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# The Radon transform on hyperbolic space

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**Abstract.** The Radon transform that integrates a function in $\mathbb{H}^n$, the $n$-dimensional hyperbolic space, over totally geodesic submanifolds with codimension 1 and the dual Radon transform are investigated in this paper. We prove inversion formulas and an inclusion theorem for the range.

## 0. Introduction

In this paper we investigate the Radon transform and its dual on the hyperbolic space. This problem has been introduced in [7] and has also been considered in [7,14]. The most interesting question concerns the inverse of the transform, and due to [7,14] this is known in odd dimension.

If $f \in L^2(\mathbb{H}^n)$, where $\mathbb{H}^n$ is the $n$-dimensional hyperbolic space, the Radon transform of $f$ is a function $Rf$ defined on the set of hyperplanes, the totally geodesic submanifolds with codimension 1. The value of $Rf$ at a given hyperplane is the integral of $f$ over that hyperplane. The dual Radon transform $R^*F$ of a function $F$ defined on the set of hyperplanes is a function on $\mathbb{H}^n$. The value of $R^*F$ at a given point $X$ is the integral of $F$ over the set of hyperplanes passing through $X$ by the surface measure of the unit sphere of the tangent space at $X$ (the normals of the hyperplanes at $X$ project the surface measure of this unit sphere to a measure on the set of the hyperplanes through $X$).

The points of the hyperplanes nearest to an arbitrarily chosen origin define a hypersurface and the 'boomerang transform' integrates a function $f$ defined on $\mathbb{H}^n$ over this hypersurface by the projected measure of the unit sphere in $T_X\mathbb{H}^n$. This means that the boomerang transform is in principle the dual Radon transform, provided $\mathbb{H}^n$ is identified (except at the origin) with the space of its hyperplanes — each one being represented by its closest point to the origin. The above defined hypersurface is thus the set of the points from which the geodesic segment

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joining the origin and the given point is seen at a right angle. For this reason our boomerang transform is an analog of the spherical means transformation of Cormack and Quinto [3].

The paper is outlined as follows.

We produce explicit formulas for the Radon and boomerang transforms on $\mathbb{H}^n$ in Section 1 and invert them in Section 2. These new inversion formulas are obtained in terms of an expansion in spherical harmonics. They give the so-called support theorems as a byproduct. Inversion formulas were known before* only in odd dimensions [8].

Our aim in Section 3 is to prove continuity results and inclusion theorems for the range of the Radon transform and for the null space of the boomerang transform on certain classes of square integrable functions.

In Section 4 we give two new closed inversion formulas valid in odd and even dimension respectively. Our odd dimensional formula exhibit some similarities to Helgason's formula in [8] and the even dimensional one involves a distribution of the Cauchy principal value type. This shows that the inversion formula is local in odd dimensions and global in even dimensions just as in the Euclidean case.

## 1. Calculation of the transforms on $\mathbb{H}^n$

For the calculation we use Poincaré's sphere model of $\mathbb{H}^n$, which is the unit ball $B^n$ in $\mathbb{R}^n(|x| < 1)$ with Riemannian metric $ds^2 = 4dx^2/(1-|x|^2)^2$. This metric has constant curvature $-1$ and the geodesics are circular arcs perpendicular to the boundary of $B^n$ ,$S^{n-1}$, and thus the totally geodesic submanifolds are the spheres intersecting $S^{n-1}$ orthogonally.

Let $\mu\colon B^n \to \mathbb{H}^n$ be the parameterization of $\mathbb{H}^n$ with $B^n$ given by the Poincaré model. Since the Euclidean and hyperbolic metrics are conformally related the Euclidean and hyperbolic geodesic polar coordinates on $B^n$ differ only in the radial coordinates. With $i_B\colon S^{n-1}\times[0,1)\to B^n$ and $i_{\mathbb{H}}\colon S^{n-1}\times\mathbb{R}_+\to\mathbb{H}^n$ denoting the respective (polar) coordinatization this means that $i_{\mathbb{H}}^{-1}\circ\mu\circ i_B(\omega,\rho) = (\omega,\bar{\mu}(\rho))$, where $\bar{\mu}\colon[0,1)\to\mathbb{R}_+$ is given as $\bar{\mu}(\rho)=\ln((1+\rho)/(1-\rho))$, and $\bar{\mu}^{-1}(h)=\operatorname{th}(h/2)$.

In what follows we shall frequently refer to the notation of Figure 1.

On the figure, the point $P$ of the geodesic $MN$ is the nearest to the origin $O$. The geodesic $MN$ is a piece of the circle centered at $O'$ with radius $r$ in $\mathbb{R}^n$. The

* We had been informed by the referee that recent results on the matter will appear in the proceedings of the AMS Arcata conference on integral geometry (June 1989). See [1,9].

