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The shadow picture problem for parallel straight lines

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Abstract. We prove that convex bodies can be distinguished by their visual angles given on any two straight lines.

1. Introduction

The *shadow picture* [3] of a compact convex set $\mathcal{B}$ is defined at each point $P \in \mathbb{R}^2 \setminus \mathcal{B}$ as the angle measure $\nu_{\mathcal{B}}(P)$ of the so called *visual angle* that $\mathcal{B}$ subtends at P. The point P and the set $\mathcal{B}$ are usually called the *source* and the *object* of shadow picture $\nu_{\mathcal{B}}(P)$, respectively. The shadow picture is often called *visual angle* or *point projection* [1].

The central problem of the subject is to show such set $\mathcal{S}$ of sources and set $\mathcal{O}$ of objects that $\mathcal{O} \ni \mathcal{B} \mapsto \nu_{\mathcal{B}}\big|_{\mathcal{S}}$ is injective. See [2], [3], [4], [5] for example only.

In this short note an earlier result of the author [5, Theorem 1], is completed.

2. Shadow pictures on two straight lines distinguish convex bodies

Let us parametrize the Grassman manifold $\mathcal{L}$ of the straight lines in the plane so that for the real number $r \in \mathbb{R}$ and unit vector $\mathbf{w} \in \mathbb{S}^1$ the straight line $l(r, \mathbf{w})$ is the one through $r\mathbf{w}$ that is perpendicular to $\mathbf{w}$. We shall use the notations $\mathbf{w}_\beta = (\cos\beta, \sin\beta)$ and $\ell(P, \beta) = l(\langle P, \mathbf{w}_\beta\rangle, \mathbf{w}_\beta)$ regularly, where $\langle\cdot,\cdot\rangle$ is the usual inner product.

Given a planar compact domain $\mathcal{D}$, we define $\bar{\mathcal{D}} \subseteq \mathcal{L}$ as the domain of those straight lines ℓ that intersect $\mathcal{D}$ and have exactly two tangents of $\mathcal{D}$ parallel to ℓ.

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Using this correspondence we can extend the meaning of the *shadow pictures* onto the planar compact domains by

$$S^{\mathcal{D}}: \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad S^{\mathcal{D}}(P) = \int_{-\pi/2}^{\pi/2} \chi_{\bar{\mathcal{D}}}(\ell(P,\beta))d\beta,$$

where $\chi_{\bar{\mathcal{D}}}$ is the indicator function of $\bar{\mathcal{D}}$. It is easily proved at the end of the proof of [5, Theorem 1] that

$$\mu(\bar{\mathcal{D}}) := \int_{\bar{\mathcal{D}}} \frac{d\mathbf{w}dr}{|\langle \mathbf{w}, \mathbf{w}_0\rangle|} = \int_{\mathbb{R}} S^{\mathcal{D}}((x,0))dx \tag{2.1}$$

if the domain $\mathcal{D}$ does not intersect the x-axis. Observe that μ becomes infinite for those domains that have non-vanishing width parallel to the y-axis, in particular, $\mu(\bar{\mathcal{B}})$ is infinite for the convex bodies $\mathcal{B}$.

Theorem 2.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be compact strictly convex bodies, and let ℓ_1 and ℓ_2 be straight lines outside of $\mathcal{B}_1 \cup \mathcal{B}_2$. If $\mathcal{B}_1$ and $\mathcal{B}_2$ subtend equal shadow pictures at each point of $\ell_1 \cup \ell_2$, then they coincide.*

Proof. For intersecting straight lines the statement is proved by [4, Theorem 1], hence we can assume that the straight lines are parallel and their indices are chosen so that ℓ_1 is closer to $\mathcal{B}_1 \cup \mathcal{B}_2$. Neither of the convex bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ can contain the other, therefore they have common tangents, i.e. straight lines that are tangent to both of them.

If a common tangent of $\mathcal{B}_1$ and $\mathcal{B}_2$ intersects the straight lines $\ell_1 \parallel \ell_2$, then the statement is proved by [5, Theorem 1].

Assume that each common tangent of $\mathcal{B}_1$ and $\mathcal{B}_2$ is parallel to $\ell_1 \parallel \ell_2$.

