q \left(\frac{G(t/r)G(r/q)+H(t/r)H(r/q)}{r}\right)\left(\frac{r}{\sqrt{q^2-r^2}\sqrt{r^2-t^2}}\right) dr,
$$

where the first factor is in $C^1([n,m]^2)$ and tends uniformly to $\frac{4}{t(\kappa-1)}$ as q goes to t. Since

$$\frac{\pi}{2} = \int_t^q \frac{r}{\sqrt{q^2-r^2}\sqrt{r^2-t^2}}\,dr \qquad ([2,4]),$$

we obtain $B(t,t) = \frac{2t\pi}{\kappa-1} \neq 0$. Further, $B \in C^1$ because G and H are C^1 functions that finishes the proof.

■

The following theorem is a simple consequence of Theorem 3 using the inversion of the plane, but it can also be proved in the way of Theorem 3.

Let $\mathcal{S}$ be an infinite curve that goes to the infinity as its parameter goes to infinity. Let S be the point of $\mathcal{S}$ nearest to the origin. Suppose that $|OS| = 1$ and $\mathcal{S}$ has two parts parameterized by $(r, \varphi(r))$ and $(r, \psi(r))$ on the ray starts from S in direction opposite to SO.

Theorem 4. *Let $\mathcal{S}$ have curvature $\kappa > -1$ at S, $\frac{\varphi(r)}{\sqrt{1-r}}, \frac{\psi(r)}{\sqrt{1-r}} \in C^2([1,\infty))$. If $f \in L^2(\mathbb{R}^2)$ has compact support and $R_{\mathcal{S}} f = 0$ for $|X| \geq m$ then f is supported in the disk of radius m.*

The following example shows that the condition (β) is necessary for the support theorem.

Example. Let $\mathcal{S}$ be the curve displayed on Figure 4. It is constructed from arcs of two concentric circles with radius 0.5 and 1. These arcs belong to angles π/k and $2\pi/k$ and are joined with rays.

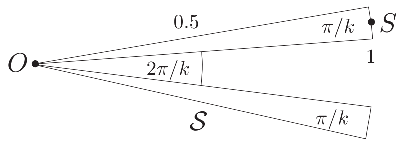

Figure 4.

Let $f(r,\alpha) = F(r)\cos(k\alpha)$, where $k \geq 2$ natural number and

$$F(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ e^{\frac{-r^2}{r-1}} & \text{if } r > 1 \end{cases}$$

Then $f \in L_1^2(\mathbb{R}^2)$ and $\mathcal{S}$ satisfies the conditions of Theorem 3 except the (β) condition. Since $f(r, \alpha + \frac{\pi}{k}) = -f(r,\alpha)$, obviously $R_{\mathcal{S}} f(X) = 0$ for $|OX| \leq 2$ hence the support theorem fails in this case.

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Note that $\kappa > 1$ is necessary consequence of our parameterization, and here $\kappa = 1$. In other words, φ is not well defined for this curve.

3. Support hypersurfaces

In this section we are going to prove the higher dimensional equivalent of Theorem 2 and the continuity of the corresponding transformation.

Let $\mathcal{S}$ be a regular hypersurface in $\mathbb{R}^n$ $(n \geq 3)$ passing through the origin, $\aleph$ be the set of the orientation preserving similarities fixing the origin and $f \in \mathrm{L}^2_*(\mathbb{R}^n, r^{n-3})$. The function

$$\mathrm{R}_{\mathcal{S}}\, f : \aleph \longrightarrow \mathbb{R} \qquad \Big(\mathrm{R}_{\mathcal{S}}\, f(\mathcal{A}) = \int_{\mathcal{A}\mathcal{S}} f(X)\, dX \Big)$$

is said to be the Radon transform of f with respect to the *support hypersurface* $\mathcal{S}$, where dX is the natural surface measure on $\mathcal{A}\mathcal{S}$.

