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Isoptic characterization of spheres

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Abstract. If a convex body in $\mathcal{K} \in \mathbb{R}^n$ subtends constant visual angles over two concentric spheres exterior to $\mathcal{K}$, then it is a ball concentric to those spheres.

1 Introduction

The *masking number*[1] $M_{\mathcal{K}}(P)$ of the convex body $\mathcal{K}$ at $P \notin \mathcal{K}$ as defined in [9, (7.1)] is the integral

$$M_{\mathcal{K}}(P) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \#(\partial\mathcal{K} \cap \ell(P, \boldsymbol{u}_{\boldsymbol{\xi}}))d\boldsymbol{\xi}, \tag{1.1}$$

where $\#$ is the counting measure, $\partial\mathcal{K}$ denotes the boundary of $\mathcal{K}$, $\boldsymbol{\xi}$ is the spherical coordinate of the unit vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$, and $\ell(P, \boldsymbol{u}_{\boldsymbol{\xi}})$ is the straight line through P having direction $\boldsymbol{u}_{\boldsymbol{\xi}}$.

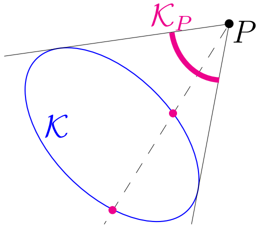

Figure 1.1: The masking number $M_{\mathcal{K}}(P)$ is twice the measure of the visual angle $\mathcal{K}_P$ of $\mathcal{K}$ at a point $P \notin \mathcal{K}$.

The set of points $P \in \mathbb{R}^n$, where a convex body $\mathcal{K} \subset \mathbb{R}^n$ has constant $\alpha \in (0, |\mathbb{S}^{n-1}|)$ masking number $M_{\mathcal{K}}(P)$ is called the α-*isomasker*[2] of the con-

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[1] This is called the point projection in [1] or shadow picture in [3].
[2] We reserve the word isoptic for the set of points where not only the measure, but also the shape of $\mathcal{K}_P$ is constant. A result toward this direction can be found in [12].

vex body $\mathcal{K}$. The α-isomasker of the convex body $\mathcal{K}$ in the plane is the set of the points where $\mathcal{K}$ subtends angles of constant $\alpha/2 \in (0, \pi)$ measure, and it is called the *α-isoptic* of $\mathcal{K}$.

Following the conjecture of Klamkin [4] Nitsche proved in [13] that if two isoptics of $\mathcal{K}$ are concentric circles, then $\mathcal{K}$ is a disc. Nitsche also asked to consider the problem in higher dimensions.

We generalize Nitsche's result to higher dimensions in Theorem 5.1 as follows: if two isomaskers of a convex body are also isomaskers of a ball with the same masking numbers, then the body is that ball. We use an integral geometric method.

2 Preliminaries

We work in the Euclidean n-space $\mathbb{R}^n$ ($n \in \mathbb{N}$). Its unit ball is $\mathcal{B} = \mathcal{B}^n$ (in the plane the unit disc is $\mathcal{D}$), its unit sphere is $\mathbb{S}^{n-1}$ and the set of its hyperplanes is $\mathbb{H}$. The ball (resp. disc) of radius $\varrho > 0$ centered at the origin $\mathbf{0}$ is denoted by $\varrho\mathcal{B} = \varrho\mathcal{B}^n$ (resp. $\varrho\mathcal{D}$). The unit sphere centered at a point P is $\mathbb{S}^{n-1}_P$.

Using spherical coordinates $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1})$ every unit vector can be written in the form $\boldsymbol{u}_{\boldsymbol{\xi}} = (\cos\xi_1, \sin\xi_1\cos\xi_2,\ \sin\xi_1\sin\xi_2\cos\xi_3, \dots)$, the i-th coordinate of which is $u^i_{\boldsymbol{\xi}} = (\prod_{j=1}^{i-1}\sin\xi_j)\cos\xi_i$ ($\xi_n := 0$). In the plane we use $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$ and $\boldsymbol{u}^\perp_\xi = \boldsymbol{u}_{\xi+\pi/2} = (-\sin\xi, \cos\xi)$. In analogy to this latter one, we introduce $\boldsymbol{\xi}^\perp = (\xi_1, \dots, \xi_{n-2}, \xi_{n-1} + \pi/2)$ for higher dimensions.

We introduce the notation $|\mathbb{S}^k| := 2\pi^{k/2}/\Gamma(k/2)$ for the standard surface measure of the k-dimensional sphere, where Γ is Euler's Gamma function.

The hyperplanes $\hbar \in \mathbb{H}$ are parametrized so that $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r)$ is orthogonal to the unit vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$ and contains the point $r\boldsymbol{u}_{\boldsymbol{\xi}}$,[3] where $r \in \mathbb{R}$. For convenience we also use $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}})$ to denote the hyperplane through the point $P \in \mathbb{R}^n$ with normal vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$. For instance, $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}}) = \hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, \langle \overrightarrow{OP}, \boldsymbol{u}_{\boldsymbol{\xi}}\rangle)$, where $O = \mathbf{0}$ is the origin and $\langle .,. \rangle$ is the usual inner product.