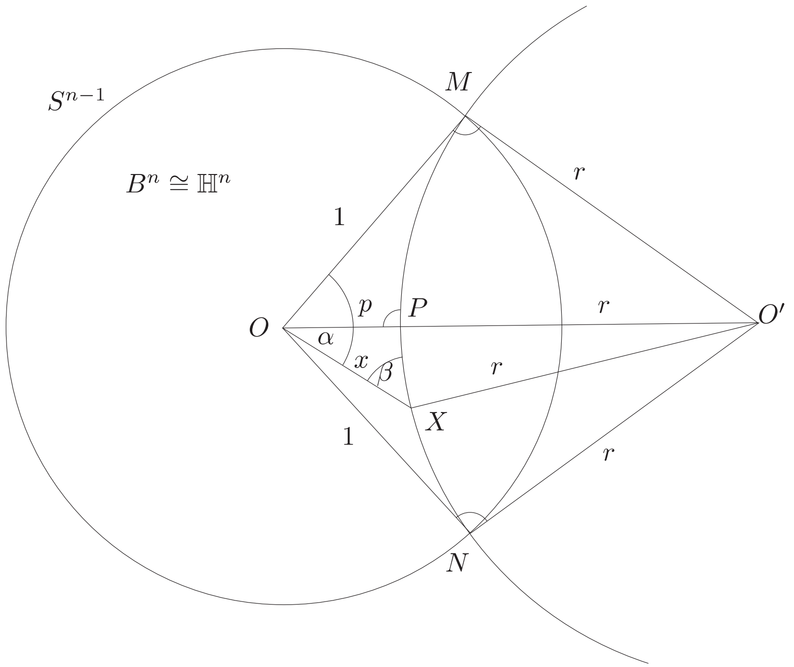


**Figure 1**

Euclidean distance of $O$ and $P$ (resp. $X$) is $p$ (resp. $x$). Here $X$ is a point on the geodesic $MN$ and $\alpha$ (resp. $\beta$) is the angle between $OX$ and $OP$ (resp. $XP$).

We start by noting the formulas

$$p = \frac{x^2+1}{2x\cos\alpha} - \sqrt{\left(\frac{x^2+1}{2x\cos\alpha}\right)^2 - 1} \tag{1.1}$$

and

$$x = \sqrt{r^2+1}\cos\alpha - \sqrt{(r^2+1)\cos^2\alpha - 1}, \tag{1.2}$$

which derive from the law of cosine applied to the Euclidean triangles $ONO'$ and $OXO'$. Let $\delta = d(P,X)$ denote the hyperbolic distance of the points $P$ and $X$. Then by the Riemannian metric $ds^2$ we have

$$\delta = \left|\ln\left(\frac{PM}{PN}\Big/\frac{XM}{XN}\right)\right|$$

and a straightforward calculation yields

$$\delta = \frac{1}{2}\left|\ln\left(\frac{r\cos\alpha + \sin\alpha}{r\cos\alpha - \sin\alpha}\right)\right|, \tag{1.3}$$

 

where, as one can easily read off from Figure 1, $r = (1/p - p)/2$. Furthermore, the law of cosine applied to the triangle $O'XO$ gives

$$\sin\beta = \frac{1 - x^2}{\sqrt{\left(\frac{x^2+1}{\cos\alpha}\right)^2 - 4x^2}}. \tag{1.4}$$

To get explicit formulas for the Radon and boomerang transforms, we parameterize the set of totally geodesic submanifolds of $\mathbb{H}^n$ by $S^{n-1} \times \mathbb{R}_+$, in such a way that $\xi(\omega, h)$ denotes the totally geodesic submanifold perpendicular to the geodesic passing through the origin with tangent vector $\omega$ at distance $h$ from the origin.

**Lemma 1.1.** *For $f \in L^2(\mathbb{H}^n)$, the Radon transform is*

$$Rf(\bar\omega, h) = \int_{S^{n-1}_{h,\omega}} f\Big(\omega, \frac{1}{2}\bar\mu\Big(\frac{\operatorname{th} h}{\langle\omega,\bar\omega\rangle}\Big)\Big) \frac{(\langle\omega,\bar\omega\rangle^2 \operatorname{cth}^2 h - 1)^{-n/2}}{\operatorname{sh} h} d\omega,$$

*where $d\omega$ is the surface measure of $S^{n-1}$, $\langle .,.\rangle$ is the standard Euclidean scalar product and $S^{n-1}_{h,\bar\omega} = \{\omega \in S^{n-1} \colon \operatorname{th} h < \langle\omega,\bar\omega\rangle\}$.*

(Here and below we use the shorthand $Rf(\bar\omega, h) = (Rf)(\xi(\bar\omega, h))$).

**Proof.** Let $(\bar\omega, h)$ be the geodesic polar coordinates of $P \in \mathbb{H}^n (p = \operatorname{th}(h/2))$ and let $X$ be a point of $\xi(\bar\omega, h)$ in direction $\omega$. (See Figure 1.)

