Funk-típusú Radon transzformációk konstans görbületű tereken

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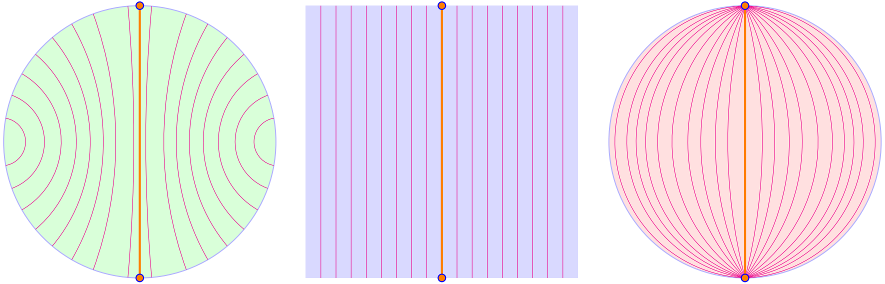

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Az animációk megtekintéséhez Adobe PDF Reader szükséges.

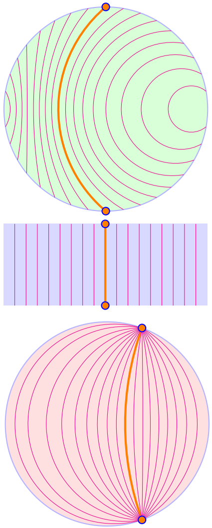
Given a totally geodesic $\mathcal{G}$ of codimension 1 in a constant curvature space $\mathbb{K}^n_\kappa$ of dimension $n \in \mathbb{N}_{2\le}$ and of curvature $\kappa \in \{1, 0, -1\}$, a connected submanifold $\mathcal{D}$ whose points have a fix distance $\varrho \geq 0$ from $\mathcal{G}$, the *axis*, is called *equidistant* of radius ϱ. We denote the set of 1-codimensional equidistants by $\mathbb{E}_\kappa$, and its subset of the totally geodesics of codimension 1 by $\mathbb{G}_\kappa$.

The *equidistant Radon transform* $\mathrm{R}^{\mathbb{E}}_\kappa$ integrates suitable functions by the natural measure over equidistants in $\mathbb{E}_\kappa$. It is injective on a large class of functions, because its restriction $\mathrm{R}^{\mathbb{G}}_\kappa := \mathrm{R}^{\mathbb{E}}_\kappa\restriction_{\mathbb{G}_\kappa}$ is injective by [19, Theorem 3.2].

Thus, if $\kappa \neq 0$, the inversion problem of the equidistant Radon transform is severely overdetermined, hence the *admissibility problem* [12, 14] arises:

(1.1) *What are the submanifolds $\mathbb{D}$ of $\mathbb{E}_\kappa$ for which the restricted equidistant Radon transform $\mathrm{R}^{\mathbb{E}}_\kappa\restriction_{\mathbb{D}}$ is injective on a reasonable space of functions?*

We call these submanifolds $\mathbb{D}$ *admissible* with respect to the chosen space of functions [12, 14], and notice that for instance $\mathbb{G}_\kappa$ is an admissible submanifold of $\mathbb{E}_\kappa$.



Equidistants of a geodesic in the Poincaré models of $\mathbb{K}^2_\kappa$ for $\kappa = 1, 0, -1$, respectively.

Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n+1}$ be an ONB of $\mathbb{R}^{n+1}$. The points $\boldsymbol{p} = p_1\boldsymbol{b}_1 + \cdots + p_n\boldsymbol{b}_n + p_{n+1}\boldsymbol{b}_{n+1}$ satisfying

(1.2) $$\kappa(p_1^2 + \cdots + p_n^2) + p_{n+1}^2 = 1$$

form the hypersurface $\mathcal{K}_\kappa^n \subset \mathbb{R}^{n+1}$ that models the constant curvature space $\mathbb{K}_\kappa^n$ [8] if it is equipped with the Riemannian metric

(1.3) $$g_{\kappa;\boldsymbol{p}} : T_{\boldsymbol{p}}\mathcal{K}_\kappa^n \times T_{\boldsymbol{p}}\mathcal{K}_\kappa^n \ni (\boldsymbol{x}, \boldsymbol{y}) \mapsto \sum_{i=1}^{n} x_i y_i + \kappa\, x_{n+1} y_{n+1}.$$

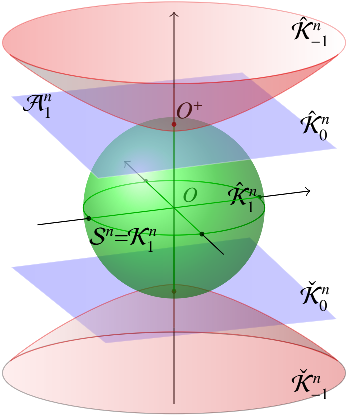
The *projective surface model* $\bar{\mathcal{K}}_\kappa^n$ arises by using the *canonical correspondence*

(1.4) $$\chi : \mathcal{K}_\kappa^n \ni E \to (E, -E) \in \bar{\mathcal{K}}_\kappa^n \cong \mathbb{K}_\kappa^n.$$

We introduce

(1.5) $$\hat{\mathcal{K}}_\kappa^n = \mathcal{K}_\kappa^n \cap \{\boldsymbol{p} : (1-\kappa)\langle \boldsymbol{p}, \boldsymbol{b}_{n+1}\rangle \geq 0\},$$

$\check{\mathcal{K}}_\kappa^n = \mathcal{K}_\kappa^n \setminus \hat{\mathcal{K}}_\kappa^n$, and $O = (0, \dots, 0)$, $O^+ = (0, \dots, 0, 1)$ and $O^- = (0, \dots, 0, -1)$.



Bijection (1.4) canonically identifies every function h on $\mathbb{K}_\kappa^n$ with the even function $\tilde{h} = h \circ \imath$ on $\mathcal{K}_\kappa^n$, so one can consider (1.1) by investigating the even functions on $\mathcal{K}_\kappa^n$. It is very well known that every totally geodesic of $\mathbb{K}_\kappa^n$ belongs to the intersection of $\mathcal{K}_\kappa^n$ with a 1-codimensional subspace of $\mathbb{R}^{n+1}$ [19]. It is almost unknown (see Lemma 5) that every equidistant of codimension 1 belongs to the intersection of $\mathcal{K}_\kappa^n$ with a hyperplane of $\mathbb{R}^{n+1}$. *Thus* (1.1) *means considering integration of functions over hyperplane sections of* $\mathcal{K}_\kappa^n$.

We call the intersections of $\mathcal{K}^n_\kappa$ with hyperplanes *slices*. The slice transform S_κ sends every suitable (not necessarily even) function f on $\mathcal{K}^n_\kappa$ to the function $\mathsf{S}_\kappa f$ on the set of slices such that it gives for every slice the integral of f over that slice.

We call a set of hyperplane sections in $\mathcal{K}^n_\kappa$ *concurrent* if the hyperplanes pass through a fixed point P of the rotational axis of $\mathcal{K}^n_\kappa$.

The *shifted Funk transform* F^P_κ is the restriction of the slice transform S_κ onto such a concurrent set with fixed point P. There are numerous results about F^P_1 on the sphere $\mathcal{K}^n_1 \cong \mathcal{S}^n$:

‹f1› *If $d(O,P) < 1$, then $\ker \mathsf{F}^P_1$ consists "odd" functions;* [23, 25, 27, 28];

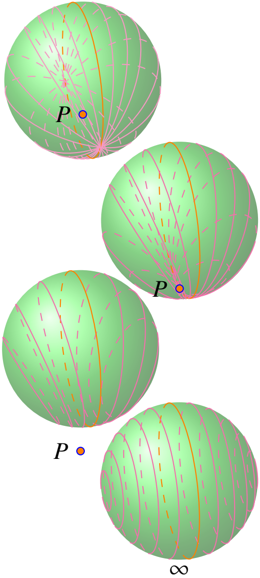
‹f2› *If $d(O,P) = 1$, then F^P_1 is invertible;* (Helgason's spherical slice transform [15, Corollary 1.26 in Chapter III], see also [1, 7, 23, 25, 28]);

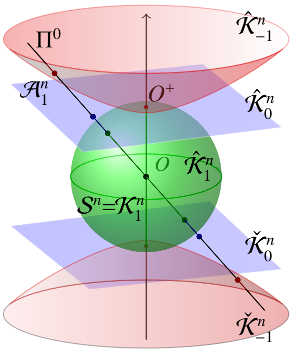
‹f3› *If $d(O,P) > 1$, then $\ker \mathsf{F}^P_1$ consists "odd" functions;* [3, 4];

‹f4› *If P is the ideal point of the rotational axis, then $\ker \mathsf{F}^P_1$ consists the functions that are odd in the third component;* ([13, 16, 33]).

Further a different result built on the previous ones:

‹f5› *If the line of the "unharmonic" points P and Q intersects $\mathcal{S}^n$, then $\ker \mathsf{F}^P_1 \cap \ker \mathsf{F}^Q_1 = \emptyset$.* (Agranovsky & Rubin [5, 6]).





The *gnomonic projection* Π^0 of $\mathcal{K}^n_\kappa$ through O to the hyperplane $\mathcal{A}^n_1$ of equation $x_{n+1} = 1$, equipped with the ideal hyperplane, gives the *Cayley–Klein models* of the constant curvature spaces.

The domain $\bar{\mathcal{M}}^n_{\kappa,1}$ of such a Cayley–Klein model, is $\mathcal{A}^n_1$ with the ideal hyperplane if $\kappa = +1$, $\mathcal{A}^n_1$ if $\kappa = 0$, and the interior of the unit ball $\mathcal{B}^n_1$ centered to O^+ in $\mathcal{A}^n_1$ if $\kappa = -1$.

The geodesics are the chords (or lines) of $\bar{\mathcal{M}}^n_{\kappa,1}$, the totally geodesics are the n-dimensional slices (or hyperplanes) of $\mathcal{B}^n_1$ or $\mathcal{A}^n_1$, respectively [8].

Since $\mathcal{K}^n_\kappa$ is a rotational manifold, it is determined by the *size function* σ_κ [17] (the geodesic sphere of radius r isometric to the Euclidean sphere of radius $\sigma_\kappa(r)$).