On $\mathbb{H}$ we use the *kinematic density* $d\hbar = dr d\boldsymbol{\xi}$ that is (up to a constant multiple) the only measure on $\mathbb{H}$ invariant with respect to the Euclidean motions [16].

By a convex body we mean a convex compact set $\mathcal{K} \subseteq \mathbb{R}^n$ with non-empty interior $\mathcal{K}^\circ$ and with piecewise C^1 boundary $\partial\mathcal{K}$. For a convex body $\mathcal{K}$ we let $p_{\mathcal{K}} \colon \mathbb{S}^{n-1} \to \mathbb{R}$ denote the support function of $\mathcal{K}$ defined by $p_{\mathcal{K}}(\boldsymbol{u}_\xi) = \sup_{\boldsymbol{x}\in\mathcal{K}}\langle \boldsymbol{u}_\xi, \boldsymbol{x}\rangle$. We also use notation $\hbar_{\mathcal{K}}(\boldsymbol{u}) = \hbar(\boldsymbol{u}, p_{\mathcal{K}}(\boldsymbol{u}))$.

If the origin is in $\mathcal{K}^\circ$, then the *support function* of $\mathcal{K}$ is positive, otherwise the zero or even negative values appear in its image according to whether the origin is

[3] Although $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r) = \hbar(-\boldsymbol{u}_{\boldsymbol{\xi}}, -r)$ this parametrization is locally bijective.

in $\partial\mathcal{K}$ or outside $\mathcal{K}$. If the origin is in $\mathcal{K}^\circ$, another useful function of a convex body $\mathcal{K}$ is its *radial function* $\varrho_{\mathcal{K}}\colon \mathbb{S}^{n-1} \to \mathbb{R}_+$ defined by $\varrho_{\mathcal{K}}(\boldsymbol{u}) = |\{r\boldsymbol{u} : r > 0\} \cap \partial K|$.

Assume that the origin $\mathbf{0}$ is an interior point of a convex body $\mathcal{K}$. Define $\mathbb{H}_0 := \{\hbar \in \mathbb{H} : \mathbf{0} \notin \hbar\}$, and let $\hat{\delta}\colon \mathbb{H}_0 \to \mathbb{R}^n$ and $\check{\delta}\colon \mathbb{R}^n \to \mathbb{H}_0$, the *dualizing maps*, be defined by

$$\hat{\delta}(\hbar(\boldsymbol{u}, r)) := -\frac{1}{r}\boldsymbol{u} \quad \text{and} \quad \check{\delta}(r\boldsymbol{u}) := \hbar\Big(-\boldsymbol{u}, \frac{1}{r}\Big), \tag{2.1}$$

respectively, where $\boldsymbol{u} \in \mathbb{S}^{n-1}$ is unit vector and $r > 0$. These functions are obviously inverses of each other, and it is an easy and well-known fact[4] that

$$\hat{\delta}(\{\hbar \in \mathbb{H} : \boldsymbol{v} \in \hbar\}) = \hbar\Big(\frac{-\boldsymbol{v}}{|\boldsymbol{v}|}, \frac{1}{|\boldsymbol{v}|}\Big) \quad \text{and} \quad \check{\delta}(\hbar(\boldsymbol{u}, r)) = \Big\{\hbar \in \mathbb{H} : \frac{-1}{r}\boldsymbol{u} \in \hbar\Big\}.$$

The dual body $\mathcal{K}^\star$ of $\mathcal{K}$ is bounded by $\partial\hat{\mathcal{K}} := \{\hat{\delta}(\hbar(\boldsymbol{u}, p_{\mathcal{K}}(\boldsymbol{u}))) : \boldsymbol{u} \in \mathbb{S}^{n-1}\}$. The dual body $\mathcal{K}^\star$, which is in fact the point reflection —to the origin $\mathbf{0}$— of the polar body $\mathcal{K}^*$ [17, Section 1.6], is convex, and its radial function is $\varrho_{\mathcal{K}^\star}(\boldsymbol{u}) = \frac{1}{p_{\mathcal{K}}(-\boldsymbol{u})}$ [17, Theorem 1.7.6]. Further, we have $(\mathcal{K}^\star)^\star = \mathcal{K}$ [17, Section 1.6].