First we assume that $n = 2$ and parameterize $S^1$ by an angle $\alpha$, with respect to some fixed direction. According to Figure 1, the point $X$ of $\xi(\bar\omega, h)$ is parameterized by $\alpha \in (-\arccos\frac{1}{p+r}, \arccos\frac{1}{p+r})$ (this comes from the triangle $OO'N$) and we can immediately write that

$$Rf(\bar\omega, h) = \int_{-\arccos(1/(p+r))}^{\arccos(1/(p+r))} f(\alpha + \bar\alpha, \bar\mu(x(\alpha))) \left|\frac{d\delta}{d\alpha}\right| d\alpha,$$

where $\bar\alpha$ is the angle of $\bar\omega$ with respect to the fixed direction and $x(\alpha)$, $\delta(\alpha)$ are given by (1.2), (1.3) for $P$ fixed and $X$ varying with $\alpha$. It is straightforward from (1.3) that

$$Rf(\bar\omega, h) = \int_{1/(p+r)<\cos(\alpha)} f\Big(\alpha + \bar\alpha, \frac{1}{2}\bar\mu\Big(\frac{(r^2+1)^{-1/2}}{\cos\alpha}\Big)\Big) \frac{r}{(r^2+1)\cos^2\alpha - 1} d\alpha.$$

One can now substitute $r = (1/p - p)/2 = (\operatorname{cth}(h/2) - \operatorname{th}(h/2))/2 = 1/\operatorname{sh} h$ to obtain the formula of the lemma.

 

The configuration relevant for the case $n > 2$ can be obtained by rotating Figure 1 around the straight line $OP$. It is well known that the surface element of a geodesic sphere with radius $\rho$ is $\operatorname{sh}^{n-1}\rho d\omega$. The reason for this is that the map $\omega \to i_{\mathbb{H}}(\omega, \rho)$ of $S^{n-1}$ onto the geodesic sphere of radius $\rho$ in $\mathbb{H}^n$ induces a map between their tangent spaces, which is a dilation by $\operatorname{sh}\rho$. This and the fact that $\xi(\bar{\omega}, h)$ is a rotational manifold imply that the surface measure on $\xi(\bar{\omega}, h)$ at the point $X$ reads as $\operatorname{sh}^{n-2}\bar{\mu}(x(\alpha))|d\delta/d\alpha||_{\cos\alpha=\langle\omega,\bar{\omega}\rangle}d\omega$. This tells us that

$$Rf(\bar{\omega}, h) = \int_{1/(p+r)<\langle\omega,\bar{\omega}\rangle} f\Big(\omega, \frac{1}{2}\bar{\mu}\Big(\frac{(r^2+1)^{-1/2}}{\langle\omega,\bar{\omega}\rangle}\Big)\Big)\frac{rsh^{n-2}\big(\frac{1}{2}\bar{\mu}\big(\frac{(r^2+1)^{-1/2}}{\langle\omega,\bar{\omega}\rangle}\big)\big)}{(r^2+1)\langle\omega,\bar{\omega}\rangle^2 - 1}d\omega.$$

Substituting $r = 1/\operatorname{sh} h$ as above we obtain the statement of the lemma.

■

**Lemma 1.2.** *For $f \in L^2(\mathbb{H}^n)$, the boomerang transform is*

$$Bf(\bar{\omega}, h) = \int_{S_0^{n-1},\bar{\omega}} f\Big(\omega, \frac{1}{2}\bar{\mu}(\langle\omega,\bar{\omega}\rangle \operatorname{th} h)\Big)\frac{(1-\langle\omega,\bar{\omega}\rangle^2\operatorname{th}^2 h)^{-1}}{\operatorname{ch} h}d\omega.$$

**Proof.** Let $(\bar{\omega}, h)$ be the geodesic polar coordinates of $X \in \mathbb{H}^n$ ($x = \operatorname{th}(h/2)$). (See Figure 1)

First we assume that $n = 2$ and parameterize $S^1$ by an angle $\alpha$, with respect to some fixed direction. According to Figure 1, the point $P$ is parameterized by $\alpha \in [-\pi/2, \pi/2]$ and we can write immediately that

$$Bf(\bar{\omega}, h) = \int_{-\pi/2}^{\pi/2} f(\alpha + \bar{\alpha}, \bar{\mu}(p(\alpha)))\left|\frac{d\beta}{d\alpha}\right| d\alpha,$$

where $\bar{\alpha}$ is the angle of $\bar{\omega}$ with respect to the fixed direction and $p(\alpha)$, $\beta(\alpha)$ are given by (1.1), (1.4) for $X$ fixed and $P$ varying with $\alpha$. It is immediate from (1.4) that

$$Bf(\bar{\omega}, h) = \int_{-\pi/2}^{\pi/2} f\Big(\alpha + \bar{\alpha}, \frac{1}{2}\bar{\mu}\Big(\frac{2x}{x^2+1}\cos\alpha\Big)\Big)\frac{1-x^4}{(x^2+1)^2 - 4x^2\cos^2\alpha}d\alpha.$$

One can now substitute $x = \bar{\mu}^{-1}(h) = \operatorname{th}(h/2)$ to obtain the formula of the lemma.

The configuration relevant for the case $n > 2$ can be obtained by rotating Figure 1 around the straight line $OX$. Thus the conformality of the model implies the lemma.

■

 

## 2. Inversion formulas and support theorems

We need the following two technical lemmas that can be easily proven from the formulas given in [2] and [4].