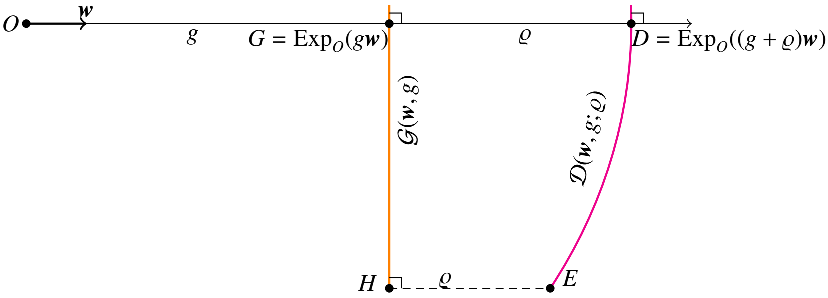
The *projector function* τ_κ defined by $\Pi^0(r\boldsymbol{w}) = \tau_\kappa(r)\boldsymbol{w}$ [19], the *injectivity radius* is ι_κ, the *geodesic injectivity radius* is ρ_κ, and we introduce $\eta_\kappa(\cdot) := \sqrt{1 - \kappa\,\sigma^2_\kappa(\cdot)}$.

$\hat{\mathcal{K}}^n_\kappa$ (type)	κ	σ_κ	τ_κ	η_κ	ρ_κ	ι_κ
$\hat{\mathcal{K}}^n_{-1} = \mathcal{H}^n_+$ (hyperbolic)	-1	sinh	tanh	cosh	∞	∞
$\hat{\mathcal{K}}^n_0 = \mathcal{A}^n_1$ (Euclidean)	0	Id	Id	1	∞	∞
$\hat{\mathcal{K}}^n_1 = \mathcal{S}^n$ (elliptic)	$+1$	sin	tan	cos	$\pi/2$	π

The *metric* d_κ on $\mathcal{K}^n_\kappa$ is determined by the Riemannian metric (1.3).

We parameterize the manifold $\hat{\mathbb{G}}_\kappa$ of the totally geodesics in $\hat{\mathcal{K}}^n_\kappa$ on $\mathcal{S}^{n-1} \times [0, \rho_\kappa)$ so that the totally geodesic $\hat{\mathcal{G}}(\boldsymbol{w}, g)$ is perpendicular to the geodesic $t \mapsto \mathrm{Exp}_{O^+}(t\boldsymbol{w})$ and contains $\mathrm{Exp}_{O^+}(g\boldsymbol{w})$. Thus $\mathcal{P}(\boldsymbol{w}, g) = \{\boldsymbol{x} : g = \langle \boldsymbol{w}, \boldsymbol{x} \rangle\}$ are the hyperplanes in $\mathbb{R}^{n+1}$, where $g \in [0, \infty)$. We parameterize on $\mathcal{S}^{n-1} \times \{(g, \varrho) : g \in [0, \rho_\kappa)$ and $\varrho + g \in (-\rho_\kappa, \rho_\kappa)\}$ the manifold $\hat{\mathbb{E}}_\kappa$ of the equidistants in $\hat{\mathcal{K}}^n_\kappa$ so that

$\hat{\mathcal{D}}(\boldsymbol{w}, g; \varrho)$ is the ϱ-equidistant of the axis $\hat{\mathcal{G}}(\boldsymbol{w}, g) \in \hat{\mathbb{G}}_\kappa$ that passes the point $\mathrm{Exp}_{O^+}(\boldsymbol{w}, g+\varrho)$.



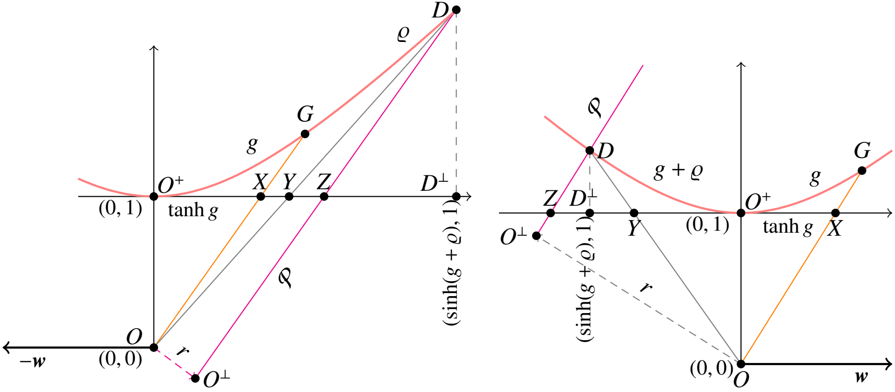
Lemma. *For any $\boldsymbol{w} = (w_1, \dots, w_{n-1}, 0) \in \mathcal{S}^{n-1}_0$ we have*

(2.1) $$\hat{\mathcal{D}}(\boldsymbol{w}, g; \varrho) = \hat{\mathcal{K}}^n_\kappa \cap \mathcal{P}\Big(\frac{\boldsymbol{w} - \tau_\kappa(g)\boldsymbol{b}_{n+1}}{\sqrt{1 + \tau^2_\kappa(g)}}, \frac{\sigma_\kappa(\varrho)}{\sqrt{\eta^2_\kappa(g) + \sigma^2_\kappa(g)}} \Big).$$

Proof. Formula (2.1) clearly holds for $\kappa \in \{0, 1\}$, so we assume $\kappa = -1$ from now on.

Firstly we determine the point D of $\hat{\mathcal{K}}^n_{-1} \cap \mathcal{P}(\frac{\boldsymbol{w}-\tanh g \boldsymbol{b}_{n+1}}{\sqrt{1+\tanh^2 g}}, r)$, closest to point O^+, where $r \in \mathbb{R}$.

Figures below show the section with the plane containing O, O^+, G and D.



Both figures show that triangles $\triangle(OXY)$ and $\triangle(DZY)$ are similar and the ratio of the similarity is $\frac{DD^\perp}{OO^+} = \cosh(g + \varrho) - 1$. This gives $r = \frac{\tanh(g+\varrho)-\tanh g}{\sqrt{1-\tanh^2(g+\varrho)}\sqrt{1+\tanh^2 g}} = \sinh\varrho \frac{\sqrt{1-\tanh^2 g}}{\sqrt{1+\tanh^2 g}}$.

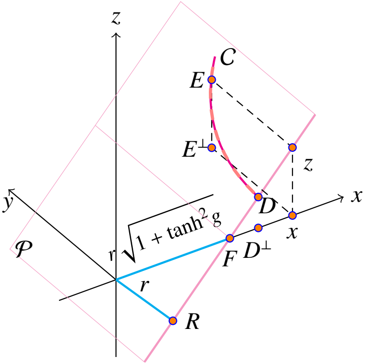
Now we determine the *slice* $C = \hat{\mathcal{K}}^n_{-1} \cap \mathcal{P}(\frac{\boldsymbol{w} - \tanh g \boldsymbol{b}_{n+1}}{\sqrt{1+\tanh^2 g}}, r)$.

For any unit vector $\boldsymbol{w}$ every point of $\mathbb{R}^{n+1}$ can be uniquely written in the form $x\boldsymbol{w} + y\boldsymbol{w}^\perp + z\boldsymbol{b}_{n+1}$, where $\boldsymbol{w}^\perp$ is a unit vector in the orthogonal complement of the plane spanned by $\boldsymbol{w}$ and $\boldsymbol{b}_{n+1}$. In this form a point is in slice C if and only if

$$x^2 + y^2 + 1 = z^2 \quad \text{and} \quad z \tanh g = x - r\sqrt{1 + \tanh^2 g},$$

hence the expression for C is

$$\begin{cases} x^2 + y^2 + 1 = (x \coth g - r\sqrt{1 + \coth^2 g})^2 & \text{if } g \neq 0, \\ r^2 + y^2 + 1 = z^2 & \text{if } g = 0. \end{cases}$$



To find the representation of C in the Poincaré model, we need the stereographic projection

$$\Pi^{-1} : x\boldsymbol{w} + y\boldsymbol{w}^\perp + z\boldsymbol{b}_{n+1} \mapsto \frac{x}{1+z}\boldsymbol{w} + \frac{y}{1+z}\boldsymbol{w}^\perp =: s\boldsymbol{w} + t\boldsymbol{w}^\perp$$

into the subspace $\mathcal{A}^n_0$ of equation $x_{n+1} = 0$.

Thus the image $\Pi^{-1}(C)$ is the solution of the equation

$$\text{(2.2)} \qquad \left(s + \frac{1}{r\sqrt{1 + \tanh^2 g} - \tanh g}\right)^2 + t^2 = \frac{r^2(1 + \tanh^2 g) - \tanh^2 g + 1}{(r\sqrt{1 + \tanh^2 g} - \tanh g)^2},$$

which is a sphere, hence, because $\Pi^{-1}(\hat{\mathcal{K}}^n_{-1})$ is the Poincaré model, C is an equidistant, so the Lemma is proved. ∎

A hyperplane, if rotations around the rotational axis are disregarded, is determined by its distance r from the origin O and by its intersection $p\boldsymbol{b}_{n+1}$ with the $(n+1)$th axis. So an equidistant $\hat{\mathcal{D}}(\boldsymbol{w}, g; \varrho)$ belongs to the hyperplane given by the pair (p, r), where

$$p = \begin{cases} \sigma_\kappa(\varrho)/\sigma_\kappa(g), & \text{if } g > 0, \\ \operatorname{sign}(\varrho)\infty, & \text{if } g = 0, \end{cases} \quad \text{and} \quad r = \frac{|\sigma_\kappa(\varrho)|}{\sqrt{\eta_\kappa^2(g) + \sigma_\kappa^2(g)}}, \quad \text{so} \quad r \le |p|. \tag{2.3}$$

Note that $g > 0$ if and only if $p \in \mathbb{R}$, $g = 0$ if and only if $p = -\infty$, and $p = 0$ if $\varrho = 0$ and $g > 0$. Notice, that every slice belongs to equidistant if $\kappa = 0$ or $\kappa = 1$, but

if $\kappa = -1$, then the slices with $r \ge |p|/\sqrt{2}$ are not equidistant.