A strictly positive integrable function $\omega\colon \mathbb{R}^n \setminus \mathcal{B} \to \mathbb{R}_+$ is called *weight* and the integral

$$V_\omega(f) := \int_{\mathbb{R}^n \setminus \mathcal{B}} f(x)\omega(x)dx$$

of an integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called the *volume of f with respect to the weight ω* or simply the *ω-volume of f*. For the volume of the indicator function $\chi_{\mathcal{S}}$ of a set $\mathcal{S} \subseteq \mathbb{R}^n$ we use the notation $V_\omega(\mathcal{S}) := V_\omega(\chi_{\mathcal{S}})$ as a shorthand. If several weights are indexed by $i \in \mathbb{N}$, then we use the even shorter notation $V_i(\mathcal{S}) := V_{\omega_i}(\mathcal{S}) = V_i(\chi_{\mathcal{S}}) := V_{\omega_i}(\chi_{\mathcal{S}})$.

Finally we introduce a utility function χ that takes relations as argument and gives 1 if its argument is fulfilled. For example $\chi(1 > 0) = 1$, but $\chi(1 \le 0) = 0$ and $\chi(x > y)$ is 1 if $x > y$ and it is zero if $x \le y$. However we still use χ also as the indicator function of the set given in its subscript.

3 Dualizing the masking function

For any point $P \in \mathbb{R}^n$ define the sets $\bar{\mathcal{K}}_P$ and $\mathcal{K}_P$ in the unit sphere $\mathbb{S}^{n-1}_P$ centered at P that contains exactly those points $X \in \mathbb{S}^{n-1}_P$ for which the hyperplane

[4] Embed the space $\mathbb{R}^n$ of $\mathcal{K}$ into $\mathbb{R}^{n+1}$ in such a way that the $(n+1)$th coordinate of every point is 1 and the $(n+1)$th coordinate axis intersects $\mathcal{K}$ in its inner point $\mathbf{0} \in \mathbb{R}^n$.

$\hbar(P,\overrightarrow{PX})$ and the straight line $\ell(P,\overrightarrow{PX})$, respectively, intersects $\mathcal{K}$. Then, by (1.1) and some easy observations we have

$$M_{\mathcal{K}}(P)=\frac{1}{2}\int_{\mathbb{S}^{n-1}}\#(\partial\mathcal{K}\cap\ell(P,\boldsymbol{u}_{\boldsymbol{\xi}}))d\boldsymbol{\xi}=\int_{\mathcal{K}_P}1\,d\boldsymbol{\xi}=\frac{1}{|\mathbb{S}^{n-2}|}\int_{\bar{\mathcal{K}}_P}1\,d\boldsymbol{\xi}$$
$$=\frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}}\chi(\hbar(P,\boldsymbol{u}_{\boldsymbol{\xi}})\cap\mathcal{K}\neq\emptyset)d\boldsymbol{\xi}.$$

From this we obtain

$$\begin{aligned}|\mathbb{S}^{n-2}|M_{\mathcal{K}}(P)&=\int_{\mathbb{S}^{n-1}}\chi(\langle\boldsymbol{u}_{\boldsymbol{\xi}},P\rangle\le p_K(\boldsymbol{u}_{\boldsymbol{\xi}}))\,d\boldsymbol{\xi}\\&=|\mathbb{S}^{n-1}|-\int_{\mathbb{S}^{n-1}}\chi(\langle\boldsymbol{u}_{\boldsymbol{\xi}},P\rangle\ge p_K(\boldsymbol{u}_{\boldsymbol{\xi}}))\,d\boldsymbol{\xi}\\&=:|\mathbb{S}^{n-1}|-M^{\star}_{\mathcal{K}}(\check{\delta}(P)).\end{aligned}\tag{3.1}$$

Assuming $\mathbf{0}\in\mathcal{K}^{\circ}$ one can reformulate the last integral to obtain

$$\begin{aligned}M^{\star}_{\mathcal{K}}(\check{\delta}(P))&=\int_{\mathbb{S}^{n-1}}\chi\Big(\langle-\boldsymbol{u}_{\boldsymbol{\xi}}\varrho_{K^\star}(-\boldsymbol{u}_{\boldsymbol{\xi}}),-\boldsymbol{u}\rangle\ge\frac{1}{r}\Big)d\boldsymbol{\xi}\\&=\int_{\mathbb{S}^{n-1}}\chi\Big(\varrho_{K^\star}(-\boldsymbol{u}_{\boldsymbol{\xi}})\ge\frac{1/r}{\langle-\boldsymbol{u}_{\boldsymbol{\xi}},-\boldsymbol{u}\rangle}\Big)d\boldsymbol{\xi}\\&=\int_{\check{\delta}(P)}\chi\big(\boldsymbol{x}\in\mathcal{K}^\star\big)\Big|\frac{d\boldsymbol{\xi}}{d\boldsymbol{x}}\Big|\,d\boldsymbol{x},\end{aligned}$$

