Quadrireciprocal and Riemannian points of a convex body

Geometric tomography and Hilbert metric

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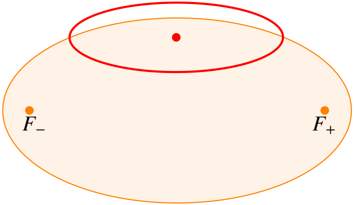

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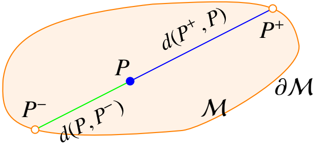
The (-1)-chord functions of a convex domain

The *i-chord function* ($i \in \mathbb{R}$, $i \neq 0$) of an open convex domain $\mathcal{M} \subset \mathbb{R}^n$ at a point P in $\mathcal{M}$ is

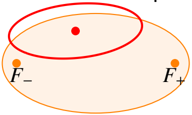
$$\rho_{\mathcal{M};i;P} : \mathcal{S}^{n-1} \to \mathbb{R}, \qquad \boldsymbol{u} \mapsto d^i(P^-, P) + d^i(P, P^+),$$

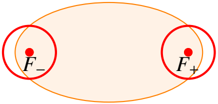
where d is the Euclidean metric, $\overline{P^-P^+} = \mathcal{M} \cap \ell_{\boldsymbol{u}}$, and $\ell_{\boldsymbol{u}}$ is the line through P with directional vector $\boldsymbol{u}$.



The *i-body*[1] of $\mathcal{M}$ at P is the star body having radial function $(\rho_{\mathcal{M};i;P}/2)^{1/i}$ at P.

Points of an ellipsoid are quadrireciprocal.





The focuses are equireciprocal.

The (-1)-body is called the *reciprocal body*. A point P is *quadrireciprocal* if $\rho_{\mathcal{M};-1;P}$ is quadratic. It is *isoreciprocal* if $\rho_{\mathcal{M};-1;P}$ is constant. A pair of isoreciprocal points are called *equireciprocal* ([4, Theorem 2] explains).

By Falconer's [4, Theorem 3], if a convex body in the plane with twice differentiable boundary has equireciprocal points, then it is an ellipse.

How many quadrireciprocal points ensure that $\mathcal{M}$ is an ellipsoid?

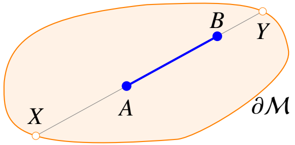
[1] The i-body is the i-chordal symmetral as defined in [6, Defintition 6.1.2].

The Hilbert metric of a convex domain

Let $\mathcal{M}$ be an open convex domain in $\mathbb{R}^n$. The function $d_{\mathcal{M}}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}$ defined by

$$d_{\mathcal{M}}(A,B)=\begin{cases}0, & \text{if } A=B,\\ \big|\ln(A,B;X,Y)\big|/2, & \text{if } A\neq B,\end{cases}$$

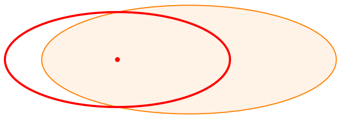
where $\overline{XY}=\mathcal{M}\cap AB$, is a metric, the *Hilbert metric*.



Hilbert geometries $(\mathcal{M},d_{\mathcal{M}})$ are Finsler manifolds [2, (29.6)]. Further, a Hilbert geometry $(\mathcal{M},d_{\mathcal{M}})$ is a *Cayley–Klein model* of the hyperbolic geometry if $\mathcal{M}$ is an ellipsoid.

A point $P\in\mathcal{M}$ is called *Riemannian* if the Finsler norm on $T_P\mathcal{M}$ is quadratic.

Every point of a Cayley–Klein model $(\mathcal{M},d_{\mathcal{M}})$ is Riemannian.



By Beltrami's (more general) theorem [1] (see also [2, (29.3)]), a Riemannian Hilbert metric has constant curvature, hence $(\mathcal{M},d_{\mathcal{M}})$ is a Cayley–Klein model of the hyperbolic geometry, i.e., $\mathcal{M}$ is an ellipsoid.

How many Riemannian points ensure that $(\mathcal{M},d_{\mathcal{M}})$ is a Cayley–Klein model?