Let S be the point of the hypersurface $\mathcal{S}$ farthest from the origin. Suppose $|OS| = 1$, $\mathcal{S}$ is axially symmetric around the axis OS and the curve given by a plane cut through the axis OS is parameterized by $(r, \varphi(r))$ on OS just like in Theorem 2.

In this case, $\mathcal{A}S = \mathcal{C}S$ implies $\mathrm{R}_{\mathcal{S}}\, f(\mathcal{A}) = \mathrm{R}_{\mathcal{S}}\, f(\mathcal{C})$, therefore we can project the transformed function onto $\mathbb{R}^n$ so that for every point $P \in \mathbb{R}^n$, $\mathrm{R}_{\mathcal{S}}\, f(P) = \mathrm{R}_{\mathcal{S}}\, f(\mathcal{A})$, where $\mathcal{A} \in \aleph$ satisfies $P = \mathcal{A}S$. Further on, we regard $R_{\mathcal{S}} f$ just in this way.

The following theorem generalizes Theorem 2.

Theorem 5. *If the curve $(r, \varphi(r))$ has curvature $\kappa > 1$ at $r = 1$, $n \geq 3$ and $\frac{\varphi(r)}{\sqrt{1-r}} \in \mathrm{C}^{[\frac{n+2}{2}]}([0,1])$ then*

i) $\mathrm{R}_{\mathcal{S}}\colon \mathrm{L}^2_s(\mathbb{R}^n, r^{n-3}) \longrightarrow \mathrm{L}^2_s(\mathbb{R}^n, r^{-n-1})$ *is continuous.*

ii) *If $f \in \mathrm{L}^2_*(\mathbb{R}^n, r^{n-3})$ and $\mathrm{R}_{\mathcal{S}}\, f(X) = 0$ for $|X| \leq s$, then $f \in \mathrm{L}^2_s(\mathbb{R}^n, r^{n-3})$.*

iii) $\mathrm{R}_{\mathcal{S}}\colon \mathrm{L}^2_*(\mathbb{R}^n, r^{n-3}) \longrightarrow \mathrm{L}^2_*(\mathbb{R}^n, r^{-n-1})$ *is one-to-one.*

Proof. We start with the second assertion (ii). Suppose that $f \in \mathrm{L}^2_p(\mathbb{R}^n, r^{n-3})$ and $p < s$. Let $f_{k,m}$ be the coefficient of $Y_{k,m}$ in the spherical harmonic expansion of f i.e.

$$f_{k,m}(r) = \int_{S^{n-1}} f(r\omega) Y_{k,m}(\omega)\, d\omega.$$

We are looking for an integral equation between $f_{k,m}$ and $(\mathrm{R}_{\mathcal{S}}\, f)_{k,m}$, the coefficient of $Y_{k,m}$ in the spherical harmonic expansion of $\mathrm{R}_{\mathcal{S}}\, f$.

We have

$$\mathrm{R}_{\mathcal{S}}\, f(r\bar{\omega}) = \int_0^r \int_{S^{n-2}\perp\bar{\omega}} f\left(t(\omega \sin\varphi(t/r) + \bar{\omega}\cos\varphi(t/r))\right) \times \\ \times (t\sin\varphi(t/r))^{n-2} \sqrt{1 + \frac{t^2}{r^2}\dot{\varphi}^2\left(\frac{t}{r}\right)}\, d\omega\, dt,$$

where $\bar{\omega}$ is a unit vector and $S^{n-2} \perp \bar{\omega}$ denotes the $(n-2)$-dimensional unit sphere in $\mathbb{R}^n$ perpendicular to $\bar{\omega}$. Using the Dirac delta distribution δ

$$\mathrm{R}_{\mathcal{S}}\, f(r\bar{\omega}) = \int_0^r \int_{S^{n-1}} f(t\omega)\delta(\langle\omega,\bar{\omega}\rangle - \cos\varphi(t/r)) \times \\ \times t^{n-2}\sin\varphi(t/r) \sqrt{1 + \frac{t^2}{r^2}\dot{\varphi}^2\left(\frac{t}{r}\right)}\, d\omega\, dt,$$

