The shadow picture problem
for parallel straight lines

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Abstract. We prove that convex bodies can be distinguished by their visual angles given on any two straight lines.

1. Introduction

The shadow picture [3] of a compact convex set $B$ is defined at each point $P \in \mathbb{R}^2 \setminus B$ as the angle measure $\nu_B(P)$ of the so-called visual angle that $B$ subtends at $P$. The point $P$ and the set $B$ are usually called the source and the object of the shadow picture $\nu_B(P)$, respectively. The shadow picture is often called visual angle or point projection [1].

The central problem of the subject is to show such set $S$ of sources and set $O$ of objects that $O \ni B \mapsto \nu_B|_S$ is injective. See [2], [3], [4], [5] for example only.

In this short note an earlier result of the author [5, Theorem 1], is completed.

2. Shadow pictures on two straight lines distinguish convex bodies

Let us parametrize the Grassman manifold $\mathcal{L}$ of the straight lines in the plane so that for the real number $r \in \mathbb{R}$ and unit vector $w \in S^1$ the straight line $l(r, w)$ is the one through $rw$ that is perpendicular to $w$. We shall use the notations $w_\beta = (\cos \beta, \sin \beta)$ and $\ell(P, \beta) = l(\langle P, w_\beta \rangle, w_\beta)$ regularly, where $\langle \cdot, \cdot \rangle$ is the usual inner product.

Given a planar compact domain $\mathcal{D}$, we define $\bar{\mathcal{D}} \subseteq \mathcal{L}$ as the domain of those straight lines $\ell$ that intersect $\mathcal{D}$ and have exactly two tangents of $\mathcal{D}$ parallel to $\ell$.

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Using this correspondence we can extend the meaning of the shadow pictures onto the planar compact domains by

\[ S^D: \mathbb{R}^2 \rightarrow \mathbb{R} \quad S^D(P) = \int_{-\pi/2}^{\pi/2} \chi_{\bar{D}}(\ell(P, \beta)) d\beta, \]

where \( \chi_{\bar{D}} \) is the indicator function of \( \bar{D} \). It is easily proved at the end of the proof of [5, Theorem 1] that

\[ \mu(\bar{D}) := \int_{\bar{D}} \frac{dw dr}{|\langle w, w_0 \rangle|} = \int_{\mathbb{R}} S^D((x, 0)) dx \]

if the domain \( D \) does not intersect the \( x \)-axis. Observe that \( \mu \) becomes infinite for those domains that have non-vanishing width parallel to the \( y \)-axis, in particular, \( \mu(\bar{B}) \) is infinite for the convex bodies \( B \).

**Theorem 2.1.** Let \( B_1 \) and \( B_2 \) be compact strictly convex bodies, and let \( \ell_1 \) and \( \ell_2 \) be straight lines outside of \( B_1 \cup B_2 \). If \( B_1 \) and \( B_2 \) sub tend equal shadow pictures at each point of \( \ell_1 \cup \ell_2 \), then they coincide.

**Proof.** For intersecting straight lines the statement is proved by [4, Theorem 1], hence we can assume that the straight lines are parallel and their indices are chosen so that \( \ell_1 \) is closer to \( B_1 \cup B_2 \). Neither of the convex bodies \( B_1 \) and \( B_2 \) can contain the other, therefore they have common tangents, i.e. straight lines that are tangent to both of them.

If a common tangent of \( B_1 \) and \( B_2 \) intersects the straight lines \( \ell_1 \parallel \ell_2 \), then the statement is proved by [5, Theorem 1].

Assume that each common tangent of \( B_1 \) and \( B_2 \) is parallel to \( \ell_1 \parallel \ell_2 \).

Let \( b \) be that one of the common tangents of \( B_1 \) and \( B_2 \) that is farthest from \( \ell_1 \). As \( B_1 \) and \( B_2 \) are strictly convex, we have their unique points \( B_i = B_i \cap b \) \((i = 1, 2)\). The other unique tangents of the domains \( B_1 \) and \( B_2 \) that are parallel to \( b \) coincide because otherwise there were a common tangent of the domains \( B_1 \) and \( B_2 \) intersecting \( b \). Let this unique tangent be \( a \), and let \( A_i = B_i \cap a \) \((i = 1, 2)\).

Choose a coordinate system so that \( \ell_1 \) is the \( x \)-axis, \( B_1 \cup B_2 \) is in the quadrant \( x, y > 0 \) of the plane and the \( x \)-coordinate of \( A_1 B_1 \) is nonnegative. Let \( y = s_a \), \( y = s_b \) and \( y = c_j \) be the equation of \( a, b \) and \( \ell_j \) \((j = 1, 2)\), respectively, where \( c_2 < 0 = c_1 < s_a < s_b \) by the above considerations.