where $P=r\boldsymbol{u}$, $r>0$, $\boldsymbol{u}\in\mathbb{S}^{n-1}$, and $|d\boldsymbol{\xi}/d\boldsymbol{x}|$ is the Jacobian of the map $\boldsymbol{x}\mapsto\boldsymbol{\xi}$ given by $\boldsymbol{x}=-|\boldsymbol{x}|\boldsymbol{u}_{\boldsymbol{\xi}}$. Let $\boldsymbol{x}=\frac{-1}{r}\boldsymbol{u}+\varrho\boldsymbol{u}_{\boldsymbol{\psi}}$, where $\boldsymbol{u}\perp\boldsymbol{u}_{\boldsymbol{\psi}}\in\mathbb{S}^{n-1}$ and $\boldsymbol{\psi}$ is a spherical coordinate on $\mathbb{S}^{n-2}$ such that $\boldsymbol{\xi}=(\xi,\boldsymbol{\psi})$. Then by rotational invariance we obtain immediately that $\big|\frac{d\boldsymbol{\xi}}{d\boldsymbol{x}}\big|=|\boldsymbol{x}|^{2-n}\big|\frac{d\xi}{d\varrho}\big|$, where $\tan\xi=\frac{\varrho}{1/r}$ and so

$$\frac{d\xi}{d\varrho}=\frac{r}{1+r^2\varrho^2}.$$

Thus, we obtain

$$\begin{aligned}M^{\star}_{\mathcal{K}}(\check{\delta}(P))&=\int_{\check{\delta}(P)}\chi\big(\boldsymbol{x}\in\mathcal{K}^\star\big)|\boldsymbol{x}|^{2-n}\frac{|P|}{1+|P|^2(|\boldsymbol{x}|^2-|P|^{-2})}\,d\boldsymbol{x}\\&=\int_{\check{\delta}(P)}\chi\big(\boldsymbol{x}\in\mathcal{K}^\star\big)\frac{1/|P|}{|\boldsymbol{x}|^n}\,d\boldsymbol{x},\end{aligned}\tag{3.2}$$

where $d\boldsymbol{x}$ is the standard surface measure on the hyperplane $\check{\delta}(P)$.

4 Measures of convex bodies

In view of (3.2) it is natural to consider the following transforms.

Let $\mathcal{M}$ and $\mathcal{K}$ be convex bodies such that $\mathbf{0} \in \mathcal{M} \subseteq \mathcal{K}^\circ$. Let $\nu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ be a *function of weights*, that is, $\nu_\hbar$ is a weight for every $\hbar \in \mathbb{H}$. Then the *weighted section function* of $\mathcal{K}$ with respect to $\mathcal{M}$, the so called *kernel*, is defined by

$$\mathrm{S}^\nu_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = \int_{\langle \boldsymbol{x},\boldsymbol{u}\rangle = p_{\mathcal{M}}(\boldsymbol{u})} \chi(\boldsymbol{x} \in \mathcal{K})\nu_{\hbar_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{x})\, d\boldsymbol{x}_{\hbar_{\mathcal{M}}(\boldsymbol{u})}, \tag{4.1}$$

where $d\boldsymbol{x}_{\hbar_{\mathcal{M}}(\boldsymbol{u})}$ is the usual surface measure on $\hbar_{\mathcal{M}}(\boldsymbol{u})$.

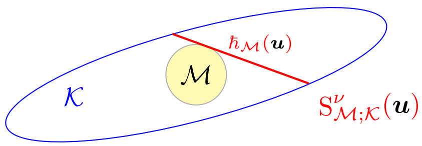

Figure 4.1: Section of $\mathcal{K}$ with respect to the kernel $\mathcal{M}$.

The function $\nu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ of weights is called *rotationally symmetric* if for every $\hbar \in \mathbb{H}$, $\boldsymbol{x} \in \hbar$ and $D \in SO(n)$ one has $\nu_{D\hbar}(D\boldsymbol{x}) = \nu_\hbar(\boldsymbol{x})$, where $D \in SO(n)$ acts naturally on $\mathbb{H}$. Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$. If $|\boldsymbol{x}| = |\boldsymbol{y}|$ and $\langle \boldsymbol{x}, \boldsymbol{u}\rangle = \langle \boldsymbol{y}, \boldsymbol{v}\rangle$, then there is a $D \in SO(n)$ such that $D\boldsymbol{x} = \boldsymbol{y}$ and $D\boldsymbol{u} = \boldsymbol{v}$. Thus we have the following lemma immediately.

Lemma 4.1. *The function ν of weights is rotationally symmetric if and only if there is a function $\bar\nu\colon \mathbb{R}^3 \to \mathbb{R}$ such that $\nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = \bar\nu(r, \langle \boldsymbol{x}, \boldsymbol{u}\rangle, |\boldsymbol{x}|)$.*

If the kernel body is a ball, i.e. $\varrho\mathcal{B}$, we use the notation $\mathrm{S}^\nu_{\varrho;\mathcal{K}} := \mathrm{S}^\nu_{\varrho\mathcal{B};\mathcal{K}}$ as a shorthand.