**Lemma 2.1.** *If $m \in \mathbb{Z}$ then $I = \pi/2$, where*

$$I = \int_t^q \frac{\cos(m \arccos(\operatorname{th} h/\operatorname{th} q))}{\sqrt{1-\operatorname{th}^2 h/\operatorname{th}^2 q}} \times \frac{\operatorname{ch}(m \operatorname{arcch}(\operatorname{th} h/\operatorname{th} t))}{\sqrt{\operatorname{th}^2 h/\operatorname{th}^2 t - 1}} \frac{dh}{\operatorname{sh} h \operatorname{ch} h}.$$

**Lemma 2.2.** *If $m \in \mathbb{Z}$, $n > 2$, $\lambda = (n-2)/2$ and $C_m^\lambda$ denotes the Gegenbauer polynomials of the first kind, then*

$$M\left(\frac{\operatorname{sh}(q-t)}{\operatorname{sh} q\ \operatorname{sh} t}\right)^{n-2} = \int_t^q \operatorname{cth}^{n-3} h\, C_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right) C_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} q}\right) \times$$

$$\times \left(\frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} - 1\right)^{\frac{n-3}{2}} \left(1 - \frac{\operatorname{th}^2 h}{\operatorname{th}^2 q}\right)^{\frac{n-3}{2}} \frac{dh}{\operatorname{sh}^2 h},$$

*where*

$$M = \pi\ 2^{3-n}\left(\frac{\Gamma(m+n-2)}{\Gamma(m+1)\Gamma(\lambda)}\right)^2 \frac{1}{\Gamma(n-1)}.$$

Now we present two propositions that describe our transformations in terms of spherical harmonics. For this purpose we recall the following facts.

A complete orthonormal system in the Hilbert space $L^2(S^{n-1})$ can be chosen consisting of spherical harmonics $Y_{l,m}$, where $Y_{l,m}$ is of degree $m$. If $Y_{l,m}$ is a member of such a system,$f \in C^\infty(S^{n-1} \times \mathbb{R}_+)$ and $p \in \mathbb{R}_+$ let the corresponding coefficients of the series in this system for $f(\omega, p)$ be $f_{l,m}(p)$. Then the series

$$\sum_{l,m}^{\infty} f_{l,m}(p) Y_{l,m}(\omega)$$

converges uniformly absolutely on compact subsets of $S^{n-1} \times \mathbb{R}_+$ to $f(\omega, p)$ [13]. Below we use the expansions

$$f(\varphi, q) = \sum_{m=-\infty}^{\infty} f_m(q) \exp(im\varphi) \quad \text{and} \quad f(\omega, q) = \sum_{l,m}^{\infty} f_{l,m}(q) Y_{l,m}(\omega)$$

in dimension 2 and in higher dimensions, respectively. These expansions will be used for the Radon and boomerang transforms $Rf$ and $Bf$ as well.

 

**Proposition 2.3.** i) *If* $f(\varphi,p)\in L^2(\mathbb{H}^2)$ *then*

$$(R)\qquad (Rf)_m(h)=2\int_h^\infty f_m(q)\frac{\cos(m\arccos(\operatorname{th}h/\operatorname{th}q))}{\operatorname{ch}h\sqrt{1-\operatorname{th}^2h/\operatorname{th}^2q}}dq.$$

ii) *If* $n>2$, $f(\omega,p)\in L^2(\mathbb{H}^n)$ *and* $\lambda=(n-2)/2$ *then*

$$(RN)\qquad (Rf)_{l,m}(h)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_h^\infty f_{l,m}(q)C_m^\lambda\left(\frac{\operatorname{th}h}{\operatorname{th}q}\right)\left(1-\frac{\operatorname{th}^2h}{\operatorname{th}^2q}\right)^{\frac{n-3}{2}}\frac{\operatorname{sh}^{n-2}q}{\operatorname{ch}h}dq,$$

*where* $|S^k|$ *is the area of* $S^k$ *and* $C_m^\lambda$ *is the Gegenbauer polynomial.*

**Proof.** If $n=2$ substituting the expansions of $f$ and $Rf$ into the formula of Lemma 1.1 we get

$$(Rf)_m(h)=\int_{\operatorname{th}h<\cos\alpha} f_m\Big(\frac{1}{2}\bar\mu\Big(\frac{\operatorname{th}h}{\cos\alpha}\Big)\Big)\frac{1/\operatorname{sh}h}{\operatorname{cth}^2h\ \cos^2\alpha-1}\exp(im\alpha)d\alpha.$$

The factor $\exp(im\alpha)$ can be replaced by its real part, because the domain of integration is symmetric and the remainder of the integrand is an even function of $\alpha$. Putting now $q=\frac{1}{2}\bar\mu(\frac{\operatorname{th}h}{\cos\alpha})$ $(\cos\alpha=\operatorname{th}h/\operatorname{th}q)$ this becomes (R).