For every $p \in \mathbb{R} \cup \{\pm\infty\}$ let $\mathcal{A}_p^n$ be the hyperplane of equation $x_{n+1} = p$, and define the *projection* $\Pi^p\colon \mathbb{R}^{n+1} \setminus \mathcal{A}_p^n \to \mathcal{A}_{p+1}^n$ by

$$\Pi^p(x_1, \ldots, x_n, x_{n+1}) = \begin{cases} \left(\frac{x_1}{x_{n+1}-p}, \ldots, \frac{x_n}{x_{n+1}-p}, p+1\right) & \text{if } p \in \mathbb{R}, \\ (x_1, \ldots, x_n, \pm\infty) & \text{if } p = \pm\infty. \end{cases}$$

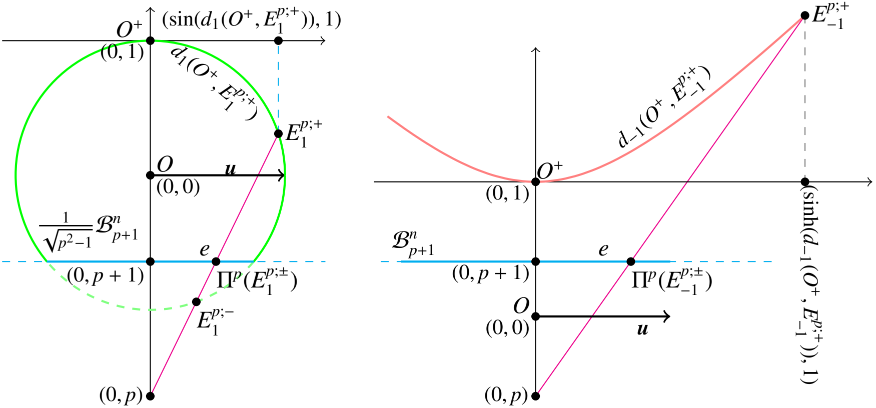
Define $\mathsf{E}\colon \mathbb{R}^n \to \mathcal{A}_0^n \subset \mathbb{R}^{n+1}$ by $\mathsf{E}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$. Then it is easy to see that

$$\begin{aligned} \bar{\mathcal{M}}_{\kappa;p}^n &:= \Pi^p(\mathcal{K}_\kappa^n) \\ &= (p+1)\boldsymbol{b}_{n+1} + \mathsf{E}(\mathcal{M}_{\kappa;p}^n). \end{aligned} \tag{2.4}$$

The *new models $\mathcal{M}_{\kappa;p}^n$ of $\mathbb{K}_\kappa^n$* represent the equidistants given by (2.3) with straight lines or chords.

$$\mathcal{M}_{\kappa;p}^n = \begin{cases} \frac{1}{\sqrt{1-p^2}}\mathcal{B}^n, & \text{if } \kappa = -1 \text{ and } |p| < 1, \\ \mathbb{R}^n, & \text{if } \kappa = -1 \text{ and } |p| \ge 1 \text{ or } p = \pm\infty, \\ \mathbb{R}^n, & \text{if } \kappa = 0 \text{ and } p \in \mathbb{R} \text{ or } p = \pm\infty, \\ \mathbb{R}^n, & \text{if } \kappa = 1 \text{ and } p \le 1, \\ \frac{1}{\sqrt{p^2-1}}\mathcal{B}^n, & \text{if } \kappa = 1 \text{ and } |p| > 1, \\ \mathcal{B}^n, & \text{if } \kappa = 1 \text{ and } p = \pm\infty. \end{cases}$$

Fix a unit vector $\boldsymbol{u} \in \mathcal{S}^n \cap \mathcal{A}_0^n$. Then every point of $\bar{\mathcal{M}}_{\kappa;p}^n = \Pi^p(\mathcal{K}_\kappa^n)$ in the subspace generated by $\boldsymbol{u}$ and $\boldsymbol{b}_{n+1}$ can be written uniquely in the form $p\boldsymbol{b}_{n+1} + e\boldsymbol{u}$, where $e \in \mathbb{R}$. So there are coefficients $\nu \in \mathbb{R}$ such that the point $E_\kappa^p = p\boldsymbol{b}_{n+1} + \nu(\boldsymbol{b}_{n+1} + e\boldsymbol{u})$ is in $\mathcal{K}_\kappa^n$.



Thus $\kappa\nu^2e^2 + (p+\nu)^2 = 1$ follows that gives two solutions

(3.1) $\nu_\kappa^{p;\pm}(e) := \dfrac{-p \pm \sqrt{1-\kappa e^2(p^2-1)}}{1+\kappa e^2}$ that define $E_\kappa^{p;\pm}(e) := p\boldsymbol{b}_{n+1} + \nu_\kappa^{p;\pm}(e)(\boldsymbol{b}_{n+1} + \nu_\kappa^{p;\pm}(e)\boldsymbol{u})$.

(3.5) and (2.4) allow us to define the mapping

$$\bar{\Pi}^{p;\pm}_{\kappa} : \bar{\mathcal{M}}^{n}_{\kappa;p} \ni x\boldsymbol{u} + (p+1)\boldsymbol{b}_{n+1} \mapsto p\boldsymbol{b}_{n+1} + \nu^{p;\pm}_{\kappa}(x)(\boldsymbol{b}_{n+1} + x\boldsymbol{u}) \in \mathcal{K}^{n}_{\kappa}. \tag{3.2}$$

Observe that by (2.1) we have

$$\mathcal{P}\Big(\frac{\boldsymbol{w} - q\boldsymbol{b}_{n+1}}{\sqrt{1+q^2}}, \frac{q|p|}{\sqrt{1+q^2}}\Big) \cap \bar{\mathcal{M}}^{n}_{\kappa;p} = \Big\{\frac{q}{\langle \boldsymbol{u}, \boldsymbol{w}\rangle}\boldsymbol{u} : \boldsymbol{u} \in \mathcal{S}^{n-1}_{0}\Big\} \cap \bar{\mathcal{M}}^{n}_{\kappa;p},$$

so we define the *special slice transforms* $\mathsf{S}^{\pm}_{\kappa}$ of a suitable function h on $C(\mathcal{K}^{n}_{\kappa})$ by

$$\mathsf{S}^{\pm}_{\kappa} h(p; \boldsymbol{w}, q) = \int_{\mathcal{S}^{n-1}_{\boldsymbol{w},q}} h\Big(\bar{\Pi}^{p;\pm}_{\kappa}(e_q(\langle \boldsymbol{w}, \boldsymbol{u}\rangle)\boldsymbol{u} + (p+1)\boldsymbol{b}_{n+1})\Big)\omega^{p;\pm}_{\kappa;q}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}, \tag{3.3}$$

where $e_q(x) = q/x$ for every $x \in (0,1]$, $\mathcal{S}^{n-1}_{\boldsymbol{w},q} = \{\boldsymbol{u} \in \mathcal{S}^{n-1}_{0} : e_q(\langle \boldsymbol{w}, \boldsymbol{u}\rangle)\boldsymbol{u} \in \bar{\mathcal{M}}^{n}_{\kappa;p}\}$, $d\boldsymbol{u}$ is the standard surface measure of $\mathcal{S}^{n-1}_{0}$, and $\omega^{p;\pm}_{\kappa;q}$ is the density pulled back by $\Pi^{p;\pm}_{\kappa}$ from the hypersurface $\hat{\mathcal{K}}^{n}_{\kappa} \cap \mathcal{P}(\frac{\boldsymbol{w} - q\boldsymbol{b}_{n+1}}{\sqrt{1+q^2}}, \frac{q|p|}{\sqrt{1+q^2}})$.

If $n = 2$, then we clearly have $\omega^{p;\pm}_{\kappa;q}(\boldsymbol{u}) = \Big|\frac{\mathrm{d}E^{p;\pm}_{\kappa}}{\mathrm{d}e}(e_q(\langle \boldsymbol{w}, \boldsymbol{u}\rangle))\Big|_{\kappa}\Big|\frac{\mathrm{d}(e_q \circ \cos)}{\mathrm{d}\varphi}(\arccos(\langle \boldsymbol{w}, \boldsymbol{u}\rangle))\Big|$.

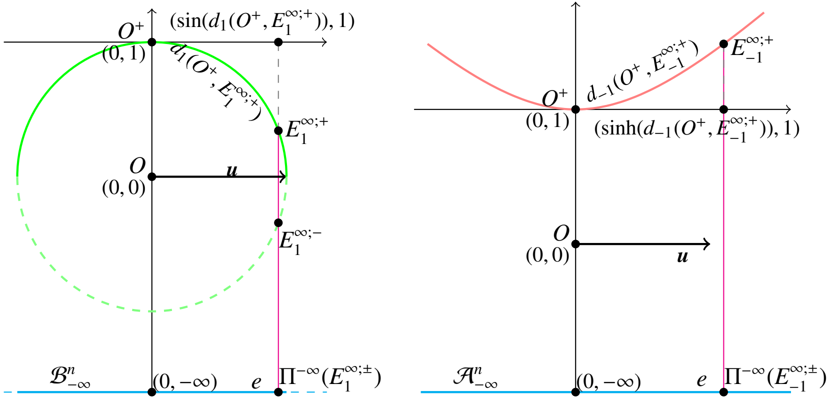
For higher dimensions this only has to be multiplied by $\sigma^{n-2}_{\kappa}(d_{\kappa}(O^{+}, E^{p;\pm}_{\kappa}(e_q(\langle \boldsymbol{w}, \boldsymbol{u}\rangle)))) = \sigma_{\kappa}(d_{\kappa}(O^{+}, E^{p;\pm}_{\kappa}(e_q))) = |\langle \boldsymbol{u}, E^{p;\pm}_{\kappa}(e_q)\rangle| = |\nu^{p;\pm}_{\kappa}(e_q)|e_q$, because $\mathcal{K}^{n}_{\kappa;p;\pm}$ is a rotational manifold.

A long and not straightforward calculation then results in

$$\omega^{p;\pm}_{\kappa;q}(\boldsymbol{u}) = \frac{\sqrt{1 + \kappa q^2(1-p^2)}}{q} \frac{|\nu^{p;\pm}_{\kappa}(e)|^{n-1} e^{n}}{\sqrt{1 - \kappa e^2(p^2 - 1)}}, \tag{3.4}$$

where $e = e_q(\langle \boldsymbol{w}, \boldsymbol{u}\rangle)$.