Let b be that one of the common tangents of $\mathcal{B}_1$ and $\mathcal{B}_2$ that is farthest from ℓ_1. As $\mathcal{B}_1$ and $\mathcal{B}_2$ are strictly convex, we have their unique points $B_i = \mathcal{B}_i \cap b$ $(i = 1, 2)$. The other unique tangents of the domains $\mathcal{B}_1$ and $\mathcal{B}_2$ that are parallel to b coincide because otherwise there were a common tangent of the domains $\mathcal{B}_1$ and $\mathcal{B}_2$ intersecting b. Let this unique tangent be a, and let $A_i = \mathcal{B}_i \cap a$ $(i = 1, 2)$.

Choose a coordinate system so that ℓ_1 is the x-axis, $\mathcal{B}_1 \cup \mathcal{B}_2$ is in the quadrant $x, y > 0$ of the plane and the x-coordinate of A_1B_1 is nonnegative. Let $y = s_a$, $y = s_b$ and $y = c_j$ be the equation of a, b and ℓ_j $(j = 1, 2)$, respectively, where $c_2 < 0 = c_1 < s_a < s_b$ by the above considerations.

Through any point P there are exactly two tangent straight lines a_i^P and b_i^P to $\mathcal{B}_i$ $(i = 1, 2)$. Let α_i^P and β_i^P denote the angles of a_i^P and b_i^P, respectively, to the positive ray of the x-axis. Assume that the notation are chosen so that

$\alpha_i^P > \beta_i^P > 0$, hence $S^{\mathcal{B}_i}(P) = \alpha_i^P - \beta_i^P$, and define the points $A_i^P = a_i^P \cap \mathcal{B}_i$ and $B_i^P = b_i^P \cap \mathcal{B}_i$ for $i = 1, 2$. For any point $P \in \ell_1 \cup \ell_2$ we clearly have $A_1^P P A_2^P \angle = B_1^P P B_2^P \angle$, $B_1^P P B_2^P \angle \neq 0 \neq A_1^P P A_2^P \angle$, and

$$\tau(A_1^P P A_2^P \triangle) = \frac{1}{2}|PA_1^P||PA_2^P| \sin A_1^P P A_2^P \angle,$$
$$\tau(B_1^P P B_2^P \triangle) = \frac{1}{2}|PB_1^P||PB_2^P| \sin B_1^P P B_2^P \angle$$

for the areas of the triangles.

Assume $B_1 \neq B_2$. Then

$$1 = \lim_{P\to\infty} \frac{\tau(A_1^P P A_2^P \triangle)}{\tau(B_1^P P B_2^P \triangle)} = \lim_{P\to\infty} \frac{\tau(A_1^P P A_1 \triangle) + \tau(A_1 P A_2 \triangle) + \tau(A_2 P A_1^P \triangle)}{\tau(B_1^P P B_1 \triangle) + \tau(B_1 P B_2 \triangle) + \tau(B_2 P B_1^P \triangle)}$$
$$= \lim_{P\to\infty} \frac{\tau(A_1 P A_2 \triangle)}{\tau(B_1 P B_2 \triangle)} = \frac{|A_1A_2|(s_a - c_j)}{|B_1B_2|(s_b - c_j)}$$

for both $j = 1, 2$. This contradicts $c_2 < c_1 = 0$, therefore $B_1 \equiv B_2$ that implies $A_1 \equiv A_2$.

Thus, $\mathcal{B}_1$ and $\mathcal{B}_2$ have exactly two common tangents a and b, $A = a \cap \mathcal{B}_1 = a \cap \mathcal{B}_2$ and $B = b \cap \mathcal{B}_1 = b \cap \mathcal{B}_2$. This implies that the boundaries $\mathcal{C}_i = \partial\mathcal{B}_i$ $(i = 1, 2)$ intersect each other only in the points A and B, because otherwise $\mathcal{B}_1$ and $\mathcal{B}_2$ will have common tangent intersecting $\ell_1 \parallel \ell_2$.