For $n>2$ write the expansions of $f$ and $Rf$ into the formula of Lemma 1.1 and use the Funk-Hecke theorem [13] to obtain

$$(Rf)_{l,m}(h)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_{\operatorname{th}h}^1 f_{l,m}\Big(\frac{1}{2}\bar\mu\Big(\frac{\operatorname{th}h}{t}\Big)\Big)\times$$
$$\times\, C_m^\lambda(t)\left(1-t^2\right)^{\frac{n-3}{2}}\frac{(t^2\operatorname{cth}^2h-1)^{-n/2}}{\operatorname{sh}h}dt.$$

Putting $q=\frac{1}{2}\bar\mu(\frac{\operatorname{th}h}{t})$ $(t=\operatorname{th}h/\operatorname{th}q)$ this becomes (RN) which was to be proved. ■

**Proposition 2.4.** i) *If* $f(\varphi,p)\in L^2(\mathbb{H}^2)$ *then*

$$(B)\qquad (Bf)_m(h)=2\int_0^h f_m(q)\frac{\cos(m\arccos(\operatorname{th}q/\operatorname{th}h))}{\operatorname{sh}h\sqrt{1-\operatorname{th}^2q/\operatorname{th}^2h}}dq.$$

ii) *If* $n>2$, $f(\omega,p)\in L^2(\mathbb{H}^n)$ *and* $\lambda=(n-2)/2$ *then*

$$(BN)\qquad (Bf)_{l,m}(h)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_0^h f_{l,m}(q)C_m^\lambda\left(\frac{\operatorname{th}q}{\operatorname{th}h}\right)\left(1-\frac{\operatorname{th}^2q}{\operatorname{th}^2h}\right)^{\frac{n-3}{2}}\frac{1}{\operatorname{sh}h}dq.$$

This proposition can be proven from Lemma 1.2 in the same way as the previous one so we leave the proof to the reader. Our following theorems give the inversion formulas. Since their proofs are very similar we only give the first proof.

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**Theorem 2.5.** i) *If* $f \in C_c^\infty(\mathbb{H}^2) \subset C_c^\infty(S^1 \times \mathbb{R}_+)$ *then*

$$\text{(RI)} \qquad f_m(t) = \frac{-1}{\pi}\frac{d}{dt}\int_t^\infty (Rf)_m(h)\frac{\operatorname{ch}(m \operatorname{arcch}(\operatorname{th} h/\operatorname{th} t))}{\operatorname{sh} h\sqrt{\operatorname{th}^2 h/\operatorname{th}^2 t - 1}}dh.$$

ii) *If* $n > 2$, $f \in C_c^\infty(\mathbb{H}^n) \subset C_c^\infty(S^{n-1} \times \mathbb{R}_+)$ *then*

$$\text{(RNI)} \qquad f_{l,m}(t) = (-1)^{n-1}\frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)}\begin{cases} \frac{d}{dt}\delta_2\delta_4\dots\delta_{n-2}F(t) & \text{if } n \text{ even} \\ \delta_1\delta_3\delta_5\dots\delta_{n-2}F(t) & \text{if } n \text{ odd,}\end{cases}$$

*where* $\delta_k = \frac{d^2}{dt^2} - k^2$ $(k \in \mathbb{N})$ *and*

$$F(t) = \int_t^\infty (Rf)_{l,m}(h)C_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\left(\frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} - 1\right)^{\frac{n-3}{2}}\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h, dh$$

**Theorem 2.6.** i) *If* $f \in C^\infty(\mathbb{H}^2) \subset C^\infty(S^1 \times \mathbb{R}_+)$ *then*

$$\text{(BI)} \qquad f_m(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t (Bf)_m(h)\frac{\operatorname{ch}(m \operatorname{arcch}(\operatorname{th} t/\operatorname{th} h))}{\operatorname{ch} h\sqrt{\operatorname{th}^2 t/\operatorname{th}^2 h - 1}}dh.$$

ii) *If* $n > 2$, $f \in C^\infty(\mathbb{H}^n) \subset C^\infty(S^{n-1} \times \mathbb{R}_+)$ *then*

$$\text{(BNI)} \qquad f_{l,m}(t) = \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)}\operatorname{ch}^{n-2} t\begin{cases} \frac{d}{dt}\delta_2\delta_4\dots\delta_{n-2}F(t) & \text{if } n \text{ even} \\ \delta_1\delta_3\delta_5\dots\delta_{n-2}F(t) & \text{if } n \text{ odd,}\end{cases}$$

*where*

$$F(t) = \int_0^t (Bf)_{l,m}(h)C_m^\lambda\left(\frac{\operatorname{th} t}{\operatorname{th} h}\right)\left(\frac{\operatorname{th}^2 t}{\operatorname{th}^2 h} - 1\right)^{\frac{n-3}{2}}\frac{\operatorname{ch}^{n-2} t}{\operatorname{ch} h}\operatorname{th}^{n-2} h\, dh.$$

**Proof.** To prove (RI) one multiplies (R) by

$$\operatorname{ch}(m \operatorname{arcch}(\operatorname{th} h/\operatorname{th} t))/\ \operatorname{sh} h\sqrt{\operatorname{th}^2 h/\operatorname{th}^2 t - 1}$$

and integrates from $t$ to infinity. Then one simplifies the integral by using Lemma 2.1 and finally differentiates with respect to $t$.