Fix a unit vector $\boldsymbol{u} \in \mathcal{S}^n \cap \mathcal{A}^n_0$. Then every point of $\Pi^\infty(\hat{\mathcal{K}}^n_\kappa)$ in the subspace generated by $\boldsymbol{u}$ and $\boldsymbol{b}_{n+1}$ can be written uniquely in the form $\infty\boldsymbol{b}_{n+1} + e\boldsymbol{u}$, where $e \in \mathbb{R}$. So there are coefficients $\nu \in \mathbb{R}$ such that the point $E^\infty_\kappa = \nu\boldsymbol{b}_{n+1} + e\boldsymbol{u}$ is in $\mathcal{K}^n_\kappa$.



Thus $\kappa e^2 + \nu^2 = 1$ follows that gives two solutions

(3.5) $\qquad \nu^{\infty;\pm}_\kappa(e) := \pm\sqrt{1-\kappa e^2}$ that define $E^{\infty;\pm}_\kappa(e) := \nu^{\infty;\pm}_\kappa(e)\boldsymbol{b}_{n+1} + e\boldsymbol{u}$.

This and (2.4) allow us to directly define the mapping

(3.6) $$\bar{\Pi}^{\infty;\pm}_\kappa \colon \bar{\mathcal{M}}^n_{\kappa;\infty} \ni x\boldsymbol{u} + \infty\boldsymbol{b}_{n+1} \mapsto \pm\sqrt{1-\kappa x^2}\,\boldsymbol{b}_{n+1} + x\boldsymbol{u} \in \mathcal{K}^n_\kappa.$$

Observe that by (2.1) we have

$$\mathcal{P}(\boldsymbol{w}, q) \cap \bar{\mathcal{M}}^n_{\kappa;\infty} = \left\{ \frac{q}{\langle \boldsymbol{u}, \boldsymbol{w} \rangle} \boldsymbol{u} : \boldsymbol{u} \in \mathcal{S}^{n-1}_0 \right\} \cap \bar{\mathcal{M}}^n_{\kappa;\infty},$$

so we define the *special slice transform* $\mathsf{S}^{\infty;\pm}_\kappa$ of a suitable function h in $C(\mathcal{K}^n_\kappa)$ by

(3.7) $$\mathsf{S}^\pm_\kappa h(\infty; \boldsymbol{w}, q) = \int_{\mathcal{S}^{n-1}_{\boldsymbol{w},q}} h\Big(\bar{\Pi}^{\infty;\pm}_\kappa (e_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)\boldsymbol{u} + \infty \boldsymbol{b}_{n+1})\Big) \omega^{\infty;\pm}_{\kappa;q}(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{u},$$

where $e_q(x) = q/x$ for every $x \in (0, 1]$, $\mathcal{S}^{n-1}_{\boldsymbol{w},q} = \{\boldsymbol{u} \in \mathcal{S}^{n-1}_0 : e_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)\boldsymbol{u} \in \bar{\mathcal{M}}^n_{\kappa;p}\}$, $d\boldsymbol{u}$ is the standard surface measure of $\mathcal{S}^{n-1}_0$, and $\omega^{\infty;\pm}_{\kappa;q}$ is the density pulled back by $\Pi^{\infty;\pm}_\kappa$ from the hypersurface $\hat{\mathcal{K}}^n_\kappa \cap \mathcal{P}(\boldsymbol{w}, q)$.

If $n = 2$, then we clearly have $\omega^{\infty;\pm}_{\kappa;q}(\boldsymbol{u}) = \left| \frac{\mathrm{d}E^{\infty;\pm}_\kappa}{\mathrm{d}e} (e_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)) \right|_\kappa \left| \frac{\mathrm{d}(e_q \circ \cos)}{\mathrm{d}\varphi} (\arccos(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)) \right|$.

For higher dimensions this only has to be multiplied by $\sigma^{n-2}_\kappa (d_\kappa(O^+, E^{\infty;\pm}_\kappa (e_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)))) = \sigma_\kappa(d_\kappa(O^+, E^{\infty;\pm}_\kappa(e_q))) = |\langle \boldsymbol{u}, E^{\infty;\pm}_\kappa(e_q) \rangle| = e_q$, because $\mathcal{K}^n_\kappa$ is a rotational manifold.

This time a straightforward calculation results in

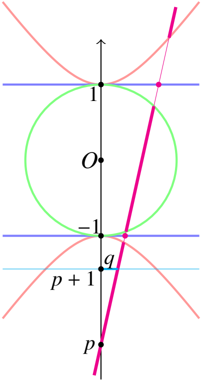
(3.8) $$\omega^{\infty;\pm}_{\kappa;q}(\boldsymbol{u}) = \frac{e^n_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle) \sqrt{1 - \kappa q^2}}{q \sqrt{1 - \kappa e^2_q(\langle \boldsymbol{w}, \boldsymbol{u} \rangle)}}.$$

We define the *slice transform* S_κ of a suitable (not necessarily even) function h on $\mathcal{K}^n_\kappa$ so that the function $\mathsf{S}_\kappa h$ on the set of hyperplanes takes the integral of h by the the natural measure over the slice $\mathcal{P} \cap \mathcal{K}_\kappa$ if $\mathcal{P} \cap \mathcal{K}_\kappa \neq$, and it takes 0 if $\mathcal{P} \cap \mathcal{K}^n_\kappa = \emptyset$.

To make the later use easier, we extend definitions (3.3) and (3.7). If $p \in \mathbb{R}$ and $q > 0$ and the hyperplane

$$\mathcal{P}\Big(\frac{\mathsf{E}(\boldsymbol{w}) - q\boldsymbol{b}_{n+1}}{\sqrt{1+q^2}}, \frac{q|p|}{\sqrt{1+q^2}}\Big) = \operatorname{span}\Big[p\boldsymbol{b}_{n+1}\,;\,(p+1)\boldsymbol{b}_{n+1} + \Big\{\frac{q}{\langle \boldsymbol{u}, \boldsymbol{w}\rangle}\boldsymbol{u} : \boldsymbol{u} \in \mathcal{S}^{n-1}_0\Big\}\Big] \tag{3.9}$$

does not intersect $\mathcal{K}^n_\kappa$, then we set $\mathsf{S}^\pm_\kappa h(p; \boldsymbol{w}, q) := 0$. Further, if $q > 0$ and the hyperplane $\mathcal{P}(\mathsf{E}(\boldsymbol{w}), q)$ does not intersect $\mathcal{K}^n_\kappa$, then we set $\mathsf{S}^\pm_\kappa h(\infty; \boldsymbol{w}, q) := 0$.



With these extended definitions we can determine the slice transform as

$$\mathsf{S}_\kappa h(p; \boldsymbol{w}, q) = \begin{cases} \mathsf{S}^+_\kappa h(p; \boldsymbol{w}, q) + \mathsf{S}^-_\kappa h(p; \boldsymbol{w}, q), & \text{if } |p| \neq 1, \\ \mathsf{S}^+_{-1} h(p; \boldsymbol{w}, q) + \mathsf{S}^-_{-1} h(p; \boldsymbol{w}, q), & \text{if } p = \pm 1 \text{ and } \kappa = -1, \\ \mathsf{S}^\mp_0 h(p; \boldsymbol{w}, q) + \mathsf{S}^\pm_0 h(-p; \boldsymbol{w}, 0), & \text{if } p = \pm 1 \text{ and } \kappa = 0, \\ \mathsf{S}^\mp_1 h(p; \boldsymbol{w}, q), & \text{if } p = \pm 1 \text{ and } \kappa = 1, \end{cases}$$

where $p \in \mathbb{R} \cup \{\pm\infty\}$, $\boldsymbol{w} \in \mathcal{S}^{n-1}$ and $q \geq 0$.

Following (2.4) we define the mappings $\mathsf{T}_\kappa^{p;\pm}\colon \mathcal{M}_{\kappa;p}^n \to \mathcal{K}_\kappa^n$ by

$$\mathsf{T}_\kappa^{p;\pm}(\boldsymbol{x}) = \begin{cases} \bar{\Pi}_\kappa^{p;\pm}(\mathsf{E}(\boldsymbol{x}) + (p+1)\boldsymbol{b}_{n+1}), & \text{if } p \in \mathbb{R}, \\ \bar{\Pi}_\kappa^{\infty;\pm}(\mathsf{E}(\boldsymbol{x}) + \infty\boldsymbol{b}_{n+1}), & \text{if } p = \infty, \end{cases} \tag{3.10}$$

where $\bar{\Pi}_\kappa^{p;\pm}$ and $\bar{\Pi}_\kappa^{\infty;\pm}$ are given by (3.2) and (3.6), respectively. Further, let $\bar{\mathsf{T}}_\kappa^{p;\pm}$ be the inverse of $\mathsf{T}_\kappa^{p;\pm}$, and define the spaces

$$\tilde{\mathcal{K}}_{\kappa;p;\pm}^n := \operatorname{Im}\mathsf{T}_\kappa^{p;\pm}, \quad \text{and} \quad \tilde{\mathcal{K}}_{\kappa;\infty;\pm}^n := \operatorname{Im}\mathsf{T}_\kappa^{\infty;\pm}.$$

The operators $\mathsf{N}_\kappa^{p;\pm}\colon C(\mathcal{M}_{\kappa;p}^n) \ni f \mapsto \mathsf{N}_\kappa^{p;\pm}f$ are defined by

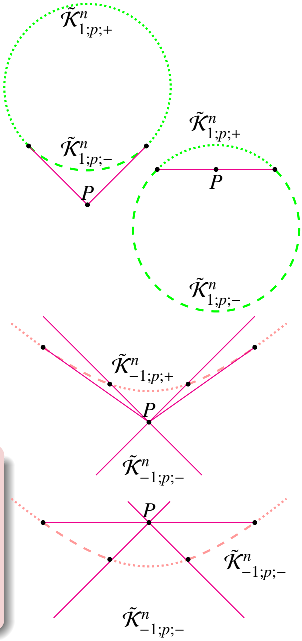
$$\mathsf{N}_\kappa^{p;\pm}f\colon \mathcal{M}_{\kappa;p}^n \ni \boldsymbol{x} \mapsto \mathsf{N}_\kappa^{p;\pm}f(\boldsymbol{x}) = \begin{cases} \dfrac{f(\boldsymbol{x})|\nu_\kappa^{p;\pm}(|\boldsymbol{x}|)|^{n-1}}{\sqrt{1-\kappa\boldsymbol{x}^2(p^2-1)}}, & \text{if } p \in \mathbb{R}^n, \\ \dfrac{f(\boldsymbol{x})}{\sqrt{1-\kappa\boldsymbol{x}^2}}, & \text{if } p = \infty, \end{cases} \tag{3.11}$$

where $\nu_\kappa^{p;\pm}$ is given by (3.5). The inverse of $\mathsf{N}_\kappa^{p;\pm}$ is $\bar{\mathsf{N}}_\kappa^{p;\pm}$.
The following *intertwining relations with the classical Radon transformation* R generalize the presenter's [19, Theorem 2.1].