where $\langle .,. \rangle$ denotes the usual inner product on $\mathbb{R}^n$. For a member $h(r\omega) = f_{k,m}(r)Y_{k,m}(\omega)$ of the spherical harmonic expansion of f we obtain

$$\mathrm{R}_{\mathcal{S}}\, h(r\bar{\omega}) = \int_0^r f_{k,m}(t)\sin\varphi(t/r)t^{n-2} \sqrt{1 + \frac{t^2}{r^2}\dot{\varphi}^2\left(\frac{t}{r}\right)} \times \\ \times \int_{S^{n-1}} \delta(\langle\omega,\bar{\omega}\rangle - \cos\varphi(t/r))Y_{k,m}(\omega)\, d\omega\, dt.$$

Using the Funk–Hecke theorem [8] for the Dirac delta δ we see

$$Y_{k,m}(\bar{\omega})\frac{|S^{n-2}|}{C_m^\lambda(1)}C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} = \int_{S^{n-1}} \delta(\langle\omega,\bar{\omega}\rangle - t)Y_{k,m}(\omega)\, d\omega,$$

where $\lambda = \frac{n-2}{2}$, $|S^{n-2}|$ is the surface volume of S^{n-2} and C_m^λ is the Gegenbauer polynomial of the first kind. This gives

$$\mathrm{R}_{\mathcal{S}}\, h(r\bar{\omega}) = Y_{k,m}(\bar{\omega})\frac{|S^{n-2}|}{C_m^\lambda(1)} \int_0^r f_{k,m}(t)t^{n-2} \sqrt{1 + \frac{t^2}{r^2}\dot{\varphi}^2\left(\frac{t}{r}\right)} \times \\ \times C_m^\lambda(\cos\varphi(t/r))\sin^{n-2}\varphi(t/r)\, dt,$$

hence

$$(\mathrm{R}_{\mathcal{S}}\, f)_{k,m}(r) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_p^r f_{k,m}(t)t^{n-2}C_m^\lambda(\cos\varphi(t/r)) \times \\ \times \sin^{n-2}\varphi(t/r) \sqrt{1 + \frac{t^2}{r^2}\dot{\varphi}^2\left(\frac{t}{r}\right)}\, dt. \tag{3}$$

Since C^{λ}_m is polynomial and $C^{\lambda}_m(1) \neq 0$, using $(*)$ and $(**)$ (in the proof of Theorem 2) we can write the kernel of this integral equation in the form $K(r,t)\sqrt{r-t}^{\,n-3}$, where $K(r,t)$ is $\mathrm{C}^{[n/2]}([p,s]^2)$ by the differentiability condition on φ and $K(t,t) = \frac{|S^{n-2}|}{2}\sqrt{\frac{2t}{\kappa-1}}^{\,n-1} \neq 0$ for $t \in [p,s]$. The integral equations of this type are proved to have unique solutions [6, pp. 515 (3.9); 7, pp.41], hence $f_{k,m}(t) = 0$ for $t \in [p,s]$.

Now we prove (i). By the definition of the norm

$$\| \mathrm{R}_{\mathcal{S}}\, f\|_{\mathrm{L}^2_s(\mathbb{R}^n, r^{-n-1})} = \int_0^\infty r^{-2} \int_{S^{n-1}} (\mathrm{R}_{\mathcal{S}}\, f(r\omega))^2 \, d\omega \, dr.$$

Substituting the spherical harmonic expansion of $\mathrm{R}_{\mathcal{S}}\, f$ into and using the orthogonality of the spherical harmonics in $\mathrm{L}^2(S^{n-1})$ we obtain

$$\| \mathrm{R}_{\mathcal{S}}\, f\|_{\mathrm{L}^2_s(\mathbb{R}^n, r^{-n-1})} = \sum_{k,m} \int_0^\infty r^{-2} ((\mathrm{R}_{\mathcal{S}}\, f)_{k,m}(r))^2 \, dr \int_{S^{n-1}} (Y_{k,m}(\omega))^2 \, d\omega.$$

