Hilbert geometries with Riemannian points

Definitive points of Hilbert geometries

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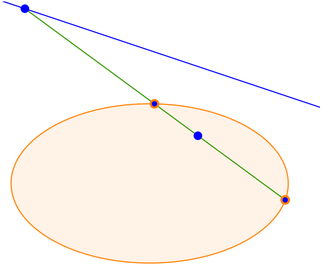

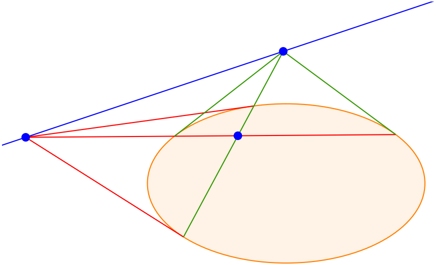

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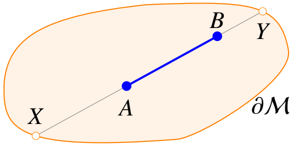
Points of a Hilbert geometry

Let $\mathcal{M}$ be an open convex domain in $\mathbb{R}^n$. The function $d_{\mathcal{M}}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}$ defined by

$$d_{\mathcal{M}}(A,B)=\begin{cases}0, & \text{if } A=B,\\ \big|\ln(A,B;X,Y)\big|/2, & \text{if } A\neq B,\end{cases}$$

where $\overline{XY}=\mathcal{M}\cap AB$, is a metric, the *Hilbert metric*.



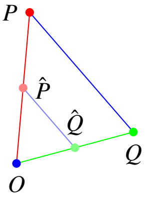
A *Hilbert geometry* $(\mathcal{M}, d_{\mathcal{M}})$ is a Finsler manifold [2, (29.6)], and it is a *Cayley–Klein model* of the hyperbolic geometry if and only if $\mathcal{M}$ is the interior of an ellipsoid [2, (29.3)].

1. Metric center: *Is there a point such that the metric reflection in the point is an isometry? What if the metric reflection in every point is an isometry?*
2. Point of curvature: *Is there a point where the Busemann curvature is non-negative? What if all the points have non-positive Busemann curvature? [3, 34th on p. 406]*
3. Radon point: *Is there a point such that the Birkhoff orthogonality is symmetric? What if the Birkhoff orthogonality is symmetric at all the points?*
4. Riemannian point: *Is there a point where the infinitesimal sphere is an ellipsoid? What if the infinitesimal sphere is an ellipsoid at all the points?*

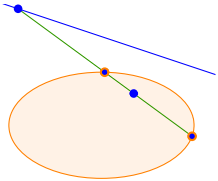
In a *Cayley–Klein model* the metric reflection is an isometry, the Busemann curvature is of non-positive, the Birkhoff orthogonality is symmetric, and the infinitesimal sphere is an ellipsoid at every point.

The Busemann curvature and the metric centers

A Hilbert geometry at a point O has *positive*, *non-negative*, *non-positive* and *negative curvature in the sense of Busemann* [3, (36.1) on p. 237] if there exists a neighborhood $\mathcal{U}$ of O such that for every pair of points $P, Q \in \mathcal{U}$ and their respective $d_{\mathcal{M}}$-midpoints $\hat{P}, \hat{Q}$ of the geodesic segments $\overline{OP}$ and $\overline{OQ}$ we have $2d_{\mathcal{M}}(\hat{P},\hat{Q}) - d_{\mathcal{M}}(P,Q)$ is positive, non-negative, non-positive and negative, respectively. Otherwise the curvature is called *indeterminate* [9, Definition 1].



No Hilbert geometry has positive or non-negative curvature at any point [12, Theorem 4.1].
A point O is a *projective center* of the set $\mathcal{M} \subseteq \mathbb{P}^n$, if there is a projectivity ϖ such that $\varpi(O)$ is the affine center of $\varpi(\mathcal{M})$ [15, p. 64].