Theorem. *If $f \in C(\mathcal{M}_{\kappa;p}^n)$ is such that $\mathsf{R}f$ exists, then the functions $h^\pm\colon \tilde{\mathcal{K}}_{\kappa;p;\pm}^n \to \mathbb{R}$ defined by $h^\pm \circ \mathsf{T}_\kappa^{p;\pm} = \bar{\mathsf{N}}_\kappa^{p;\pm}f$ satisfy for every $q \neq 0$ that*

$$\mathsf{S}_\kappa^\pm h^\pm(p;\boldsymbol{w},q) = \sqrt{1+\kappa q^2(1-p^2)}\,\mathsf{R}f(\boldsymbol{w},q). \tag{3.12}$$

$$\mathsf{S}_\kappa^\pm h^\pm(\infty;\boldsymbol{w},q) = \sqrt{1-\kappa q^2}\,\mathsf{R}f(\boldsymbol{w},q). \tag{3.13}$$

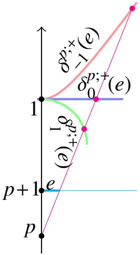


If $\mathcal{L} \subset \mathcal{K}_\kappa^n$ is a bounded open convex domain, then $C_m(\mathcal{K}_\kappa^n, \mathcal{L})$ $(m \in \mathbb{N})$ is the set of all continuous functions $h \in C(\mathcal{K}_\kappa^n)$ that satisfy the *decay conditions* $h(E) = O(1)\,\sigma_\kappa^{-m}(d_\kappa(E, O))$ as $d(E, O) \to \iota_\kappa$ and $h(E) = O(1)\,\sigma_\kappa^{-m}(d_\kappa(E, \partial\mathcal{L}))$ as $E \to \partial\mathcal{L}$, where $O \in \mathcal{K}_\kappa^n$ is any fixed point, and the usual big-O notation is in use. We use the abbreviation $C_m(\mathcal{K}_\kappa^n) = C_m(\mathcal{K}_\kappa^n, \emptyset)$, and the notations $C_m(\mathcal{K}_\kappa^n, p) := C_m(\mathcal{K}_\kappa^n, \tilde{\mathcal{K}}_{\kappa;p;+}^n) = C_m(\mathcal{K}_\kappa^n, \tilde{\mathcal{K}}_{\kappa;p;-}^n)$ $(p \in \mathbb{R} \cup \{\pm\infty\})$.

Support Theorem of Helgason [15, Theorem 2.6 of Chapter I].
If $f \in C_\infty(\mathbb{R}^n)$, and there exists a constant $A > 0$ such that $\mathsf{R}f(\mathcal{P})$ vanishes for every hyperplane farther from the origin than A, then $f(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > A$.
(Counter examples show that the decay condition can not be dropped [15, Remark 2.9 of Chapter I].*)*

As $\mathcal{K}_\kappa^n$ is a rotational manifold, we can define the non-negative functions

$$\begin{gathered}\delta_\kappa^{p;\pm}\colon [0, \ell_p) \to \mathbb{R}_+,\\ \delta_\kappa^{p;\pm}(|\boldsymbol{x}|) = d_\kappa(O^\pm, \mathsf{T}_\kappa^{p;\pm}(\boldsymbol{x})),\end{gathered} \quad \text{where } \ell_p = \begin{cases} \infty, & \text{if } |p| < \infty \text{ and } \kappa p^2 \le \kappa,\\ 1/\sqrt{|p^2 - 1|}, & \text{if } |p| < \infty \text{ and } \kappa p^2 > \kappa,\\ 1, & \text{if } |p| = \infty \text{ and } \kappa = 1,\\ \infty, & \text{if } |p| = \infty \text{ and } \kappa \le 0.\end{cases}$$

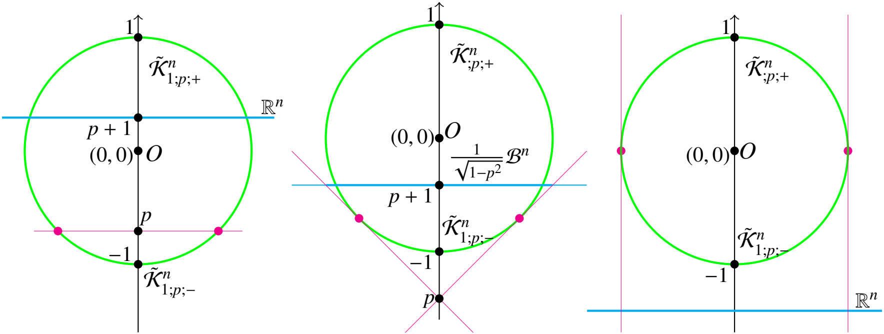


We have $\sigma_\kappa(\delta_\kappa^{p;\pm}(e)) = \nu_\kappa^{p;\pm}(e)e$ and $\eta_\kappa(\delta_\kappa^{p;\pm}(e)) = p + \nu_\kappa^{p;\pm}(e)$ if $p \in \mathbb{R}$,
while $\sigma_\kappa(\delta_\kappa^{\infty;\pm}(e)) = e$ and $\eta_\kappa(\delta_\kappa^{\infty;\pm}(e)) = \sqrt{1 - \kappa e^2}$ if $p = \infty$.

We define for every $p \in \mathbb{R} \cup \{\pm\infty\}$ the p-*shifted Funk transform* (This term follows the phrasing in [5].) of a suitable function h on $\mathcal{K}_\kappa^n$ by $\mathsf{F}_\kappa^p h\colon \mathcal{S}^{n-1} \times \mathbb{R}_{>0} \ni (\boldsymbol{w}, q) \mapsto \mathsf{S}_\kappa h(p; \boldsymbol{w}, q)$.

As in [19, Theorem 3.2], pulling the Support Theorem of Helgason back to $\mathcal{K}^n_\kappa$ through the adequate intertwining relation of (3.12) and (3.13) is again efficient.

- **‹se1›** *If* $h \in C_\infty(\mathcal{K}^n_1, \pm 1)$ *and* $\mathsf{F}^{\pm 1}_1 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\mp}(e\boldsymbol{u}))$ *vanishes for every* $e < 2\arctan s$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹se2›** *If* $|p| < 1$, $h \in C_\infty(\mathcal{K}^n_1, p)$ *vanishes on* $\tilde{\mathcal{K}}^n_{1;p;\mp}$, *and* $\mathsf{F}^p_1 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > \delta^{p;\pm}_1(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹se3›** *If* $|p| > 1$, $h \in C(\mathcal{K}^n_1)$ *vanishes on* $\tilde{\mathcal{K}}^n_{1;p;\mp}$, *and* $\mathsf{F}^p_1 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > \delta^{p;\pm}_1(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹se4›** *If* $h \in C(\mathcal{K}^n_1)$ *vanishes on* $\tilde{\mathcal{K}}^n_{1;\infty;\mp}$, *and* $\mathsf{F}^\infty_1 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > \arcsin(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sp1›** *If* $h \in C_\infty(\mathcal{K}^n_0, \pm 1)$ *and* $\mathsf{F}^{\pm 1}_0 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\mp}(e\boldsymbol{u}))$ *vanishes for every* $e > 2s$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sp2›** *If* $|p| \neq 1$, $h \in C_\infty(\mathcal{K}^n_0, p)$ *vanishes on* $\tilde{\mathcal{K}}^n_{0;p;\mp}$, *and* $\mathsf{F}^p_0 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > |-p \pm 1|q$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sp3›** *If* $h \in C_\infty(\mathcal{K}^n_0)$ *vanishes on* $\tilde{\mathcal{K}}^n_{0;\infty;\mp}$, *and* $\mathsf{F}^\infty_0 h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > s$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sh1›** *If* $h \in C_n(\mathcal{K}^n_{-1}, 0)$ *vanishes either on* $\check{\mathcal{K}}^n_{-1}$ *or on* $\hat{\mathcal{K}}^n_{-1}$, *and* $\mathsf{F}^0_{-1} h(\boldsymbol{w}, q) = 0$, *for every* $q > s \in (0, 1)$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > 2\,\mathrm{artanh}(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sh2›** *If* $|p| \neq 0$, *and* $h \in C_{n-1}(\mathcal{K}^n_{-1}, p)$ *vanishes on* $\check{\mathcal{K}}^n_{-1}$ *or on* $\hat{\mathcal{K}}^n_{-1}$ *according to* $p < 0$ *or* $p > 0$, *respectively, and* $\mathsf{F}^p_{-1} h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > \delta^{p;\pm}_{-1}(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.
- **‹sh3›** *If* $h \in C_\infty(\mathcal{K}^n_{-1})$ *vanishes either on* $\check{\mathcal{K}}^n_{-1}$ *or on* $\hat{\mathcal{K}}^n_{-1}$, *and* $\mathsf{F}^\infty_{-1} h(\boldsymbol{w}, q) = 0$, *for every* $q > s > 0$ *and* $\boldsymbol{w} \in \mathcal{S}^{n-1}$, *then* $h(\mathrm{Exp}_{O^\pm}(e\boldsymbol{u}))$ *vanishes for every* $e > \mathrm{arsinh}(s)$ *and* $\boldsymbol{u} \in \mathcal{S}^{n-1}$.

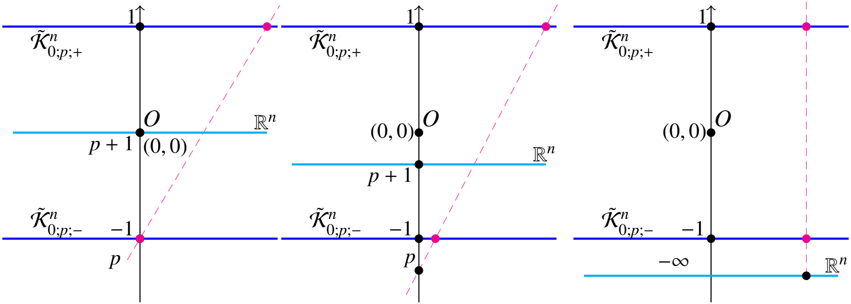


Theorem.