Lemma 4.2. *Let the convex body $\mathcal{K}$ contain the ball $\varrho\mathcal{B}$. Then for any rotationally symmetric function ν of weights we have*

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^\nu_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = |\mathbb{S}^{n-2}| \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \bar\nu(\varrho, \varrho, |\boldsymbol{x}|)\frac{(|\boldsymbol{x}|^2 - \varrho^2)^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{n-2}}\, d\boldsymbol{x}, . \tag{4.2}$$

Proof. Define the function μ^ε of weights by

$$\mu^\varepsilon_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) := \nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x} + (r - \langle \boldsymbol{x}, \boldsymbol{u}\rangle)\boldsymbol{u})\chi(0 \le \langle \boldsymbol{x}, \boldsymbol{u}\rangle - r \le \varepsilon),$$

where $\varepsilon > 0$. Now we can write[5]

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\zeta}})d\boldsymbol{\zeta} &= \int_{\mathbb{S}^{n-1}} \int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle=\varrho} \nu_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\chi(\boldsymbol{x}\in\mathcal{K})\,d\boldsymbol{x}_{\hbar}\,d\boldsymbol{\zeta}\\
&= \int_{\mathbb{S}^{n-1}} \lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle\ge\varrho} \mu^{\varepsilon}_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\chi(\boldsymbol{x}\in\mathcal{K})\,d\boldsymbol{x}\Big)\,d\boldsymbol{\zeta}\\
&= \lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{\mathbb{S}^{n-1}}\int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle\ge\varrho} \mu^{\varepsilon}_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\chi(\boldsymbol{x}\in\mathcal{K})\,d\boldsymbol{x}\,d\boldsymbol{\zeta}\Big)\\
&= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle\ge\varrho} \mu^{\varepsilon}_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\,d\boldsymbol{\zeta}\Big)\,d\boldsymbol{x}.\end{aligned}$$

As ν is rotationally symmetric, $\nu_{\hbar(\boldsymbol{u},\langle\boldsymbol{x},\boldsymbol{u}\rangle)}(\boldsymbol{x}) = \bar\nu(\langle\boldsymbol{x},\boldsymbol{u}\rangle,\langle\boldsymbol{x},\boldsymbol{u}\rangle,|\boldsymbol{x}|)$, and this implies $\mu^{\varepsilon}_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x}) = \bar\nu(\varrho,\varrho,|\boldsymbol{x}|)\chi(0\le\langle\boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle-\varrho\le\varepsilon)$. Therefore, letting $|\boldsymbol{x}|\boldsymbol{u}_{\boldsymbol{\xi}} = \boldsymbol{x}$, where $\boldsymbol{u}_{\boldsymbol{\xi}}\in\mathbb{S}^{n-1}$, the calculation above continues as

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\zeta}})d\boldsymbol{\zeta}\\
&\qquad= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \bar\nu(\varrho,\varrho,|\boldsymbol{x}|)\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle\ge\varrho}\chi(0\le\langle\boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle-\varrho\le\varepsilon)\,d\boldsymbol{\zeta}\Big)\,d\boldsymbol{x}.\end{aligned}$$

As

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(\frac{1}{\varepsilon}\int_{\langle \boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle\ge\varrho}\chi(0\le\langle\boldsymbol{x},\boldsymbol{u}_{\boldsymbol{\zeta}}\rangle-\varrho\le\varepsilon)\,d\boldsymbol{\zeta}\Big)\\
&\quad= \lim_{\varepsilon\to 0}\Big(\frac{|\mathbb{S}^{n-2}|/|\boldsymbol{x}|}{\varepsilon/|\boldsymbol{x}|}\int_{\varrho/|\boldsymbol{x}|}^{(\varrho+\varepsilon)/|\boldsymbol{x}|}\sqrt{1-\lambda^2}^{\,n-3}\,d\boldsymbol{\lambda}\Big) = \frac{|\mathbb{S}^{n-2}|}{|\boldsymbol{x}|}\sqrt{1-\Big(\frac{\varrho}{|\boldsymbol{x}|}\Big)^2}^{\,n-3},\end{aligned}$$

the lemma is proved. ∎

Although the following lemma was already proved as Lemma 5.3 in [11], we present it here for the sake of completeness with its short proof.

Lemma 4.3. *Let ω_i $(i=1,2)$ be weights, let $\mathcal{K}$ and $\mathcal{L}$ be convex bodies containing the unit ball $\mathcal{B}$, and let $c\ge 1$.*

(1) *If $cV_1(\mathcal{K})\le V_1(\mathcal{L})$ and there is a constant $c_{\mathcal{K}}$ such that*

$$\begin{aligned}\omega_2(X)&\ge c_{\mathcal{K}}\omega_1(X), &&\text{if } X\notin\mathcal{K},\\
\omega_2(X)&= c_{\mathcal{K}}\omega_1(X), &&\text{if } X\in\partial\mathcal{K},\\
\omega_2(X)&\le c_{\mathcal{K}}\omega_1(X), &&\text{if } X\in\mathcal{K},\end{aligned}$$

where equality may occur only in a set of measure zero, then $cV_2(\mathcal{K})\le V_2(\mathcal{L})$.