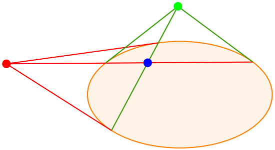
A point $O \in \mathcal{M}$ is a metric center of $(\mathcal{M}, d_{\mathcal{M}})$ if and only if it is a projective center of $\mathcal{M}$. (Kelly & Strauss [10, Theorem XXXX])

A point in a Hilbert geometry is of non-positive curvature if and only if it is a projective center. (KÁ [12, Theorem 4.2]),

If the border of a Hilbert geometry is twice differentiable and has two metric centers, then it is a Cayley–Klein model of the hyperbolic geometry. (Kelly & Strauss [10, Theorem 3])

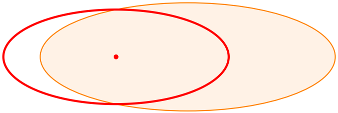
Radon points and Riemannian points

In a Hilbert geometry *perpendicularity is reversible* for every pair of lines through a point O if and only if the perpendicularity is reversible with respect to the local Minkowski geometry at O (Kay [7, Theorem 2]). This means that the infinitesimal circle at O is a *Radon curve* [14], so we call such a point a *Radon point* (K.Á. [13, (5.1)]).



If every point of a Hilbert geometry is a Radon point, then the Hilbert geometry is a Cayley–Klein model of the hyperbolic geometry. (Kelly & Paige [8])

A point of a Hilbert geometry is a *Riemannian point*, if its local Minkowski geometry is Euclidean (K.Á. [13]). Every point of a Cayley–Klein model is Riemannian.



By Beltrami's (more general) theorem [1] a Riemannian Hilbert metric has constant curvature, hence it is a Cayley–Klein model of the hyperbolic geometry, i.e., its domain is the interior of an ellipsoid.

How many Radon points ensure that $(\mathcal{M}, d_{\mathcal{M}})$ is a Cayley–Klein model?

How many Riemannian points ensure that $(\mathcal{M}, d_{\mathcal{M}})$ is a Cayley–Klein model?

Preliminaries

From now on, we only work in the plane unless explicitely said otherwise.

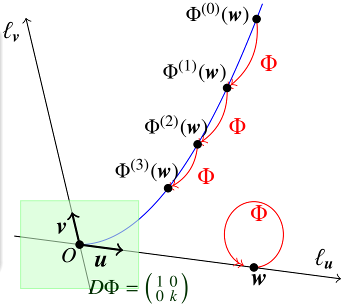
Identifying the tangent spaces $T_P\mathcal{M}$ of $(\mathcal{M}, d_{\mathcal{M}})$ with $\mathbb{R}^2$ by the map $\iota_P : \boldsymbol{v} \mapsto P + \boldsymbol{v}$, the Finsler function $F_{\mathcal{M}} : \mathcal{M} \times \mathbb{R}^2 \to \mathbb{R}$ associated with $d_{\mathcal{M}}$ at a point $P \in \mathcal{M}$ is determined by

$$F_{\mathcal{M}}(P, \boldsymbol{v}) = \frac{1}{2}\Big(\frac{1}{\lambda_{\boldsymbol{v}}^-} + \frac{1}{\lambda_{\boldsymbol{v}}^+}\Big), \tag{2.1}$$

where $\boldsymbol{v} \in T_P\mathcal{M}$, and $\lambda_{\boldsymbol{v}}^{\pm} \in (0, \infty]$ is such that $P_{\boldsymbol{v}}^{\pm} := P \pm \lambda_{\boldsymbol{v}}^{\pm}\boldsymbol{v} \in \partial\mathcal{M}$ [2, (50.4)]. The indicatrixes of $F_{\mathcal{M}}$ are called *infinitesimal circles*, and denoted by $C_P^{\mathcal{M}}$.
A point $P \in \mathcal{M}$ is Riemannian if $C_P^{\mathcal{M}}$ is an ellipse, that is, if the Finsler function is quadratic.

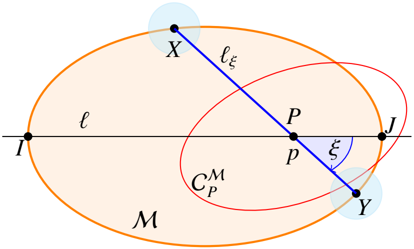
Stable Manifold Theorem. **([4, p. 114] and [5, Theorem 4.1]).**