Therefore, to prove (i) we only have to show

$$\int_0^\infty r^{-2} ((\mathrm{R}_{\mathcal{S}}\, f)_{k,m}(r))^2 \, dr \leq C \int_0^\infty r^{2n-4} (f_{k,m}(r))^2 \, dr \tag{4}$$

for some constant C independent from k and m. To this end, we estimate the left hand side after rewriting (3) into (4). We know that $\sin^{n-2}\varphi(p)\sqrt{1+p^2\dot{\varphi}^2(p)}$ is continuous on the interval $[0,1]$ by $(*)$, $(**)$ and $n \geq 3$ and therefore it has maximum M. Observing also that $|C^{\lambda}_m(x)| \leq |C^{\lambda}_m(1)|$ for $x \in [-1,1]$, we see that

$$\int_0^\infty \left(\frac{M|S^{n-1}|}{r} \int_s^r |f_{k,m}| t^{n-2} \, dt \right)^2 dr \tag{5}$$

is more than the left hand side of (4). Using Hardy's inequality

$$\left\| \frac{1}{\nu} \int_0^\nu g(u) \, du \right\|_{\mathrm{L}^2(\mathbb{R}_+)} \leq 2\|g\|_{\mathrm{L}^2(\mathbb{R}_+)}$$

for the function $|f_{k,m}(t)|t^{n-2}$ one obtains just the result requested for (4).

■

We are closing the paper by discussing further possible generalizations of our results and their relations to other problems.

Considering the conditions in Theorem 2 and Theorem 3, it may be worth to note that $\mathcal{S}$ may intersect itself and even may wind around the origin. One can also observe that the differentiability condition for φ and ψ are most restricting at the point S, because of the factor $\frac{1}{\sqrt{1-r}}$. This locality one can feel from these can be made even more explicite using the referee's "onion peeling" idea that we present here. Let $\varepsilon > 0$ and assume the curve $\mathcal{S}$ satisfies all the conditions of Theorem 3 in the ring $\frac{1}{1+\varepsilon} < |X| \leq 1$. If $f \in \mathrm{L}^2_s(\mathbb{R}^2)$ and $\mathrm{R}_{\mathcal{S}}\, f$ is zero for $|X| < s(1+\varepsilon)$ then the nonzero part of the integration for $\mathrm{R}_{\mathcal{S}}\, f(X)$ uses at most only the part of $\mathcal{S}$ lying in the ring $\frac{1}{1+\varepsilon} < |X| \leq 1$, hence Theorem 3 gives $f \in \mathrm{L}^2_{s+\varepsilon s}(\mathbb{R}^2)$. If $\mathrm{R}_{\mathcal{S}}\, f$ is zero for $x < p$ then we have to use this step $\log_{1+\varepsilon} p/s$-times to get the result of Theorem 3. Note that this "onion peeling" trick allows us to consider curves not going through the origin. (For example, the support theorem is valid for ellipsoids having one of their focuses in the origin [7, Sect.1].) The same can be done in higher dimensions.

We considered the natural measure on the curves (resp. hypersurfaces) but without any further change the proofs can be generalized by multiplying the arclength with any rotation invariant C^∞ function [6]. This can be done even together with the above mentioned generalizations.

Comparing our results with the analogous statements for the rotational invariant Radon transform given in [6] we find a difference in the case of the plane. This happens because Quinto used the group $O(n)$ to define the rotational invariance for the Radon transform, while we used the group $SO(n)$ to generate the family of curves. For $n \geq 3$ this does not make any difference, because both groups are transitive on the pairs of the unit vectors. But $SO(2)$ is not transitive on the pairs of unit vectors in the plane while $O(2)$ is transitive. Therefore Quinto's Proposition 2.2 [6] changes only in dimension two, namely for $\mu(x,\omega,p) = U(x-p\omega,p)$, if we use $SO(2)$ instead of $O(2)$. This new situation can be handled with our method in Theorem 3, thus proving Quinto's result for the group $SO(2)$.

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