Through any point \( P \) there are exactly two tangent straight lines \( a_i^P \) and \( b_i^P \) to \( B_i \) \((i = 1, 2)\). Let \( \alpha_i^P \) and \( \beta_i^P \) denote the angles of \( a_i^P \) and \( b_i^P \), respectively, to the positive ray of the \( x \)-axis. Assume that the notation are chosen so that

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$\alpha_i^p > \beta_i^p > 0$, hence $S^{B_i}(P) = \alpha_i^p - \beta_i^p$, and define the points $A_i^p = a_i^p \cap B_i$ and $B_i^p = b_i^p \cap B_i$ for $i = 1, 2$. For any point $P \in \ell_1 \cup \ell_2$ we clearly have $A_i^p PA_j^p \angle = B_i^p PB_j^p \angle$, $B_i^p PB_j^p \angle \neq 0 \neq A_i^p PA_j^p \angle$, and
\[
\tau(A_i^p PA_j^p \angle) = \frac{1}{2}|PA_i^p||PA_j^p| \sin A_i^p PA_j^p \angle, \\
\tau(B_i^p PB_j^p \angle) = \frac{1}{2}|PB_i^p||PB_j^p| \sin B_i^p PB_j^p \angle
\]
for the areas of the triangles.

Assume $B_1 \neq B_2$. Then
\[
1 = \lim_{P \to \infty} \frac{\tau(A_1^p PA_2^p \angle)}{\tau(B_1^p PB_2^p \angle)} = \lim_{P \to \infty} \frac{\tau(A_1^p PA_1^p \angle) + \tau(A_1^p PA_2^p \angle) + \tau(A_2^p PA_2^p \angle)}{\tau(B_1^p PB_1^p \angle) + \tau(B_1^p PB_2^p \angle) + \tau(B_2^p PB_2^p \angle)} = \lim_{P \to \infty} \frac{\tau(A_1^p PA_2^p \angle)}{\tau(B_2^p PB_2^p \angle)} = \frac{|A_1^p A_2^p|}{|B_1^p B_2^p|}(s_a - c_j)
\]
for both $j = 1, 2$. This contradicts $c_2 < c_1 = 0$, therefore $B_1 \equiv B_2$ that implies $A_1 \equiv A_2$.

Thus, $B_1$ and $B_2$ have exactly two common tangents $a$ and $b$, $A = a \cap B_1 = a \cap B_2$ and $B = b \cap B_1 = b \cap B_2$. This implies that the boundaries $\mathcal{C}_i = \partial B_i$ ($i = 1, 2$) intersect each other only in the points $A$ and $B$, because otherwise $B_1$ and $B_2$ will have common tangent intersecting $\ell_1 \parallel \ell_2$.

Let $B = B_1 \setminus B_2$ and $C = B_2 \setminus B_1$, and define $\phi_j: \overline{C} \cup \overline{D} \to \overline{C} \cup \overline{D}$ so that $\phi_j(\ell)$ is the straight line through the point $P = \ell \cap \ell_j$ that closes angle $S^{B_i}(P)$ with $\ell$. It is easy to prove, that $\phi_j(\overline{C}) = \overline{D}$ and $\phi_j(\overline{D}) = \overline{C}$, and it is easily proved at the end of the proof of [5, Theorem 1] that $\phi_j (j = 1, 2)$ preserves the measure $\mu$.

Let $\xi_j: \overline{C} \cup \overline{D} \to \mathbb{R}$ map the straight lines $\ell$ to the $x$-coordinate $\ell_j$ of the point $\ell \cap \ell_j$. Choose the real number $x_1$ so that $5\pi/6 < \beta_1((x, 0))$ for all $x > x_1$, and observe that then we also have $S^{B_i}((x, 0)) < \pi/6$ for all $x > x_1$ and $i \in \{1, 2\}$. Let $L_1 = \{ \ell \in \overline{D} : x_1 < \xi_1(\ell) \}$ and construct the sets $L'_k = \phi_1(L_k)$ and $L_{k+1} = \phi_2(L'_k)$ for every $k \in \mathbb{N}$. Clearly, $L_k \subseteq \overline{D}$ and $L'_k \subseteq \overline{C}$.

Let $x' = \xi_2(\phi_1(\ell((x, 0), \beta)))$ and $x'' = \xi_1(\phi_2(\phi_1(\ell((x, 0), \beta))))$ as on Figure 1.

Figure 1. Operation of $\phi_1$ and $\phi_2$ results $x'' = \xi_1(\phi_2(\phi_1(\ell))) > \xi_1(\ell) = x$

We have here $S^{B_i}((x', 0)) < S^{B_i}((x, 0)) < \pi/6$, and the strictly increasing sequence $x_k := \inf_{\ell \in L_k} \xi_1(\ell)$, and therefore the strictly shrinking sequence $\mathcal{L}_k$.
It is shown at the end in the proof of [5, Theorem 1] that $\mu(\bar{B}_1 \triangle \bar{B}_2)$ is finite, therefore $\mu(\mathcal{L}_k)$ is finite for every $k \in \mathbb{N}$. Further, as $\phi_j$ $(j = 1, 2)$ preserves the measure $\mu$, we have $\mu(\mathcal{L}_1) = \cdots = \mu(\mathcal{L}_k) = \cdots$ also. These imply
\[
\mu(\{\ell \in \bar{D} : x_k \leq \xi_1(\ell) \leq x_{k+1}\}) \leq \mu(\mathcal{L}_k \setminus \mathcal{L}_{k+1}) = \mu(\mathcal{L}_k) - \mu(\mathcal{L}_{k+1}) = 0,
\]
hence the set $\{\ell \in \bar{D} : x_k \leq \xi_1(\ell) \leq x_{k+1}\}$ is empty. This contradicts the fact that $x_k < x_{k+1}$, therefore our assumption can not be valid.

The theorem is proved.

As a corollary one can remove the extra convexity condition in [5, Theorem 4].

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References


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