‹ks1› *If* $|p| < 1$ *and* $h \in C_\infty(\mathcal{K}^n_1, p)$*, then* $\mathsf{F}^p_1 h$ *vanishes if and only if there is a function* $f \in C_\infty(\mathcal{M}^n_{1;p})$ *such that* $\pm h^\pm_p \circ \mathsf{T}^{p;\pm}_1 = \bar{\mathsf{N}}^{p;\pm}_1 f$.

‹ks2› *If* $|p| > 1$ *and* $h \in C(\mathcal{K}^n_1, p)$*, then* $\mathsf{F}^p_1 h$ *vanishes if and only if there is a function* $f \in C(\mathcal{M}^n_{1;p})$ *such that* $\pm h^\pm_p \circ \mathsf{T}^{p;\pm}_1 = \bar{\mathsf{N}}^{p;\pm}_1 f$.

‹ks3› *If* $h \in C(\mathcal{K}^n_1, \infty)$*, then* $\mathsf{F}^\infty_1 h$ *vanishes if and only if there is a function* $f \in C(\mathcal{M}^n_{1;\infty})$ *such that* $\pm h^\pm_\infty \circ \mathsf{T}^{\infty;\pm}_1 = f$.

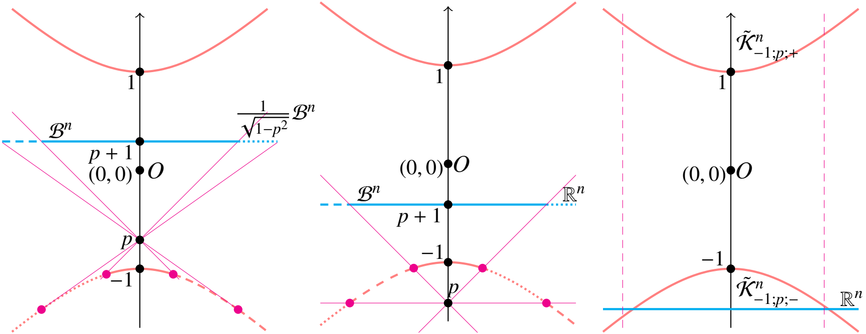


Theorem.

‹kp1› *If $p=\pm 1$ and $h\in C_\infty(\mathcal{K}^n_0)$, then $\mathsf{F}^p_1 h$ vanishes if and only if h vanishes on $\tilde{\mathcal{K}}^n_{0;p;\mp}$, and the integrals of h over the lines through $O^\pm$ in $\tilde{\mathcal{K}}^n_{0;-p;\mp}$ vanish.*

‹kp2› *If $|p|\neq 1$ and $h\in C_\infty(\mathcal{K}^n_0)$, then $\mathsf{F}^p_1 h$ vanishes if and only if there is a function $f\in C_\infty(\mathcal{M}^n_{0;p})$ such that $\pm h^\pm_p\circ \mathsf{T}^{p;\pm}_0=\bar{\mathsf{N}}^{p;\pm}_0 f$.*

‹kp3› *If $h\in C_\infty(\mathcal{K}^n_0)$, then $\mathsf{F}^\infty_0 h$ vanishes if and only if there is a function $f\in C(\mathcal{M}^n_{0;\infty})$ such that $\pm h^\pm_\infty\circ \mathsf{T}^{\infty;\pm}_0=f$.*



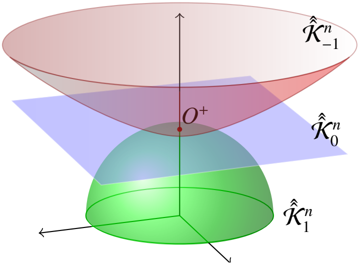
Theorem.

‹kh1› *If* $h \in C_n(\mathcal{K}^n_{-1})$*, then* $\mathsf{F}^0_{-1}h$ *vanishes if and only if there is a function* $f \in C(\mathcal{M}^n_{-1;0})$ *such that* $\pm h^\pm_0 \circ \mathsf{T}^{0;\pm}_{-1} = \bar{\mathsf{N}}^{0;\pm}_{-1} f$.

‹kh2› *If* $|p| \in (0,1)$ *and* $h \in C_{n-1}(\mathcal{K}^n_{-1}, p)$*, then* $\mathsf{F}^p_{-1}h$ *vanishes if and only if there is a function* $f \in C_0(\mathcal{M}^n_{-1;p}) \cap C_0(\mathcal{M}^n_{-1;0})$ *such that* $\pm h^\pm_p \circ \mathsf{T}^{p;\pm}_{-1} = \bar{\mathsf{N}}^{p;\pm}_{-1} f$.

‹kh3› *If* $|p| > 1$ *and* $h \in C_\infty(\mathcal{K}^n_{-1}, p)$*, then* $\mathsf{F}^p_{-1}h$ *vanishes if and only if there is a function* $f \in C(\mathcal{M}^n_{-1;p})$ *such that* $\pm h^\pm_p \circ \mathsf{T}^{p;\pm}_{-1} = \bar{\mathsf{N}}^{p;\pm}_{-1} f$.

‹kh4› *If* $h \in C_\infty(\mathcal{K}^n_{-1})$*, then* $\mathsf{F}^\infty_{-1}h$ *vanishes if and only if there is a function* $f \in C(\mathcal{M}^n_{-1;\infty})$ *such that* $\pm h^\pm_\infty \circ \mathsf{T}^{\infty;\pm}_{-1} = f$.



The double covering of $\mathbb{K}^n_\kappa$ given by (1.4) can be reduced by taking the spaces $\hat{\hat{\mathcal{K}}}^n_\kappa = \tilde{\mathcal{K}}^n_{\kappa;-\infty;+} \setminus \mathcal{A}^n_0$, and the *identifying mapping*

$$\hat{\hat{\chi}}_\kappa \colon \hat{\hat{\mathcal{K}}}^n_\kappa \ni E \to (E, -E) \in \bar{\mathcal{K}}^n_\kappa \cong \mathbb{K}^n_\kappa.$$

Then $\hat{\hat{\chi}}_\kappa$ is bijective for $\kappa \le 0$, but it is only injective for $\kappa = 1$, because $\chi_\kappa(S^n \cap \mathcal{A}^n_0)$ is not in its image.

The *equidistant Radon transform of Funk-type* $\hat{\hat{\mathsf{R}}}^p_\kappa$, the restriction of the shifted Funk transform to the set of hyperplanes intersecting $\hat{\hat{\mathcal{K}}}^n_\kappa$ in equidistants, sends a suitable function $h \in C(\mathbb{K}^n_\kappa)$ to

(5.1) $\hat{\hat{\mathsf{R}}}^p_\kappa h(\boldsymbol{w}, g) = \mathsf{F}^p_\kappa \hat{\hat{h}}(\boldsymbol{w}, \tau_\kappa(g)),$ if $p \in \mathbb{R}$,

(5.2) $\hat{\hat{\mathsf{R}}}^\infty_\kappa h(\boldsymbol{w}, \varrho) = \mathsf{F}^\infty_\kappa \hat{\hat{h}}(\boldsymbol{w}, \sigma_\kappa(\varrho)),$ if $p = \infty$,

where $\boldsymbol{w} \in \mathcal{S}^{n-1}$, $g, \varrho \in [0, \rho_\kappa)$, and

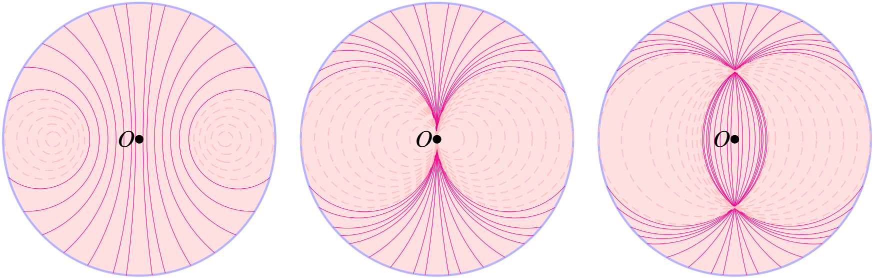
$$\hat{\hat{h}} \colon \mathcal{K}^n_\kappa \ni E \to \hat{\hat{h}}(E) = \begin{cases} h(\hat{\hat{\chi}}_\kappa(E)), & \text{if } E \in \hat{\hat{\mathcal{K}}}^n_\kappa, \\ 0, & \text{otherwise.} \end{cases}$$

In the *elliptic case*, every non-degenerate slice of $\hat{\hat{\mathcal{K}}}^n_1$ is a part of an equidistant in $\mathcal{K}^n_1$, so the properties of $\hat{\hat{\mathsf{R}}}^p_1$ are essentially similar to that of F^p_1.

In the *parabolic case*, every slice of $\hat{\hat{\mathcal{K}}}^n_0$ is an equidistant, so the properties of $\hat{\hat{\mathsf{R}}}^p_0$ are exactly the same as the properties of F^p_0 (i.e. essentially a reparameterization the classic Radon transform).

The *hyperbolic case* differs significantly, because a slice is not necessarily an equidistant.

In *the hyperbolic case*, if a normal vector $\boldsymbol{n}$ of a hyperplane $\mathcal{P}$ fulfills $\langle \boldsymbol{n}, \boldsymbol{b}_{n+1}\rangle > 1/\sqrt{2}$, then the intersection $\mathcal{P}\cap\mathcal{K}^n_{-1}$ is not an equidistant, because there does not exist a totally geodesic of co-dimension 1 whose hyperplane's normal vector is parallel with $\boldsymbol{n}$. These hyperplane sections of $\hat{\mathcal{K}}^n_{-1}$ and the corresponding hypersurfaces in $\mathbb{K}^n_{-1}$, that are not equidistant, are called *virtual equidistants*. (Virtual equidistants are the horocycles (paracycles) and circles in the hyperbolic plane. Equidistants are the hypercycles. See [36].)



Equidistants (continuous magenta arcs) and virtual equidistants (dashed pink circles) in the Poincaré model of the hyperbolic plane for $p \in (0, 1)$, $p = 1$, $p > 1$, respectively.

As figures clearly show one can not have support theorems if $p = 1$, but support theorems do exist for functions having *support in the ball of radius $|\ln p|$ with center at O if $p \neq 1$*.

For properly formulated theorems in all three cases $\kappa \in \{1, 0, -1\}$ see [20].