To prove (RNI) one multiplies (RN) by

$$C_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\left(\frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} - 1\right)^{\frac{n-3}{2}}\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h$$

 

and integrates from $t$ to infinity again. Then Lemma 2.2 leads to

$$F(t) = M\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_t^\infty f_{l,m}(q)\,\mathrm{sh}^{n-2}(q-t)dq.$$

To finish the proof it is enough to observe that

$$\frac{d^2}{dt^2}\,\mathrm{sh}^k(q-t) = k^2\,\mathrm{sh}^k(q-t) + k(k-1)\,\mathrm{sh}^{k-2}(q-t). \qquad \blacksquare$$

The following two corollaries are direct consequences of the above theorems. The first one has a stronger version in [8,Theorem III.1.2].

**Corollary 2.7.** *If $f \in C_c^\infty(\mathbb{H}^n)$ and $A > 0$ then the values of $Rf(\omega,p)$ for $p \geq A$ determine $f(\omega,p)$ for $p \geq A$. If $Rf(\omega,p) = 0$ on this domain, then $f(\omega,p) = 0$ too.*

**Corollary 2.8.** *If $f \in C^\infty(\mathbb{H}^n)$ and $A > 0$ then the values of $Bf(\omega,p)$ for $0 \leq p \leq A$ determine $f(\omega,p)$ for $0 \leq p \leq A$. If $Bf(\omega,p) = 0$ on this domain, then $f(\omega,p) = 0$ too.*

## 3. Null spaces and ranges

Our first proposition in this section establishes the continuity of the Radon and boomerang transforms. Let $\psi(\omega,p)$ denote the hypersurface from the points of which the geodesic segment joining $O$ and $i_{\mathbb{H}}(\omega,p)$ is seen at a right angle. To calculate the boomerang transform at the point $i_{\mathbb{H}}(\omega,p)$ one needs only integrate on $\psi(\omega,p)$.

**Proposition 3.1.** *Let $\mathbb{S}$ be a measurable set in $\mathbb{H}^n$ and $n \geq 3$. The maps $R\colon L^2(\mathbb{S}, \mathrm{sh}^{n-1}\delta_x dx) \to L^2(\mathbb{S}^R)$ and $B\colon L^2(\mathbb{S}, \mathrm{sh}^{1-n}\delta_x dx) \to L^2(\mathbb{S}^B)$ are continuous, where*

$$\mathbb{S}^R = \{(\omega,p) \in S^{n-1}\times\mathbb{R}_+\colon \xi(\omega,p)\cap\mathbb{S} \neq \emptyset\},$$
$$\mathbb{S}^B = \{(\omega,p) \in S^{n-1}\times\mathbb{R}_+\colon \psi(\omega,p)\cap\mathbb{S} \neq \emptyset\},$$

*$\delta_x$ is the distance of $x$ from the origin and $dx$ is the Lebesgue measure on $\mathbb{H}^n$.*

**Proof.** We can assume $\mathbb{S} = \mathbb{H}^n$ without loss of generality. By the orthogonality of the spherical harmonics $Y_{l,m}$ it is enough to prove that for $f(\omega, p) = g(p)Y_{l,m}(\omega)$

$$\|Rf\|^2_{L^2(S^{n-1}\times\mathbb{R}_+)} \le 16|S^{n-2}|^2\|f\|^2_{L^2(\mathbb{H}^n,\operatorname{sh}^{n-1}\delta_x dx)}.$$

Using (RN) we see that

$$\begin{aligned}\|Rf\|^2_{L^2(S^{n-1}\times\mathbb{R}_+)} &= 4\|(Rf)_{l,m}\|^2_{L^2(\mathbb{R}_+)}\\ &= \frac{4|S^{n-2}|^2}{(C^\lambda_m(1))^2}\times\int_0^\infty \frac{1}{\operatorname{ch}^2 h}\Big(\int_h^\infty g(q)C^\lambda_m\Big(\frac{\operatorname{th} h}{\operatorname{th} q}\Big)\Big(1-\frac{\operatorname{th}^2 h}{\operatorname{th}^2 q}\Big)^{\frac{n-3}{2}}\operatorname{sh}^{n-2} q\, dq\Big)^2 dh.\end{aligned}$$

Since $|C^\lambda_m(t)| \le |C^\lambda_m(1)|$ for $t \in [0,1]$, $\operatorname{ch}^2 h \ge 1$ and $|1-\operatorname{th}^2 h/\operatorname{th}^2 q| \le 1$ we get

$$\begin{aligned}\|Rf\|^2_{L^2(S^{n-1}\times\mathbb{R}_+)} &\le 4|S^{n-2}|^2\int_0^\infty\left(\int_h^\infty |g(q)|\operatorname{sh}^{n-2} q\, dq\right)^2 dh\\ &= 4|S^{n-2}|^2\int_0^\infty\left(\frac{1}{h}\int_0^h \left|g\left(\frac{1}{q}\right)\right|\operatorname{sh}^{n-2}\left(\frac{1}{q}\right)\frac{dq}{q^2}\right)^2 dh.\end{aligned}$$