Connecting the problems

Identifying the tangent spaces $T_P\mathcal{M}$ of $(\mathcal{M}, d_{\mathcal{M}})$ with $\mathbb{R}^n$ by the map $\iota_P\colon \boldsymbol{v} \mapsto P+\boldsymbol{v}$, the Finsler function $F_{\mathcal{M}}\colon \mathcal{M}\times\mathbb{R}^n \to \mathbb{R}$ associated with $d_{\mathcal{M}}$ can be given at a point $P \in \mathcal{M}$ by

(1.1)
$$F_{\mathcal{M}}(P,\boldsymbol{v}) = \frac{1}{2}\Big(\frac{1}{\lambda_{\boldsymbol{v}}^-} + \frac{1}{\lambda_{\boldsymbol{v}}^+}\Big),$$

where $\boldsymbol{v} \in T_P\mathcal{M}$, and $\lambda_{\boldsymbol{v}}^{\pm} \in (0,\infty]$ is such that $P_{\boldsymbol{v}}^{\pm} := P \pm \lambda_{\boldsymbol{v}}^{\pm}\boldsymbol{v} \in \partial\mathcal{M}$ [2, (50.4)]. So map ι_P assigns the indicatrixes of $F_{\mathcal{M}}$ and the reciprocal bodies of $\mathcal{M}$ to each other. In this context reciprocal bodies are called *infinitesimal spheres*. We have the following result:

Riemannian points of $d_{\mathcal{M}}$ correspond to the quadrireciprocal points of $\mathcal{M}$, and vice versa.

This allows to rephrase the partial results mentioned earlier:

The reciprocal bodies of a convex body $\mathcal{M}$ are all ellipsoids if and only if $\partial\mathcal{M}$ is an ellipsoid.

Two reciprocal bodies of a convex plane body $\mathcal{M}$ with twice differentiable boundary $\partial\mathcal{M}$ are both circular if and only if $\partial\mathcal{M}$ is an ellipse and the points are the focuses.

The remaining problem is formulated in both scenarios as follows:

Do two Riemannian points ensure that $(\mathcal{M}, d_{\mathcal{M}})$ is a Cayley–Klein model?
Do two quadrireciprocal points ensure that $\mathcal{M}$ is an ellipsoid?

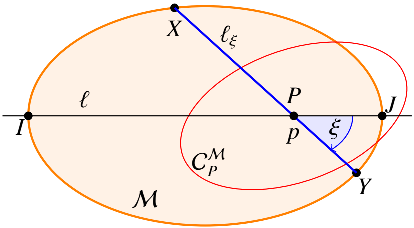
Answering these questions needs both points of view alternatively.

Preliminaries

From now on, we only work in the plane unless explicitely said otherwise.

In this case infinitesimal sphere is called *infinitesimal circle* and denoted by $C_P^{\mathcal{M}}$.

This presentation uses the following *setup*:
Q and P are Riemannian points of $(\mathcal{M}, d_{\mathcal{M}})$; $\ell = PQ$ is a straight line that intersects $\partial\mathcal{M}$ in points I and J; *a* coordinate system is fixed so that $I = (-1,0)$, $J = (1,0)$; then $Q = (q,0)$ and $P = (p,0)$, where $-1 < q < p < 1$; the Euclidean metric d_e is fixed so that $\{(1,0),(0,1)\}$ is an orthonormal basis; ℓ_ξ is the straight line through P with directional vector $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$;



If not otherwise specified, X and Y are the points where ℓ_ξ intersects $\partial\mathcal{M}$.
For iterations of maps like $X \mapsto Y$ the theory dynamical systems helps:

Stable Manifold Theorem. ([4, p. 114] and [5, Theorem 4.1]).

Let $\mathcal{N}_0 \subset \mathbb{R}^2$ be a neighborhood of the origin $\mathbf{0}$, and let the mapping $\Phi\colon \mathcal{N}_0 \to \mathbb{R}^2$ be of class C^l $(l \in [1,\infty])$. If there are linearly independent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\Phi(\boldsymbol{w}) = \boldsymbol{w}$ for every $\boldsymbol{w} \in \ell_{\boldsymbol{u}} \cap \mathcal{N}_0$, and $\mathrm{D}\Phi_{(0,0)}\boldsymbol{v} = k\boldsymbol{v}$ for some $k \in (0,1)$, then in some neighborhood $\mathcal{N} \subseteq \mathcal{N}_0$ of $\mathbf{0}$, the set $\{\boldsymbol{w} \in \mathcal{N} : \Phi^{(r)}(\boldsymbol{w}) \to \mathbf{0}$ as $r \to \infty\}$ is the graph of a C^l function from $\ell_{\boldsymbol{v}} \cap \mathcal{N}$ to $\ell_{\boldsymbol{u}} \cap \mathcal{N}$.