[5]Similar calculation is given in [11]. It is given here for the sake of completeness.

(2) *If $V_1(\mathcal{K}) \le cV_1(\mathcal{L})$ and there is a constant $c_{\mathcal{L}}$ such that*

$$\begin{aligned} \omega_2(X) &\le c_{\mathcal{L}}\omega_1(X), && \text{if } X \notin \mathcal{L}, \\ \omega_2(X) &= c_{\mathcal{L}}\omega_1(X), && \text{if } X \in \partial\mathcal{L}, \\ \omega_2(X) &\ge c_{\mathcal{L}}\omega_1(X), && \text{if } X \in \mathcal{L}, \end{aligned}$$

where equality may occur only in a set of measure zero, then $V_2(\mathcal{K}) \le cV_2(\mathcal{L})$.

In both cases equality in the resulted inequality implies $\mathcal{K} = \mathcal{L}$ and $c = 1$.

Proof. In both statements $\mathcal{K}\triangle\mathcal{L} = \emptyset$ implies $V_1(\mathcal{K}) = V_1(\mathcal{L})$, hence $c = 1$ and $V_1(\mathcal{K}) = V_1(\mathcal{L})$.

Assume from now on that $\mathcal{K}\triangle\mathcal{L} \ne \emptyset$.

We prove here only (1) since the verification of (2) is similar.

Having (1) we proceed as

$$\begin{aligned} &V_2(\mathcal{L}) - cV_2(\mathcal{K}) \\ &= V_2(\mathcal{L}) - V_2(\mathcal{K}) + (1-c)V_2(\mathcal{K}) = V_2(\mathcal{L}\setminus\mathcal{K}) - V_2(\mathcal{K}\setminus\mathcal{L}) + (1-c)V_2(\mathcal{K}) \\ &= \int_{\mathcal{L}\setminus\mathcal{K}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx - \int_{\mathcal{K}\setminus\mathcal{L}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx + (1-c)V_2(\mathcal{K}) \\ &> c_{\mathcal{K}}(V_1(\mathcal{L}\setminus\mathcal{K}) - V_1(\mathcal{K}\setminus\mathcal{L})) + (1-c)V_2(\mathcal{K}) = c_{\mathcal{K}}(V_1(\mathcal{L}) - V_1(\mathcal{K})) + (1-c)V_2(\mathcal{K}) \\ &\ge (c-1)(c_{\mathcal{K}}V_1(\mathcal{K}) - V_2(\mathcal{K})) = (c-1)\Big(\int_{\mathcal{K}}\Big(c_{\mathcal{K}} - \frac{\omega_2(x)}{\omega_1(x)}\Big)\omega_1(x)dx\Big) \ge 0. \end{aligned}$$

This implies $V_2(\mathcal{L}) - cV_2(\mathcal{K}) > 0$.

The lemma is proved. ■

5 Spherical isomaskers

First we calculate the integral of the masking function $M_{\mathcal{K}}$ of the convex body $\mathcal{K} \subset \bar{r}\mathcal{B}^n$ over the sphere $\bar{r}\mathbb{S}^{n-1}$ ($\bar{r} > 0$). Starting with equation (3.1) we get

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(\bar{r}\boldsymbol{u}_{\boldsymbol{\xi}})\,d\boldsymbol{\xi} &= \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} |\mathbb{S}^{n-1}| - M^{\star}_{\mathcal{K}}(\check{\delta}(\bar{r}\boldsymbol{u}_{\boldsymbol{\xi}}))\,d\boldsymbol{\xi} \\ &= \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} M^{\star}_{\mathcal{K}}(\check{\delta}(\bar{r}\boldsymbol{u}_{\boldsymbol{\xi}}))\,d\boldsymbol{\xi}. \end{aligned}$$

Assuming $\mathbf{0} \in \mathcal{K}^{\circ}$ we can continue by using (3.2) and (4.1) and obtain

$$\int_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(r\boldsymbol{u}_{\boldsymbol{\xi}})\,d\boldsymbol{\xi} = \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}}\int_{\hbar(-\boldsymbol{u}_{\boldsymbol{\xi}},1/r)} \chi\big(\boldsymbol{x} \in \mathcal{K}^{\star}\big)\frac{1/\bar{r}}{|\boldsymbol{x}|^n}\,d\boldsymbol{x}\,d\boldsymbol{\xi}.$$

This means

$$\int_{\mathbb{S}^{n-1}} M_{\mathcal{K}}(r\boldsymbol{\xi})\, d\boldsymbol{\xi} = \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\bar r};\mathcal{K}^\star}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi}, \tag{5.1}$$

where $\nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = r|\boldsymbol{x}|^{-n}$. Having this we are ready to prove the following generalization of Nitsche's result [13].