Let $\mathcal{N}_0 \subset \mathbb{R}^2$ be a neighborhood of the origin $\mathbf{0}$, and let the mapping $\Phi : \mathcal{N}_0 \to \mathbb{R}^2$ be of class C^l ($l \in [1, \infty]$). If there are linearly independent vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\Phi(\boldsymbol{w}) = \boldsymbol{w}$ for every $\boldsymbol{w} \in \ell_{\boldsymbol{u}} \cap \mathcal{N}_0$, and $\mathrm{D}\Phi_{(0,0)}\boldsymbol{v} = k\boldsymbol{v}$ for some $k \in (0, 1)$, then in some neighborhood $\mathcal{N} \subseteq \mathcal{N}_0$ of $\mathbf{0}$, the set $\{\boldsymbol{w} \in \mathcal{N} : \Phi^{(r)}(\boldsymbol{w}) \to \mathbf{0}$ as $r \to \infty\}$ is the graph of a C^l function from $\ell_{\boldsymbol{v}} \cap \mathcal{N}$ to $\ell_{\boldsymbol{u}} \cap \mathcal{N}$.



Setup and tools

This presentation uses the following *setup*:
Q and P are Riemannian points of $(\mathcal{M}, d_{\mathcal{M}})$; $\ell = PQ$ is a straight line that intersects $\partial\mathcal{M}$ in points I and J; a coordinate system is fixed so that $I = (-1, 0)$, $J = (1, 0)$; then $Q = (q, 0)$ and $P = (p, 0)$, where $-1 < q < p < 1$; the Euclidean metric d_e is fixed so that $\{(1, 0), (0, 1)\}$ is an orthonormal basis; ℓ_ξ is the straight line through P with directional vector $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$;



Observe that (2.1) gives $2F_{\mathcal{M}}(P, X - P) - 1 = 1/\lambda^-_{X-P} > 0$ for $X \in \partial\mathcal{M}$, so, as a continuous function takes its minimal value, there is a suitably small $\varepsilon > 0$ such that the map

(2.2) $$\Phi_P \colon Z \mapsto \Phi_P(Z) = P + (P - Z)\frac{1}{2F_{\mathcal{M}}(P, Z - P) - 1}$$

is well defined on the Minkowski sum $\mathcal{M}^\varepsilon := \partial\mathcal{M} + \varepsilon\mathcal{B}^2$, where $\mathcal{B}^2$ is the unit ball at $(0, 0)$.

Observe that the curve $\partial\mathcal{M}$ is invariant under Φ_P, and Φ_P^2 is the identity on $\partial\mathcal{M}$.

If not otherwise specified, X and Y are the points of ℓ_ξ near at where ℓ_ξ intersects $\partial\mathcal{M}$.

We parameterize $C_P^{\mathcal{M}}$ in polar coordinates with center P by $\boldsymbol{r} \colon [-\pi, \pi) \ni \xi \mapsto r(\xi)\boldsymbol{u}_\xi \in \mathbb{R}^2$. Then

(2.3) $$\frac{1}{|XP|} + \frac{1}{|PY|} = \frac{2}{r(\xi)}, \qquad \text{where } \{X, Y\} = \ell_\xi \cap \partial\mathcal{M}.$$

Approximation lemma

For points X, Y of $\mathcal{M}^\varepsilon$ let $(x, y) = X - I$ and $(u, v) = J - Y$.

Assume that $\partial\mathcal{M}$ is twice differentiable. Then (2.3) shows that the radius function r of $C_P^{\mathcal{M}}$ is *twice differentiable*.

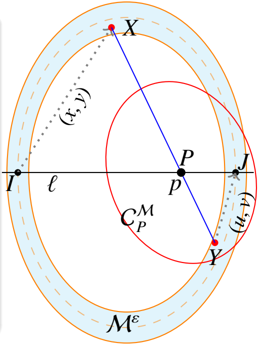
Approximation lemma. (K.Á.: [13, Lemma 3.2]).

If $X \in I + \varepsilon\mathcal{B}^2$, and $Y = \Phi_P(X)$, then

$$v\Big(1 + \frac{u}{1-p} + O(u^2)\Big) = y\Big(\frac{1-p}{1+p} + x\frac{1-p}{(1+p)^2} + O(x^2)\Big), \tag{2.4}$$

and

$$\begin{aligned}-u = {} & x\frac{(1-p)^2}{(1+p)^2} - y\frac{2r'(0)}{(1+p)^3} + x^2\frac{2(1-p)^2}{(1+p)^4} - xy\frac{r'(0)2(3-p)}{(1+p)^5} + \\ & + y^2\frac{1}{(1+p)^3}\Big(-(1-p) + \frac{2(r'(0))^2}{(1+p)^3} + \frac{r''(0)}{1+p}\Big) + \\ & + O(x^3) + O(x^2y) + o(y^2).\end{aligned} \tag{2.5}$$



From now on, we assume that P and Q are Riemannian points of $(\mathcal{M}, d_{\mathcal{M}})$.