At the same time Hardy's inequality,

$$\left\|\frac{1}{v}\int_0^v k(u)du\right\|_{L^2(\mathbb{R}_+)} \le 2\|k\|_{L^2(\mathbb{R}_+)},$$

gives

$$\|Rf\|^2_{L^2(S^{n-1}\times\mathbb{R}_+)} \le 16|S^{n-2}|^2\left\|g\left(\frac{1}{q}\right)\operatorname{sh}^{n-2}\left(\frac{1}{q}\right)q^{-2}\right\|^2_{L^2(\mathbb{R}_+)}.$$

Putting $p = 1/q$, then using $q \le \operatorname{sh} q$ we find the theorem for $R$. The proof for the boomerang transform is very similar and is left to the reader.

■

**Theorem 3.2.** *Let* $\mathbb{S}$ *be a measurable set in* $\mathbb{H}^n$ *(may be* $\mathbb{H}^n$*),* $n \ge 3$ *and* $g_{j,l,m}(\omega,h) = \frac{\operatorname{th}^j h}{\operatorname{ch} h}Y_{l,m}(\omega)$ *for* $j,l,m \in \mathbb{N}$*, where* $0 \le j < m$ *and* $(m-j)$ *is even. The Radon transform is an injection of* $L^2(\mathbb{S}, \operatorname{sh}^{n-1}\delta_x dx)$ *into*

$$\mathcal{A} = (\operatorname{Cl}\operatorname{Sp}\{g_{j,l,m}(\omega,h)\})^\perp \cap L^2(\mathbb{S}^R),$$

*where* Cl Sp *means the closure of the span of the set of functions indicated.*

**Proof.** $g_{j,l,m} \in L^2(S^{n-1} \times \mathbb{R}_+)$ is proved by the fact that

$$\|g_{j,l,m}\|_{L^2(S^{n-1}\times\mathbb{R}_+)} = 2/\sqrt{2j+1}.$$

Now let

$$f(\omega, q) = v(q)Y_{i,k}(\omega) \in L^2(\mathbb{S}, \operatorname{sh}^{n-1} \delta_x dx)$$

and

$$g^*(\omega, h) = g(\operatorname{th} h)Y_{l,m}(\omega)/\operatorname{ch} h.$$

We get immediately from (RN) by changing the order of integrations that

$$\begin{aligned}\langle g^*, Rf\rangle_{L^2(S^{n-1}\times\mathbb{R}_+)} \\ = \delta_{l,i}\delta_{m,k}\frac{4|S^{n-2}|}{C_m^\lambda(1)}\int_0^\infty v(q)\frac{\operatorname{sh}^{n-1} q}{\operatorname{ch} q}\int_0^1 g(x\operatorname{th} q)C_m^\lambda(x)\left(1-x^2\right)^{\frac{n-3}{2}} dx\, dq,\end{aligned}$$

where $\delta_{m,k}$ is the Kronecker delta. This means that $Rf$ is orthogonal to $g^*$ for all $f \in L^2(\mathbb{S}, \operatorname{sh}^{n-1} \delta_x dx)$ if and only if

$$\int_0^1 g(xy)C_m^\lambda(x)\left(1-x^2\right)^{\frac{n-3}{2}} dx \equiv 0.$$

By Lemma 5.1 and Theorem 5.2 of [11] and Theorem 3.2 of [10] this is equivalent to $g$ being in the closure of the span of the functions $x^j$, where $0 \le j < m$ and $m - j$ is even. This proves that the range of the Radon transform is in $\mathcal{A}$.

Now we prove the injectivity of $R$. We are looking for a function $f(\omega, q) = v(q)Y_{i,k}(\omega) \in L^2(\mathbb{S}, \operatorname{sh}^{n-1} \delta_x dx)$, which has zero Radon transform. Thus we should find a function $v \in L^2(\mathbb{R}_+, \operatorname{sh}^{2n-2} q\, dq)$ which satisfies the equation

$$0 \equiv \int_h^\infty v(q)C_m^\lambda\Big(\frac{\operatorname{th} h}{\operatorname{th} q}\Big)\Big(1-\frac{\operatorname{th}^2 h}{\operatorname{th}^2 q}\Big)^{\frac{n-3}{2}} \operatorname{sh}^{n-2} q\, dq.$$

Assuming $v(q) = w(\operatorname{cth} q)\operatorname{sh}^{1-n} q/\operatorname{ch} q$ and changing the variables, $s = \operatorname{th} q$ and $t = \operatorname{th} h$, we get the integral equation

$$0 \equiv \int_t^1 \frac{1}{s} w\left(\frac{1}{s}\right) C_m^\lambda\left(\frac{t}{s}\right)\left(1-\frac{t^2}{s^2}\right)^{\frac{n-3}{2}} ds,$$

which has to be satisfied by $w$ for all $t \in [0,1]$. Since $w(1/s) \in L^2([0,1])$ this Volterra integral equation is of the type (3.7) of [12], so its only solution is the zero function, which completes the proof.