Theorem 5.1. *Let $\varrho_2 > \varrho_1 > \bar r > 0$ and let $\mathcal{K}$ be a convex body contained in the interior of $\varrho_1\mathcal{B}^n$. If the sphere $\varrho_1\mathbb{S}^{n-1}$ is the common α-isomasker and $\varrho_2\mathbb{S}^{n-1}$ is the common β-isomasker of the convex body $\mathcal{K}$ and $\bar r\mathcal{B}$, then $\mathcal{K} = \bar r\mathcal{B}$.*

Proof. By the conditions we have $M_{\mathcal{K}}(\varrho_1\boldsymbol{u}) = \alpha = M_{\bar r\mathcal{B}^n}(\varrho_1\boldsymbol{u})$ and $M_{\mathcal{K}}(\varrho_2\boldsymbol{u}) = \beta = M_{\bar r\mathcal{B}^n}(\varrho_2\boldsymbol{u})$ for every $\boldsymbol{u} \in \mathbb{S}^{n-1}$.

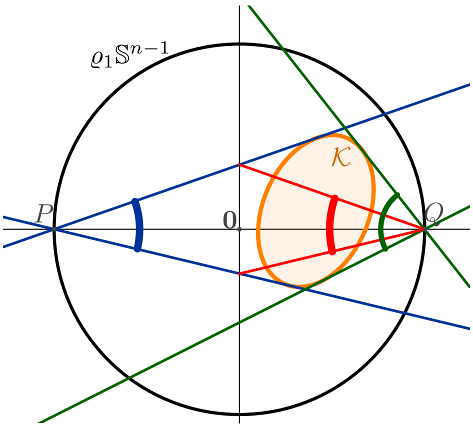

Figure 5.1: $M_{\mathcal{K}}(P)$ is clearly smaller than $M_{\mathcal{K}}(Q)$.

By some elementary observations and reasoning illustrated in Figure 5.1 it follows that $\mathcal{K}^\circ$ contains the common center $\mathbf{0}$ of the balls $\bar r\mathcal{B}$, $\varrho_1\mathcal{B}^n$ and $\varrho_2\mathcal{B}^n$.

Now equation (5.1) implies

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\mathcal{K}^\star}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};(\bar r\mathcal{B}^n)^\star}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\frac{1}{\bar r}\mathcal{B}^n}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi},$$

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\mathcal{K}^\star}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};(\bar r\mathcal{B}^n)^\star}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\frac{1}{\bar r}\mathcal{B}^n}(\boldsymbol{u}_{\boldsymbol{\xi}})\, d\boldsymbol{\xi}.$$

As the function ν of weights having $\bar\nu(\varrho_2, \varrho_2, r) = \varrho_2 r^{-n}$ is obviously rotational invariant, (4.2) implies

$$\int_{\mathcal{K}^\star\setminus\frac{1}{\varrho_2}\mathcal{B}^n}\frac{(|\boldsymbol{x}|^2-\varrho_2^{-2})^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{2n-2}}\,d\boldsymbol{x}=\int_{\frac{1}{\bar r}\mathcal{B}^n\setminus\frac{1}{\varrho_2}\mathcal{B}^n}\frac{(|\boldsymbol{x}|^2-\varrho_2^{-2})^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{2n-2}}\,d\boldsymbol{x},$$

and

$$\int_{\mathcal{K}^\star\setminus\frac{1}{\varrho_1}\mathcal{B}^n}\frac{(|\boldsymbol{x}|^2-\varrho_1^{-2})^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{2n-2}}\,d\boldsymbol{x}=\int_{\frac{1}{\bar r}\mathcal{B}^n\setminus\frac{1}{\varrho_1}\mathcal{B}^n}\frac{(|\boldsymbol{x}|^2-\varrho_1^{-2})^{\frac{n-3}{2}}}{|\boldsymbol{x}|^{2n-2}}\,d\boldsymbol{x}.$$

Let $\bar\omega_1(r):=r^{2-2n}(r^2-\varrho_1^{-2})^{\frac{n-3}{2}}$, $\bar\omega_2(r):=r^{2-2n}(r^2-\varrho_2^{-2})^{\frac{n-3}{2}}$, and let $\omega_1(\boldsymbol{x}):=\bar\omega_1(|\boldsymbol{x}|)$, $\omega_2(\boldsymbol{x}):=\bar\omega_2(|\boldsymbol{x}|)$. Then $\frac{\omega_1}{\omega_2}$ is clearly a constant, say $c_{\mathcal{L}}$, on $\frac{1}{\bar r}\mathcal{B}^n$, and

$$\frac{\bar\omega_1(r)}{\bar\omega_2(r)}=\frac{(r^2-\varrho_1^{-2})^{\frac{n-3}{2}}}{(r^2-\varrho_2^{-2})^{\frac{n-3}{2}}}=\left(1-\frac{\varrho_1^{-2}-\varrho_2^{-2}}{r^2-\varrho_1^{-2}}\right)^{\frac{n-3}{2}}$$

shows that $\frac{\bar\omega_1}{\bar\omega_2}$ is strictly monotone increasing.