Specializing the configuration

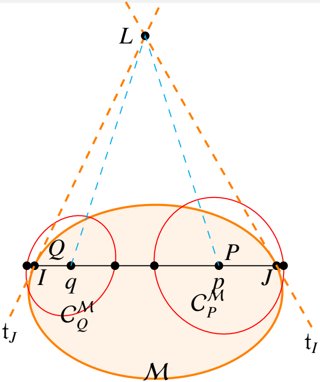
Let t_I and t_J be the tangents of $\mathcal{M}$ at I and J, respectively. Let $L = \mathrm{t}_I \cap \mathrm{t}_J$ (maybe ideal point).
Choose a straight line l through L that avoids $\mathcal{M}$, and let ϖ *be a perspectivity* that takes l to the ideal line. Then its derivative $\dot{\varpi}$ makes $\dot{\varpi}(C_Q^{\mathcal{M}}) \equiv C_{\varpi(Q)}^{\varpi(\mathcal{M})}$, and $\dot{\varpi}(C_P^{\mathcal{M}}) \equiv C_{\varpi(P)}^{\varpi(\mathcal{M})}$. As $\dot{\varpi}$ *is an affine map*, it keeps quadraticity, $\varpi(Q)$ and $\varpi(P)$ are Riemannian points in $(\varpi(\mathcal{M}), d_{\varpi(\mathcal{M})})$.

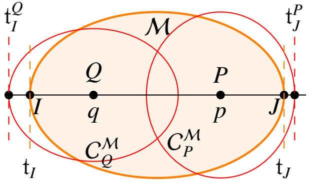
Thus $\mathrm{t}_I \parallel \mathrm{t}_J$ *can be assumed without loss of generality*.

Let t_I^Q and t_J^P be the tangents of $C_Q^{\mathcal{M}}$ and $C_P^{\mathcal{M}}$, respectively, where ℓ intersects the infinitesimal circles. It is an easy *consequence of* [2, (28.11)], that the tangents t_I^Q and t_J^P are parallel to LQ and LP, respectively.

Thus $\mathrm{t}_I^Q \parallel \mathrm{t}_I \parallel \mathrm{t}_J \parallel \mathrm{t}_J^P$, and we *choose d_e so that $\ell \perp \mathrm{t}_I$*.
So $C_P^{\mathcal{M}}$ and $C_Q^{\mathcal{M}}$ are ellipses with *polar equations* of the form $\frac{1}{r^2(\varphi)} = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}$ at centers P and Q, respectively. This implies

$$(2.6) \qquad r'(0) = 0 \quad \text{and} \quad r''(0) = r^3(0)\Big(\frac{1}{r^2(0)} - \frac{1}{r^2(\pi/2)}\Big).$$



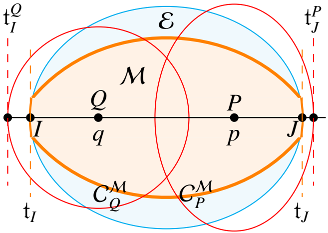


Finding the fitting ellipse

Lemma. *If $\partial\mathcal{M}$ is twice differentiable at I and J, then there is a unique ellipse $\mathcal{E}$ touching $\mathcal{M}$ at I, J such that $C_Q^{\overline{\mathcal{E}}} \equiv C_Q^{\mathcal{M}}$ and $C_P^{\overline{\mathcal{E}}} \equiv C_P^{\mathcal{M}}$.*

Proof. Fix the Euclidean metric d in which $C_Q^{\mathcal{M}}$ is a circle. Assume that $X \in \partial\mathcal{M}$, hence also $Y = \Phi_P(X) \in \partial\mathcal{M}$. Basic differential geometry gives that the respective curvatures of $\partial\mathcal{M}$ at I and J are