Other way to overcome the double covering of $\mathbb{K}^n_\kappa$ is to restrict the function space under investigation to the even functions on $\mathcal{K}^n_\kappa$.

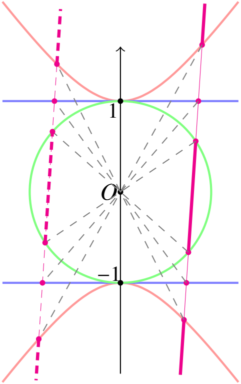
The *isodistant Radon transform of Funk-type* R^p_κ, the restriction of the shifted Funk transform of even functions on $\mathcal{K}^n_\kappa$ to the set of hyperplanes intersecting $\mathcal{K}^n_\kappa$ in equidistants, sends a suitable function $h \in C(\mathbb{K}^n_\kappa)$ to

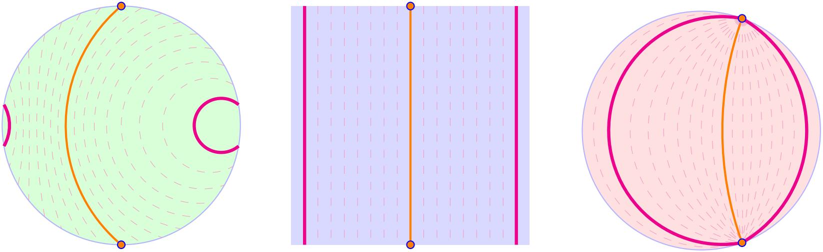
(5.3) $\mathsf{R}^p_\kappa h(\boldsymbol{w}, g) = \mathsf{F}^p_\kappa \tilde{h}(\boldsymbol{w}, \tau_\kappa(g))$ if $p \in \mathbb{R}$, $\boldsymbol{w} \in \mathcal{S}^{n-1}$, $g \in [0, \rho_\kappa)$,

(5.4) $\mathsf{R}^\infty_\kappa h(\boldsymbol{w}, \varrho) = \mathsf{F}^\infty_\kappa \tilde{h}(\boldsymbol{w}, \sigma_\kappa(\varrho))$ if $p = \infty$, $\boldsymbol{w} \in \mathcal{S}^{n-1}$, $\varrho \in [0, \rho_\kappa)$,

where $\tilde{h} \colon \mathcal{K}^n_\kappa \ni E \to \tilde{h}(E) = h(\chi_\kappa(E))$.

This means integration over the isodistants of $\mathbb{K}^n_\kappa$.



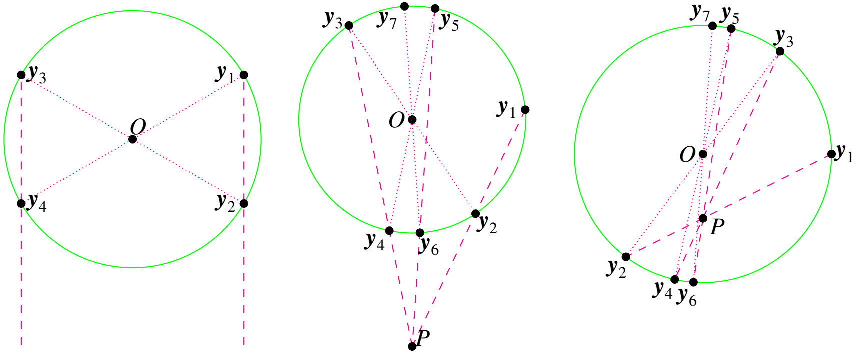


(Isodistants of a geodesic in the Poincaré models of $\mathbb{K}^2_\kappa$ for $\kappa = 1, 0, -1$, respectively.)

Observe that from any point $\boldsymbol{y}_1 = \mathsf{T}_\kappa^{p;+}(\boldsymbol{x}_0) \in \hat{\mathcal{K}}_\kappa^n$ ($p \in \mathbb{R} \cup \{\pm\infty\}$) the recursion

(5.5) $$\boldsymbol{y}_{2i+2} := \mathsf{T}_\kappa^{p;-}(\boldsymbol{x}_i),\ \boldsymbol{y}_{2i+3} := -\boldsymbol{y}_{2i+2},\ \boldsymbol{x}_{i+1} := \bar{\mathsf{T}}_\kappa^{p;+}(\boldsymbol{y}_{2i+3}) \in \mathcal{M}_{\kappa;p}^n$$

generates points for every $i = 1, 2, \ldots$. This sequence of points $\boldsymbol{y}_i$ is finite if $|p| = 1$ or $p = \pm\infty$. Otherwise we get an infinite sequence, and it is easy to see that the open segment of $\boldsymbol{x}_{2i}$ and $\boldsymbol{x}_{2i+1}$ contains $\mathbf{0}$ for every $i \in \mathbb{N}$, and we have $\boldsymbol{x}_{2i} \to \mathbf{0}$ and $\boldsymbol{x}_{2i+1} \to \mathbf{0}$ as $i \to \infty$.



The first points generated by (5.5) if $p = \infty$, $|p| > 1$ and $|p| < 1$.

This means that the sequences $\boldsymbol{y}_{2i+1}$ and $\boldsymbol{y}_{2i}$ tend to the intersections $O^\pm = \mathcal{K}_\kappa^n \cap OP$, respectively, where $P = p\boldsymbol{b}_{n+1}$ as usual.

Theorem. *The isodistant Radon transform of Funk-type on the elliptic space.*

‹ie1› *If* $h \in C(\mathbb{K}_1^n)$*, then* $\mathsf{R}_1^\infty h$ *vanishes if and only if there is an odd function* $f \in C(\mathcal{M}_{1;\infty}^n)$ *such that* $\pm\tilde{h}_\infty^\pm \circ \mathsf{T}_1^{\infty;\pm} = \bar{\mathsf{N}}_1^{p;\pm} f$*, where* $\tilde{h} = h \circ \chi_\kappa$*.*

‹ie2› *If* $|p| > 1$*, then* R_1^p *is injective on* $C(\mathbb{K}_1^n, p)$*.*

‹ie3› *If* $|p| \le 1$*, then* R_1^p *is injective on* $C_\infty(\mathbb{K}_1^n, p)$*.*

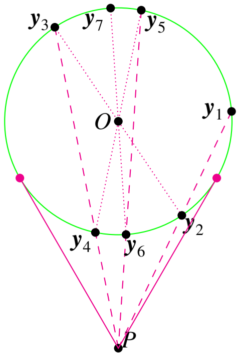
Sketch of the proof. ‹ie2›: By the symmetry of $\mathcal{S}^n = \mathcal{K}_1^n$ we can assume $p < -1$.

Assume that $h \in C(\mathbb{K}_1^n, p)$ is in the kernel of R_1^p. This means that $\mathsf{F}_1^p \tilde{h}$ vanishes for the even function $\tilde{h} \in C(\mathcal{K}_1^n, p)$. Thus there exists a function $f \in C(\mathcal{M}_{1;p}^n)$ such that $\pm\tilde{h}_p^\pm \circ \mathsf{T}_1^{p;\pm} = \bar{\mathsf{N}}_1^{p;\pm} f$.

Since $\tilde{h}$ is even, we have $\tilde{h}(\boldsymbol{y}_{2i+1}) = \tilde{h}(\boldsymbol{y}_{2i})$ for every $i \in \mathbb{N}$, so

$$(5.6) \quad \frac{\tilde{h}(\boldsymbol{y}_{2i+2})}{\tilde{h}(\boldsymbol{y}_{2i})} = \frac{\tilde{h}(\boldsymbol{y}_{2i+2})}{\tilde{h}(\boldsymbol{y}_{2i+1})} = \frac{\tilde{h}_p^-(\boldsymbol{y}_{2i+2})}{\tilde{h}_p^+(\boldsymbol{y}_{2i+1})} = \frac{-\bar{\mathsf{N}}_1^{p;-} f(\boldsymbol{x}_i)}{\bar{\mathsf{N}}_1^{p;+} f(\boldsymbol{x}_i)} = -\frac{|\nu_1^{p;+}(|\boldsymbol{x}_i|)|^{n-1}}{|\nu_1^{p;-}(|\boldsymbol{x}_i|)|^{n-1}}.$$

However (3.5) gives $\phi_p := \lim_{e\to 0}\left|\frac{\nu_1^{p;+}(e)}{\nu_1^{p;-}(e)}\right| = \lim_{e\to 0}\left|\frac{-p+\sqrt{1-e^2(p^2-1)}}{-p-\sqrt{1-e^2(p^2-1)}}\right| = \frac{-p+1}{|-p-1|} > 1$, and therefore $\lim_{i\to\infty}\frac{\tilde{h}(\boldsymbol{y}_{2i+4})}{\tilde{h}(\boldsymbol{y}_{2i})} = \phi_p^{2(n-1)} > 1$. Thus $\tilde{h}(\boldsymbol{y}_1) \ne 0$ implies that $|\tilde{h}(O^\pm)| = \infty$, a contradiction, hence $\tilde{h}(\boldsymbol{y}_1)$ vanishes that, as $\boldsymbol{y}_1$ was chosen arbitrarily, proves ‹ie2›. ∎



Theorem. *The isodistant Radon transform of Funk-type on the parabolic space.*

‹ip1› *If $h \in C_\infty(\mathbb{K}^n_0)$, then $\mathrm{R}^\infty_0 h$ vanishes if and only if h is an odd function.*

‹ip2› *If $p \in \mathbb{R}$, then R^p_0 is injective on $C_\infty(\mathbb{K}^n_0)$.*

Observe that

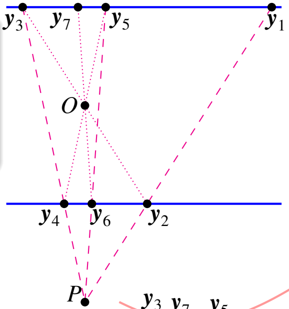
$$\mathrm{R}^p_0 h(\boldsymbol{w}, t) = \mathrm{R}h(\boldsymbol{w}, t(1-p)) + \mathrm{R}h(\boldsymbol{w}, t(1+p)) \tag{5.7}$$

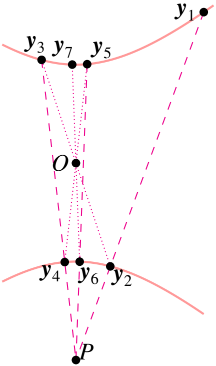
Theorem. *The isodistant Radon transform of Funk-type on the hyperbolic space.*

‹ih1› *If $h \in C_\infty(\mathbb{K}^n_{-1})$, then $\mathrm{R}^\infty_{-1} h$ vanishes if and only if there is an odd function $f \in C(\mathcal{M}^n_{-1;\infty})$ such that $\pm\tilde{h}^\pm_\infty \circ \mathrm{T}^{\infty;\pm}_{-1} = \bar{\mathrm{N}}^{p;\pm}_{-1} f$, where $\tilde{h} = h \circ \chi_\kappa$.*

‹ih2› *Let $|p| > 0$, $|p| \neq 1$, and let $h \in C(\mathbb{K}^n_{-1}, p)$ is such that $h(\mathrm{Exp}_{O^+}(e\boldsymbol{u}))$ vanishes for every $e \geq |\ln|p||$ and $\boldsymbol{u} \in \mathcal{S}^{n-1}$. If $\mathrm{R}^p_{-1} h$ vanishes, then h vanishes too.*

‹ih3› *If R^0_{-1} is injective on $C_n(\mathbb{K}^n_{-1})$.*





Pulling and applying other, already known results, perhaps most importantly the range descriptions, through our intertwining relations will give numerous new results about the shifted Funk transforms.