Let $\mathcal{C} = \mathcal{B}_1 \setminus \mathcal{B}_2$ and $\mathcal{D} = \mathcal{B}_2 \setminus \mathcal{B}_1$, and define $\phi_j : \bar{\mathcal{C}} \cup \bar{\mathcal{D}} \to \bar{\mathcal{C}} \cup \bar{\mathcal{D}}$ so that $\phi_j(\ell)$ is the straight line through the point $P = \ell \cap \ell_j$ that closes angle $S^{\mathcal{B}_i}(P)$ with ℓ. It is easy to prove, that $\phi_j(\bar{\mathcal{C}}) = \bar{\mathcal{D}}$ and $\phi_j(\bar{\mathcal{D}}) = \bar{\mathcal{C}}$, and it is easily proved at the end of the proof of [5, Theorem 1] that ϕ_j $(j = 1, 2)$ preserves the measure μ.

Let $\xi_j : \bar{\mathcal{C}} \cup \bar{\mathcal{D}} \to \mathbb{R}$ map the straight lines ℓ to the x-coordinate $\xi_j(\ell)$ of the point $\ell \cap \ell_j$. Choose the real number x_1 so that $5\pi/6 < \beta_1((x, 0))$ for all $x > x_1$, and observe that then we also have $S^{\mathcal{B}_i}((x, 0)) < \pi/6$ for all $x > x_1$ and $i \in \{1, 2\}$. Let $\mathcal{L}_1 = \{\ell \in \bar{\mathcal{D}} : x_1 < \xi_1(\ell)\}$ and construct the sets $\mathcal{L}'_k = \phi_1(\mathcal{L}_k)$ and $\mathcal{L}_{k+1} = \phi_2(\mathcal{L}'_k)$ for every $k \in \mathbb{N}$. Clearly, $\mathcal{L}_k \subseteq \bar{\mathcal{D}}$ and $\mathcal{L}'_k \subseteq \bar{\mathcal{C}}$.

Let $x' = \xi_2(\phi_1(\ell((x, 0), \beta)))$ and $x'' = \xi_1(\phi_2(\phi_1(\ell((x, 0), \beta))))$ as on Figure 1.

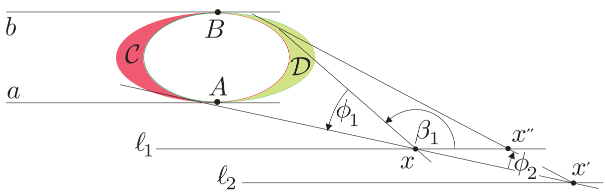

Figure 1. Operation of ϕ_1 and ϕ_2 results $x'' = \xi_1(\phi_2(\phi_1(\ell))) > \xi_1(\ell) = x$

We have here $S^{\mathcal{B}_i}((x', 0)) < S^{\mathcal{B}_i}((x, 0)) < \pi/6$, and the strictly increasing sequence $x_k := \inf_{\ell \in \mathcal{L}_k} \xi_1(\ell)$, and therefore the strictly shrinking sequence $\mathcal{L}_k$.

It is shown at the end in the proof of [5, Theorem 1] that $\mu(\bar{\mathcal{B}}_1 \triangle \bar{\mathcal{B}}_2)$ is finite, therefore $\mu(\mathcal{L}_k)$ is finite for every $k \in \mathbb{N}$. Further, as ϕ_j $(j = 1, 2)$ preserves the measure μ, we have $\mu(\mathcal{L}_1) = \cdots = \mu(\mathcal{L}_k) = \cdots$ also. These imply

$$\mu(\{\ell \in \bar{\mathcal{D}} : x_k \leq \xi_1(\ell) \leq x_{k+1}\}) \leq \mu(\mathcal{L}_k \setminus \mathcal{L}_{k+1}) = \mu(\mathcal{L}_k) - \mu(\mathcal{L}_{k+1}) = 0,$$

hence the set $\{\ell \in \bar{\mathcal{D}} : x_k \leq \xi_1(\ell) \leq x_{k+1}\}$ is empty. This contradicts the fact that $x_k < x_{k+1}$, therefore our assumption can not be valid.

The theorem is proved.

■

As a corollary one can remove the extra convexity condition in [5, Theorem 4].

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