Step 1

Observe that (1.1) gives $2F_{\mathcal{M}}(P, X-P) - 1 = 1/\lambda^{-}_{X-P} > 0$ for $X \in \partial\mathcal{M}$, so, as a continuous function takes its minimal value, there is a suitably small $\varepsilon > 0$ such that the map

$$\Phi_P \colon Z \mapsto \Phi_P(Z) = P + (P-Z)\frac{1}{2F_{\mathcal{M}}(P, Z-P) - 1} \tag{2.1}$$

is well defined on the Minkowski sum $\mathcal{M}^{\varepsilon} := \partial\mathcal{M} + \varepsilon\mathcal{B}^2$, where $\mathcal{B}^2$ is the unit ball at $(0,0)$. Parameterize $C^{\mathcal{M}}_P$ in polar coordinates with center P by $\boldsymbol{r} \colon [-\pi, \pi) \ni \xi \mapsto r(\xi)\boldsymbol{u}_{\xi} \in \mathbb{R}^2$. Then $\frac{1}{|XP|} + \frac{1}{|PY|} = \frac{2}{r(\xi)}$, so *$r$ is twice differentiable if $\partial\mathcal{M}$ is twice differentiable*.

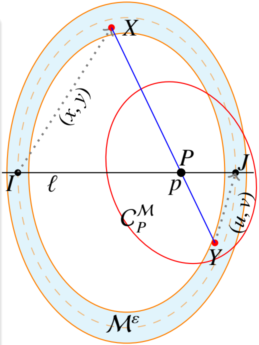
Approximation lemma. (K.Á.: [9, Lemma 3.2]).

Assume that $\partial\mathcal{M}$ is twice differentiable. Let $X \in I + \varepsilon\mathcal{B}^2$, and set $Y = \Phi_P(X)$. Let $(x,y) = X - I$ and $(u,v) = J - Y$. Then

$$v\Big(1 + \frac{u}{1-p} + O(u^2)\Big) = y\Big(\frac{1-p}{1+p} + x\frac{1-p}{(1+p)^2} + O(x^2)\Big), \tag{2.2}$$

and

$$\begin{aligned} -u = {} & x\frac{(1-p)^2}{(1+p)^2} - y\frac{2r'(0)}{(1+p)^3} + x^2\frac{2(1-p)^2}{(1+p)^4} - xy\frac{r'(0)2(3-p)}{(1+p)^5} + \\ & + y^2\frac{1}{(1+p)^3}\Big(-(1-p) + \frac{2(r'(0))^2}{(1+p)^3} + \frac{r''(0)}{1+p}\Big) + \\ & + O(x^3) + O(x^2y) + o(y^2). \end{aligned} \tag{2.3}$$



Step 2

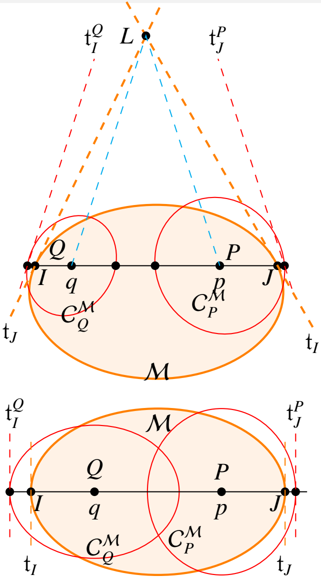
Let t_I and t_J be the tangents of $\mathcal{M}$ at I and J, respectively. Let $L = \mathrm{t}_I \cap \mathrm{t}_J$ (maybe ideal point). Let t_I^Q and t_J^P be the tangents of $C_Q^{\mathcal{M}}$ and $C_P^{\mathcal{M}}$, respectively, where ℓ intersects the infinitesimal circles.

Choose a straight line l through L that avoids $\mathcal{M}$, and let *ϖ be a perspectivity* that takes l to the ideal line. Then its derivative $\dot{\varpi}$ makes $\dot{\varpi}(C_Q^{\mathcal{M}}) \equiv C_{\varpi(Q)}^{\varpi(\mathcal{M})}$, and $\dot{\varpi}(C_P^{\mathcal{M}}) \equiv C_{\varpi(P)}^{\varpi(\mathcal{M})}$. As *$\dot{\varpi}$ is an affine map*, hence keeps quadraticity, $\varpi(Q)$ and $\varpi(P)$ are Riemannian points in $(\varpi(\mathcal{M}), d_{\varpi(\mathcal{M})})$. Thus $\mathrm{t}_I \parallel \mathrm{t}_J$ can be assumed without loss of generality.