The above observations show that the conditions in (2) of Lemma 4.3 are satisfied for $\mathcal{K}^\star$, $\mathcal{L}:=\frac{1}{\bar r}\mathcal{B}^n$ and $c=1$, hence $V_2(\mathcal{K}^\star)\le V_2(\mathcal{L})$, and equality implies $\mathcal{K}^\star=\mathcal{L}$ and $c=1$.

As $\mathcal{K}=(\mathcal{K}^\star)^\star=(\mathcal{L})^\star=\bar r\mathcal{B}^n$, the theorem is proved. ■

6 Discussion

To have a complete generalization of Nitsche's result [13] from the point of view of Theorem 5.1, one should prove that if a convex body $\mathcal{K}$ has two spherical isomaskers of values $\alpha_1\ne\alpha_2$, then there is a ball $\bar r\mathcal{B}^n$ with the same α_1- and α_2-isomaskers of radius $\varrho_1\ne\varrho_2$. Although Nitsche proved this in the plane, the authors conjecture that this is no longer valid in higher dimensions.

Conjecture 6.1. *There are positive values $\alpha_1\ne\alpha_2$ and $\varrho_1\ne\varrho_2$ such that there is a non-spherical convex body $\mathcal{K}\subset\mathbb{R}^n$ the α_1- and α_2-isomaskers of which are spheres of radius $\varrho_1\ne\varrho_2$, respectively.*

However note that it is proved in [7] that if two convex bodies in the plane have rotational symmetry of angle $2(\pi-\nu)$ and have common ν-isoptic, then that ν-isoptic is a circle.

In higher dimensions the only positive result the authors know about is the surprisingly easy [5, Theorem 2]. It states that if a convex body $\mathcal{K}\subset\mathbb{R}^n$ has an isoptic $\mathcal{I}$ in the sense of a k-dimensional angles for any $1<k<n-1$, then $\mathcal{K}$ is reconstructible from $\mathcal{I}$.

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▲

References

[1] R. J. Gardner, *Geometric tomography* (second edition), Encyclopedia of Math. and its Appl. **58**, Cambridge University Press, Cambridge, 2006 (first edition in 1996).

[2] J. W. Green, Sets subtending a constant angle on a circle, *Duke Math. J.*, **17** (1950), 263–267.

[3] J. Kincses and Á. Kurusa, Can you recognize the shape of a figure from its shadows?, *Beiträge zur Alg. und Geom.*, **36** (1995), 25–34.

[4] M. S. Klamkin, Conjectured isoptic characterization of a circle, *Amer. Math. Monthly*, **95** (1988), 845.

[5] Á. Kurusa, You can recognize the shape of a figure by its shadows!, *Geom. Dedicata*, **59** (1996), 103–112.

[6] Á. Kurusa, The shadow picture problem for nonintersecting curves, *Geom. Dedicata*, **59** (1996), 113–125.

[7] Á. Kurusa, Is a convex plane body determined by an isoptic?, *Beitr. Alg. Geom.*, **53:2** (2012), 281–294; doi: 10.1007/s13366-011-0074-2.

[8] Á. Kurusa, The shadow picture problem for parallel straight lines, *J. Geom.*, **103:3** (2012), 515–518; doi: 10.1007/s00022-012-0137-z.

[9] Á. Kurusa, The masking function of multicurves, *manuscript* (2014), submitted.

[10] Á. Kurusa and T. Ódor, Characterizations of balls by sections and caps, *Beitr. Alg. Geom.* (2014), to appear; doi: 10.1007/s13366-014-0203-9.

[11] Á. Kurusa and T. Ódor, Spherical floating body, *manuscript* (2014), submitted.

[12] S. Matsuura, A problem in solid geometry, *J. Math. Osaka City Univ.*, **A 12** (1961), 89–95.

[13] J. C. C. Nitsche, Isoptic characterization of a circle, (Proof of a conjecture of M.S. Klamkin), *Amer. Math. Monthly*, **97** (1990), 45–47.

[14] T. Ódor, *Rekonstrukciós, karakterizációs és extrémum problémák a geometriában*, PhD dissertation, Budapest, 1994 (in hungarian; title in english: *Problems of reconstruction, characterization and extremum in geometry*).

[15] T. Ódor, *Ball characterizations by visual angles and sections*, manuscript, (2002), unpublished.

[16] L. A. Santaló, *Integral Geometry and Geometric Probability*, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 2004.

[17] R. Schneider, *Convex bodies: The Brunn–Minkowski Theory*, Encyclopedia of Math. and its Appl., Cambridge Univ. Press, Cambridge, 1993; *second edition*, 2013.

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