(2.7) $$\kappa_I := \lim_{x\to 0}\frac{2x}{y^2} \quad \text{and} \quad \kappa_J := \lim_{u\to 0}\frac{2u}{v^2}.$$



Using the formulas of the *Approximation Lemma* in conjunction with the quadraticity (2.6) leads to

$$\kappa_J = \lim_{u\to 0}\frac{2u}{v^2} = \lim_{u\to 0}\frac{-2x}{y^2} + \frac{2}{r(0)} - 2r(0)\Big(\frac{1}{r^2(0)} - \frac{1}{r^2(\pi/2)}\Big) = -\kappa_I + \frac{2r(0)}{r^2(\pi/2)}.$$

Repeating the same calculation for Φ_Q gives $\kappa_J = -\kappa_I + \frac{2}{1-q^2}$, hence $r(\frac{\pi}{2}) = \sqrt{1-q^2}\,\sqrt{1-p^2}$. Now easy calculation shows that $(q,0)$ is a focus of the ellipse $x^2 + \frac{y^2}{1-q^2} = 1$, and the infinitesimal circle at $(p,0)$ is the ellipse $\frac{(x-p)^2}{(1-p^2)^2} + \frac{y^2}{(1-q^2)(1-p^2)} = 1$. Thus choosing the ellipse $x^2 + \frac{y^2}{1-q^2} = 1$ for $\mathcal{E}$ proves the lemma. ∎

Coincidence in a neighborhood

Lemma. *If $\partial\mathcal{M}$ is C^2 at I and J, then $\mathcal{E}$ coincides with $\partial\mathcal{M}$ in a neighborhood of I and J.*

Proof. According to the last formula in the proof of the previous lemma, the infinitesimal circles $C_P^{\overline{\mathcal{E}}} \equiv C_P^{\mathcal{M}}$ and $C_Q^{\overline{\mathcal{E}}} \equiv C_Q^{\mathcal{M}}$ can be represented by polar equations of form

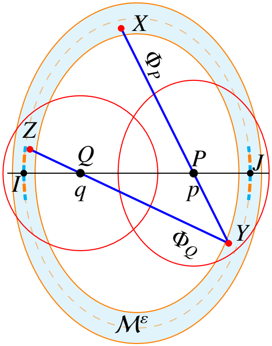
$$\frac{1}{r^2(\varphi)} = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}, \quad \text{and} \quad \frac{1}{r_q^2(\varphi)} = \frac{1}{r_q^2(0)},$$

respectively. Substitution of these into (2.2) shows that Φ_P and Φ_Q are real analytic mappings on $\mathcal{M}^\varepsilon$.

Thus $\Phi := \Phi_Q \circ \Phi_P \colon X \mapsto Y \mapsto Z$ is also a real analytic mapping. The *Approximation Lemma* and a (long) calculation gives that

$$\Phi^\Psi(z,y) := \Psi^{-1} \circ \Phi \circ \Psi(z,y) = (z + o(1), yk + o(y^2)),$$

where $\Psi\colon (z,y) \mapsto (zy^2, y)$, $y \neq 0$, $k = \frac{1-p}{1+p}\frac{1+q}{1-q} < 1$, and z is close to $\kappa_I/2$. So defining $\Phi^\Psi(z,0) := (z,0)$ extends Φ^Ψ to a real analytic mapping around $(\kappa_I/2, 0)$. As Φ^Ψ fixes the points $(z,0)$ near $(\kappa_I/2,0)$, and it has the derivative $\dot{\Phi}^\Psi(\kappa_I/2,0) = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ at $(\kappa_I/2,0)$, the *Stable Manifold Theorem* applies. Thus a neighborhood $\mathcal{N}$ of $(\kappa_I/2,0)$ exists such that $C = \{\boldsymbol{w} \in \mathcal{N} : (\Phi^\Psi)^{(r)}(\boldsymbol{w}) \to (\kappa_I/2,0) \text{ as } r \to \infty\}$ is the graph of a C^1 function $z \mapsto y$. As Φ fixes $\partial\mathcal{M}$, this proves the lemma. ∎