Following the method of the presenter's [19, Theorem 3.1] will extend and sharpen the support theorems to shifted Funk transforms of lower dimensional equidistants.

Observation (5.7) raises the question

(6.1) *For what kind of transforms* M, N *on the Grassmann manifold of hyperplanes does* $\mathsf{R}^{\mathsf{M},\mathsf{N}} f = (\mathsf{R}f) \circ \mathsf{M} + (\mathsf{R}f) \circ \mathsf{N}$ *determine the function* f *taken from a reasonable big function space?*

For an instant answer we can generalize (5.7) by considering the transformation $f \mapsto \mathsf{R}^{\boldsymbol{v}} f(\boldsymbol{w}, t) := \mathsf{R}f(\boldsymbol{w}, v_1 t) + \mathsf{R}f(\boldsymbol{w}, v_2 t)$, where $\boldsymbol{v} = (v_1, v_2)$ and $0 < v_1 < v_2$.

Theorem. *Let* $f \in C(\mathbb{R}^n)$ *be such that for every* $k \in \mathbb{N}$ *the function* $\boldsymbol{x} \mapsto f(\boldsymbol{x})|\boldsymbol{x}|^k$ *is bounded. If* $\mathsf{R}^{\boldsymbol{v}} f(\boldsymbol{w}, t)$ *vanishes for every* $t > 1$*, then the support of* f *is in the unit ball* $\mathcal{B}^n$.

Proof. Since $\mathsf{R}f(\boldsymbol{w}, ct) = c^{n-1}\mathsf{R}f_c(\boldsymbol{w}, t)$, where $f_c \colon \boldsymbol{x} \mapsto f(c\boldsymbol{x})$, we have $\mathsf{R}^{\boldsymbol{v}} f = \mathsf{R}(v_1^{n-1} f_{v_1} + v_2^{n-1} f_{v_2})$. So the Support Theorem of Helgason gives that $v_1^{n-1} f_{v_1} + v_2^{n-1} f_{v_2}$ vanishes outside the unit ball. Thus, $f(\boldsymbol{x}) = -\frac{v_1^{n-1}}{v_2^{n-1}} f(\boldsymbol{x}\frac{v_1}{v_2})$ if $|\boldsymbol{x}| > 1$, hence $f(\boldsymbol{x}) = (-\frac{v_1^{n-1}}{v_2^{n-1}})^k f(\boldsymbol{x}(\frac{v_1}{v_2})^k)$ for every $k \in \mathbb{N}$. As f satisfies the infinite decay condition, we have $(-\frac{v_1^{n-1}}{v_2^{n-1}})^k f(\boldsymbol{x}(\frac{v_1}{v_2})^k) \to 0$ if $k \to \infty$, hence $f(\boldsymbol{x}) = 0$ follows. ■

It seems also interesting to consider the manifold of equidistants whose hyperplanes have constant distance $r \in (0,1)$ from the origin of $\mathbb{R}^{n+1}$. Schneider proved the following "freaky" (this term is from Ungar [32]) result in [30]:

The manifold of hyperplanes touching $r\mathcal{S}^n$ $(r \in [0,1))$ is

‹s1› *admissible* if r is not a root of any Gegenbauer polynomial of weight $(1-x^2)^{\frac{n-3}{2}}$;

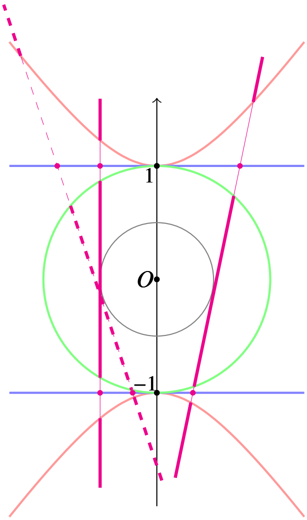
‹s2› *inadmissible* if r is a root of an even Gegenbauer polynomial of weight $(1-x^2)^{\frac{n-3}{2}}$.

Considering the Euclidean with respect to the family of isodistants given by the manifold of hyperplanes that are tangent to $r\mathcal{S}^n$ $(r \in [0,1))$ leads again to problem (6.1) about the Radon transform

$$f \mapsto \mathsf{R}^{(r)}f(\boldsymbol{w},t) = \mathsf{R}f(\boldsymbol{w}, t - r\sqrt{1+t^2}) + \mathsf{R}f(\boldsymbol{w}, t + r\sqrt{1+t^2}).$$

One could also ask for determining the surface to which the hyperplanes of an admissible family are tangent:

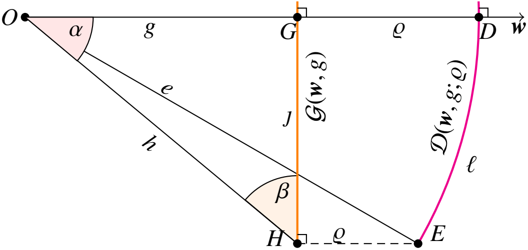
‹t1› *The restriction of $\mathsf{R}_\kappa^{\mathbb{E}}$ onto the slices the hyperplanes of which are tangent to a fixed spheroid in the interior of $\mathcal{B}^n$ that touches exactly one pair of antipodal points is invertible.* (Salman [29]);



The inverse $\bar{\mathsf{T}}_\kappa^{p;\pm}$ of the map $\mathsf{T}_\kappa^p\colon \mathcal{M}_{\kappa;p}^n \to \mathcal{K}_\kappa^n$ given in (3.10) embeds a model of $\mathbb{K}_\kappa^n$ into $\mathbb{R}^n$. *These models are mostly unknown*, but

- ‹i› it is the *Cayley–Klein disc model* [34] with the gnomonic mapping $\bar{\mathsf{T}}_\kappa^0$ [35] if $p = 0$,
- ‹ii› it is the *Poincaré disc model* [36] with the stereographic mapping $\bar{\mathsf{T}}_\kappa^{\pm 1}$ [37] if $p = \pm 1$,
- ‹iii› it is the *Gans model* [11] with $\bar{\mathsf{T}}_\kappa^{\pm\infty}$ is the orthogonal projection if $p = \pm\infty$.

Finally we show how to calculate directly the equidistant Radon transform on $\mathbb{K}_\kappa^n$. Let D be the point in $\mathcal{D}(\boldsymbol{w}, g; \varrho)$ closest to O, and let $E \in \mathcal{D}(\boldsymbol{w}, g; \varrho)$. Let G and H be the orthogonal projections of D and E into $\mathcal{G}(\boldsymbol{w}, g)$, respectively.



A qualitative depiction of an equidistant with its geodesic axis

Then G is the point of $\mathcal{G}(\boldsymbol{w}, g)$ closest to O, $g = d_\kappa(O, G)$, $\varrho = d_\kappa(G, D)$, $\jmath = d_\kappa(G, H)$, $h = d_\kappa(O, H)$ and $e = d_\kappa(O, E)$. Further the angle of the geodesics OH and OG at O is $\alpha \in (-\pi, \pi)$, and let ℓ be the length of the shortest arc in $\mathcal{D}(\boldsymbol{w}, g; \varrho)$ that connects E to D.

It is well known for the curvatures $\kappa = +1, 0$, but also proved for $\kappa = -1$ in [31, p. 68], that $\ell = \jmath\,\eta_\kappa(\varrho)$.

KÖSZÖNÖM A FIGYELMET!

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A connected submanifold in a constant curvature space is called equidistant if its points are in equal distances from a totally geodesic. The equidistant Radon transform on a constant curvature space integrates suitable real functions by the canonical measure over equidistants. Inverting the equidistant Radon transform is severely overdetermined, because the totally geodesic Radon transform is a restriction of the equidistant Radon transform, and it is known to be invertible on some large classes of functions. The admissible problem [12, 14] is about finding a set equidistants such that the restriction of the equidistant Radon transform on this set is injective and is not overdetermined. One of the main results of this paper is that the Funk-type sets of equidistants are admissible. To reach this we establish that every equidistant is the intersection of the quadratic hypersurface $\mathcal{K}_\kappa$ modeling the space of constant curvature $\kappa = 0, \pm 1$. Then after introducing the slice transform, which is the integration over the intersections of hyperplanes with the quadratic hypersurface $\mathcal{K}_\kappa$, we prove intertwining relations between the slice transform and the classical Euclidean Radon transform. Having a fix point P on the rotational axis of $\mathcal{K}_\kappa$, the shifted Funk transform F_κ^P is the integration over the hyperplane slices of $\mathcal{K}_\kappa$ that pass P. We prove numerous sharp support theorems and several descriptions of the kernels for shifted Funk transforms that for the case of $\kappa = 1$ not only summarize and often sharpen several other recent results in [3, 4, 6, 7, 13, 15, 16, 23, 25, 27, 28, 33] and not so recent results in [1, 13, 15], but also bring to light new results for the cases $\kappa = 0, -1$. Applying these to the Riemannian spaces of constant curvature Funk-type equidistant and Funk-type isodistant Radon transforms appear for which we prove sharp support theorems and also describe kernels. These considerably generalize the presenter's earlier results in [19].