It is an easy *consequence of* [2, (28.11)], that the tangents t_I^Q and t_J^P are parallel to LQ and LP, respectively. Thus $\mathrm{t}_I^Q \parallel \mathrm{t}_I \parallel \mathrm{t}_J \parallel \mathrm{t}_J^P$, and we choose d_e so that $\ell \perp \mathrm{t}_I$.

So $C_P^{\mathcal{M}}$ and $C_Q^{\mathcal{M}}$ are ellipses with *polar equations* of the form $\frac{1}{r^2(\varphi)} = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}$ at centers P and Q, respectively. This implies

$$(2.4) \qquad r'(0) = 0 \ \text{ and } \ r''(0) = r^3(0)\Big(\frac{1}{r^2(0)} - \frac{1}{r^2(\pi/2)}\Big).$$

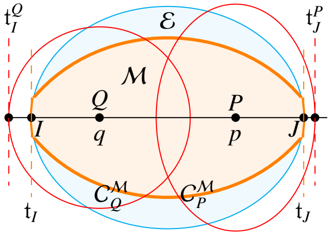


Step 3

Lemma. *If $\partial\mathcal{M}$ is twice differentiable at I and J, then there is a unique ellipse $\mathcal{E}$ touching $\mathcal{M}$ at I, J such that $C_Q^{\bar{\mathcal{E}}} \equiv C_Q^{\mathcal{M}}$ and $C_P^{\bar{\mathcal{E}}} \equiv C_P^{\mathcal{M}}$.*

Proof. Fix the Euclidean metric d in which $C_Q^{\mathcal{M}}$ is a circle. Assume that $X \in \partial\mathcal{M}$, hence also $Y = \Phi_P(X) \in \partial\mathcal{M}$. Basic differential geometry gives that the respective curvatures of $\partial\mathcal{M}$ at I and J are

(2.5) $$\kappa_I := \lim_{x\to 0}\frac{2x}{y^2} \quad \text{and} \quad \kappa_J := \lim_{u\to 0}\frac{2u}{v^2}.$$



Using the formulas of the *Approximation Lemma* in conjunction with the quadraticity (2.4) leads to

$$\kappa_J = \lim_{u\to 0}\frac{2u}{v^2} = \lim_{u\to 0}\frac{-2x}{y^2} + \frac{2}{r(0)} - 2r(0)\Big(\frac{1}{r^2(0)} - \frac{1}{r^2(\pi/2)}\Big) = -\kappa_I + \frac{2r(0)}{r^2(\pi/2)}.$$

Repeating the same calculation for Φ_Q gives $\kappa_J = -\kappa_I + \frac{2}{1-q^2}$, hence $r(\frac{\pi}{2}) = \sqrt{1-q^2}\,\sqrt{1-p^2}$. Now easy calculation shows that $(q, 0)$ is a focus of the ellipse $x^2 + \frac{y^2}{1-q^2} = 1$, and the infinitesimal circle at $(p, 0)$ is the ellipse $\frac{(x-p)^2}{(1-p^2)^2} + \frac{y^2}{(1-q^2)(1-p^2)} = 1$. Thus choosing the ellipse $x^2 + \frac{y^2}{1-q^2} = 1$ for $\mathcal{E}$ proves the lemma. ■

Step 4

Lemma. *If $\partial\mathcal{M}$ is C^2 at I and J, then $\mathcal{E}$ coincides with $\partial\mathcal{M}$ in a neighborhood of I and J.*

Proof. According to the last formula in the proof of the previous lemma, the infinitesimal circles $C_P^{\overline{\mathcal{E}}} \equiv C_P^{\mathcal{M}}$ and $C_Q^{\overline{\mathcal{E}}} \equiv C_Q^{\mathcal{M}}$ can be represented by polar equations of form

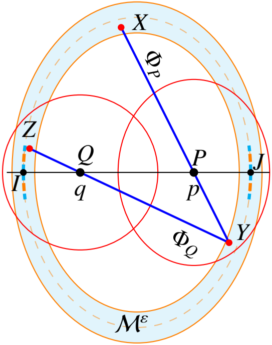
$$\frac{1}{r^2(\varphi)} = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}, \quad \text{and} \quad \frac{1}{r_q^2(\varphi)} = \frac{1}{r_q^2(0)},$$

respectively. Substitution of these into (2.1) shows that Φ_P and Φ_Q are real analytic mappings on $\mathcal{M}^\varepsilon$.