Full extension of the coincidence

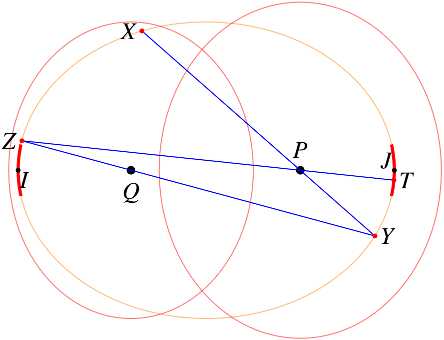
Lemma. *If Q and P are common Riemannian points of the Hilbert geometries $(\mathcal{L}, d_{\mathcal{L}})$ and $(\mathcal{M}, d_{\mathcal{M}})$, and the boundaries $\partial\mathcal{L}$ and $\partial\mathcal{M}$ coincide in a neighborhood of the line PQ, then $\mathcal{L} \equiv \mathcal{M}$.*

Proof. Let $\mathcal{N}$ be a neighborhood of line PQ such that $\partial\mathcal{L} \cap \mathcal{N} \equiv \mathcal{N} \cap \partial\mathcal{M}$.

Observe that $C_Q^{\mathcal{L}} \equiv C_Q^{\mathcal{M}}$ and $C_P^{\mathcal{L}} \equiv C_P^{\mathcal{M}}$, because the common arcs of $\partial\mathcal{L}$ and $\partial\mathcal{M}$ determine small common arcs of the quadratic infinitesimal circles near line QP.

Thus both Φ_P and Φ_Q map any common arc of $\partial\mathcal{L}$ and $\partial\mathcal{M}$ to a common arc of $\partial\mathcal{L}$ and $\partial\mathcal{M}$.

See the *proof without words* on the right! ∎



The results

Theorem. (K.Á.: [13, Theorem 4.4]).

If a Hilbert geometry has two Riemannian points, and its boundary is twice differentiable where it is intersected by the line joining those Riemannian points, then it is a Cayley–Klein model of the hyperbolic plane.

The same in the language of geometric tomography [6] reads as:

Theorem. (K.Á.: [13, Theorem 5.1]).

Let Q and P be two interior points of a convex compact domain $\mathcal{M}$. Assume that the boundary $\partial\mathcal{M}$ is twice differentiable where it intersects line QP. If the (-1)-chord function at Q and P are quadratic, then $\partial\mathcal{M}$ is an ellipse.

This generalizes Falconer's [4, Theorem 3], where only circles were considered.
However, Falconer's [4, Theorem 4] gives that for any two fixed points P, Q, a bunch of strictly convex bounded open domains $\mathcal{M}$ exist such that $P, Q \in \mathcal{M}$ are equireciprocal, the boundary $\partial\mathcal{M}$ is differentiable at $I, J \in PQ \cap \partial\mathcal{M}$ and twice differentiable everywhere in $\partial\mathcal{M} \setminus \{I, J\}$, BUT *$\partial\mathcal{M}$ is not an ellipse*.
Observe that in such an $\mathcal{M}$ there can not exist a third inner point with quadratic (-1)-chord function, because then $\partial\mathcal{M}$ should be an ellipse by the above theorem.

Riemannian points in higher dimensions

Our results do not imply similar results for higher dimensions directly, so we still do not know

How many Riemannian points are needed to deduce the hyperbolicity of a Hilbert geometry in dimension $n > 2$?

My *belief* is that $n + 1$ Riemannian points in general position is enough. A braver tip would be that n is enough if the boundary is twice differentiable.

Although it was the real motivation behind this work, and I am indebted to *Tibor Ódor* for that discussion where the problem, now formulated with the notion of Radon point, arisen, no result for general Radon points was reached. So we still curious about:

How many Radon points are needed to deduce the hyperbolicity of a Hilbert geometry in dimension 2?

(Notice that in dimensions $n \geq 3$ the Radon points are Riemannian points.)

However [13, Theorem 4.4] supports my *conjecture* that

Conjecture. The existence of two Radon points implies the hyperbolicity of a Hilbert geometry if the boundary is twice differentiable.

If twice differentiability fails, then we know that even two Riemannian points do not guarantee the hyperbolicity of the Hilbert geometry.

THANK YOU FOR YOUR ATTENTION!

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Structure of the talk

1. **Points of a Hilbert geometry**
   - The Busemann curvature and the metric centers
   - Radon points and Riemannian points
2. **Two Riemannian points**
   - Preliminaries
   - Preparations
   - Proof
   - Results
3. **Discussions**