Thus $\Phi := \Phi_Q \circ \Phi_P \colon X \mapsto Y \mapsto Z$ is also a real analytic mapping. The *Approximation Lemma* and a (long) calculation gives that

$$\Phi^\Psi(z,y) := \Psi^{-1} \circ \Phi \circ \Psi(z,y) = (z + o(1), yk + o(y^2)),$$

where $\Psi\colon (z,y) \mapsto (zy^2, y)$, $y \neq 0$, $k = \frac{1-p}{1+p}\frac{1+q}{1-q} < 1$, and z is close to $\kappa_I/2$. So defining $\Phi^\Psi(z,0) := (z,0)$ extends Φ^Ψ to a real analytic mapping around $(\kappa_I/2, 0)$. As Φ^Ψ fixes the points $(z,0)$ near $(\kappa_I/2,0)$, and it has the derivative $\dot\Phi^\Psi(\kappa_I/2,0) = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ at $(\kappa_I/2,0)$, the *Stable Manifold Theorem* applies. Thus a neighborhood $\mathcal{N}$ of $(\kappa_I/2,0)$ exists such that $C = \{\boldsymbol{w} \in \mathcal{N} : (\Phi^\Psi)^{(r)}(\boldsymbol{w}) \to (\kappa_I/2,0) \text{ as } r \to \infty\}$ is the graph of a C^1 function $z \mapsto y$. As Φ fixes $\partial\mathcal{M}$, this proves the lemma. ■



Step 5

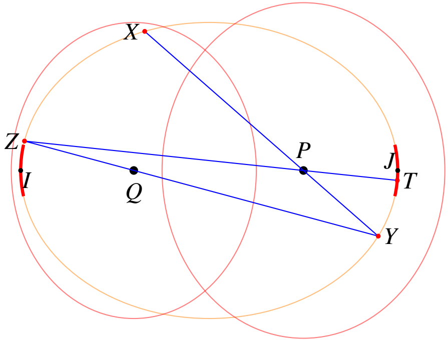
Lemma. *If Q and P are common Riemannian points of the Hilbert geometries $(\mathcal{L}, d_{\mathcal{L}})$ and $(\mathcal{M}, d_{\mathcal{M}})$, and the boundaries $\partial\mathcal{L}$ and $\partial\mathcal{M}$ coincide in a neighborhood of the line PQ, then $\mathcal{L} \equiv \mathcal{M}$.*

Proof. Let $\mathcal{N}$ be a neighborhood of line PQ such that $\partial\mathcal{L} \cap \mathcal{N} \equiv \mathcal{N} \cap \partial\mathcal{M}$.

Observe that $C_Q^{\mathcal{L}} \equiv C_Q^{\mathcal{M}}$ and $C_P^{\mathcal{L}} \equiv C_P^{\mathcal{M}}$, because the common arcs of $\partial\mathcal{L}$ and $\partial\mathcal{M}$ determine small common arcs of the quadratic infinitesimal circles near line QP.

Thus both Φ_P and Φ_Q map any common arc of $\partial\mathcal{L}$ and $\partial\mathcal{M}$ to a common arc of $\partial\mathcal{L}$ and $\partial\mathcal{M}$.

See the *proof without words* on the right! ■



The results

Theorem. (K.Á.: [9, Theorem 4.4]).

If a Hilbert geometry has two Riemannian points, and its boundary is twice differentiable where it is intersected by the line joining those Riemannian points, then it is a Cayley–Klein model of the hyperbolic plane.

The same in the language of geometric tomography [6] reads as:

Theorem. (K.Á.: [9, Theorem 5.1]).

Let Q and P be two interior points of a convex compact domain $\mathcal{M}$. Assume that the boundary $\partial\mathcal{M}$ is twice differentiable where it intersects line QP. If the (-1)-chord functions at Q and P are quadratic, then $\partial\mathcal{M}$ is an ellipse.

This generalizes Falconer's [4, Theorem 3], where only circles were considered.
However, Falconer's [4, Theorem 4] gives that for any two fixed points P, Q, a bunch of strictly convex bounded open domains $\mathcal{M}$ exist such that $P, Q \in \mathcal{M}$ are equireciprocal, the boundary $\partial\mathcal{M}$ is differentiable at $I, J \in PQ \cap \partial\mathcal{M}$ and twice differentiable everywhere in $\partial\mathcal{M} \setminus \{I, J\}$, BUT *$\partial\mathcal{M}$ is not an ellipse*.
Observe that in such an $\mathcal{M}$ there can not exist a third inner point with quadratic (-1)-chord function, because then $\partial\mathcal{M}$ should be an ellipse by the above theorem.

Riemannian points in higher dimensions

Unfortunately, our results do not imply similar statements for higher dimensions directly. Looking for possible higher dimensional analogs one can use [3, (16.12), p. 91], which says that

> *a convex body in $\mathbb{R}^n$ ($n \geq 3$) is an ellipsoid if and only if for a fixed $k \in \{2, \ldots, n-1\}$ every k-plane through an inner point intersects it in a k-dimensional ellipsoid.*

This immediately implies the following generalization.

Theorem. *If a Hilbert geometry has twice differentiable boundary, and has a Riemannian point P such that for some fixed $k \in \{2, \ldots, n-1\}$ on every k-plane through P there is a distinct Riemannian point, then it is a Cayley–Klein model of the hyperbolic space.*

The question arises:

How many Riemannian points are needed to deduce the hyperbolicity of a Hilbert geometry in dimension $n > 2$?

My *believe* is that $n + 1$ Riemannian points in general position guarantees the hyperbolicity of the Hilbert geometry. A more brave tip is that n is enough if the boundary is twice differentiable.

Radon points in dimension 2

It is proved in [7, Theorem 2] that perpendicularity in a Hilbert geometry is reversible for two lines if the perpendicularity of these two lines is also reversible with respect to the local Minkowski geometry at the intersection point of the lines. Thus, perpendicularity is reversible at a point if and only if the infinitesimal circle is a *Radon curve* [10].

Calling such points *Radon points*, the question[2] arises:

How many Radon points are needed to deduce the hyperbolicity of a Hilbert geometry in dimension 2?

Kelly and Paige proved in [8] that a Hilbert geometry is a Cayley–Klein model of the hyperbolic geometry if every point is a Radon point.

As Riemannian points are Radon points, my [9, Theorem 4.4] supports my *conjecture* that

Conjecture. The existence of two Radon points implies the hyperbolicity of a Hilbert geometry if the boundary is twice differentiable.

If twice differentiability fails, then we know that even two Riemannian points do not guarantee the hyperbolicity of the Hilbert geometry.

[2] The author thanks Tibor Ódor for a discussion where this question was arisen.

THANK YOU FOR YOUR ATTENTION!

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Let $\mathcal{M}$ be a strictly convex body with non-empty interior.
At a point P in the interior of $\mathcal{M}$ the i-chord function $\rho_{\mathcal{M};i;P}$ is defined [6, Definition 6.1.1] for every unit vector $\mathbf{u}$ so that if P divides into two segments the chord in which $\mathcal{B}$ intersects the line ℓ of direction $\mathbf{u}$ passing through P, then $\rho_{\mathcal{M};i;P}(\mathbf{u})$ is the i-power sum of the lengths of the segments.
A point P is called *quadrireciprocal* if at P the (-1)-chord function is quadratic.
The Hilbert metric [2, page 297]

$$d\colon \mathcal{M}\times\mathcal{M} \mapsto d(A,B) = \begin{cases} 0, & \text{if } A = B,\\ \frac{1}{2}\left|\ln(A,B;C,D)\right|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{M}\cap AB \end{cases}$$

is Finslerian in the interior of the strictly convex body $\mathcal{M}$.
A point P in the interior of $\mathcal{M}$ is called *Riemannian* if at P the infinitesimal sphere of the Hilbert metric is quadratic.
In this talk we show that *a point is quadrireciprocal if and only if it is also Riemannian*, and this duality is used to prove that

> *a twice differentiable convex body is an ellipsoid if and only if it has two quadrireciprocal points,*

and the other way around that

> *a Hilbert geometry of twice differentiable boundary is the hyperbolic geometry if and only if it has two Riemannian points.*

The proof uses tools of geometric tomography [6] and of Hilbert geometries [2] as well.
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Structure of the talk

1. **The questions**
   - (-1)-chord functions
   - Hilbert metric
   - The connection
2. **The answer**
   - Preliminaries
   - Preparations
   - The results
3. **Further questions**
   - Riemannian points in higher dimensions
   - Radon points in dimension 2