■

A similar method can be used for getting the corresponding result for the boomerang transform.

**Theorem 3.3.** *Let $\mathbb{S}$ be a measurable set in $\mathbb{H}^n$ (may be $\mathbb{H}^n$), $n \geq 3$ and $g_{j,l,m}(\omega, h) = \frac{\operatorname{th}^j h}{\operatorname{ch}^2 h} Y_{l,m}(\omega)$ for $j, l, m \in \mathbb{N}$, where $0 \leq j < m$ and $(m - j)$ is even. The kernel of the boomerang transform in $L^2(\mathbb{S}, \operatorname{sh}^{1-n} \delta_x dx)$ is* Cl Sp$\{g_{j,l,m}(\omega, h)\}$.

## 4. Closed inversion formula

**Theorem 4.1.** *Let $n \geq 2$, $\lambda = (n-2)/2$ and $f \in C_c^\infty(\mathbb{H}^n)$. If $n$ is odd then*

$$f(\bar{\omega}, t) = (-1)^{\frac{n-1}{2}} \frac{2^{1-n}}{\pi^{n-1}} \delta_1 \delta_3 \ldots \delta_{n-2} \Big( B \Big( Rf(\omega, h) \frac{\operatorname{cth}^{2\lambda} h}{\operatorname{sh} h} \Big) (\bar{\omega}, t) \operatorname{sh}^{n-1} t \Big).$$

*If $n$ is even then*

$$f(\bar{\omega}, t) = (-1)^{\frac{n}{2}} \frac{2^{1-n}}{\pi^n} \frac{d}{dt} \delta_2 \delta_4 \ldots \delta_{n-2} \Big( B \Big( \mathcal{H} \Big\langle Rf(\omega, h) \frac{\operatorname{cth}^{2\lambda} h}{\operatorname{sh} h} \Big\rangle \Big) (\bar{\omega}, t) \operatorname{sh}^{n-1} t \Big),$$

*where the $\mathcal{H}$ distribution is*

$$\mathcal{H} f(\omega, h) = \frac{1}{\operatorname{ch}^2 h} \int_{-\infty}^{\infty} f(\omega, r) \frac{1}{\operatorname{th} r - \operatorname{th} h} dr.$$

(We use here the natural identification $f(\omega, r) = f(-\omega, -r)$ for $r < 0$.)

**Proof.** We start with the odd-dimensional case, where Theorem 2.5 tells us that

$$f_{l,m}(t) = \mathcal{C}\mathcal{D} \int_t^\infty (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\operatorname{th} h}{\operatorname{th} t} \right) \left( \frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h} \operatorname{cth}^{n-2} h \, dh,$$

where $\mathcal{C} = \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)}$ and $\mathcal{D} = \delta_1 \delta_3 \ldots \delta_{n-2}$. The integral $\int_t^\infty$ can be modified by making use of $\int_t^\infty = \int_0^\infty - \int_0^t$ which yields

(∗)

$$f_{l,m}(t) = I + \mathcal{C}(-1)^{n-1/2} \mathcal{D} \int_0^t (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\operatorname{th} h}{\operatorname{th} t} \right) \times \\ \times \left( 1 - \frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} \right)^{\frac{n-3}{2}} \frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h} \operatorname{cth}^{n-2} h \, dh,$$

where

$$I = \mathcal{C}\mathcal{D} \int_0^\infty (Rf)_{l,m}(h) C_m^\lambda \left( \frac{\operatorname{th} h}{\operatorname{th} t} \right) \left( \frac{\operatorname{th}^2 h}{\operatorname{th}^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h} \operatorname{cth}^{n-2} h \, dh.$$

Using (RN) and reversing the order of integrations it turns out that $I$ is proportional to

$$\int_0^\infty f_{l,m}(q)\,\mathrm{sh}^{2\lambda} q \int \Psi_0^q C_m^\lambda\left(\frac{\mathrm{th}\,h}{\mathrm{th}\,q}\right)\left(1-\frac{\mathrm{th}^2 h}{\mathrm{th}^2 q}\right)^{\frac{n-3}{2}} \times$$
$$\times\,\mathcal{D}\left(C_m^\lambda\left(\frac{\mathrm{th}\,h}{\mathrm{th}\,t}\right)\left(\frac{\mathrm{th}^2 h}{\mathrm{th}^2 t}-1\right)^{\frac{n-3}{2}}\mathrm{sh}^{2\lambda} t\right)\frac{\mathrm{cth}^{n-1} h}{\mathrm{ch}^2 h}dh\,dq.$$

Substituting $x=\mathrm{th}\,h/\,\mathrm{th}\,q$ in $J$, the integral with respect to $h$, and using that the integrand is an even function we obtain

$$J=\frac{\mathrm{cth}^{2\lambda} q}{2}\int_{-1}^1 C_m^\lambda(x)\left(1-x^2\right)^{\frac{n-3}{2}} \times$$
$$\times\,\mathcal{D}\left(C_m^\lambda\left(\frac{\mathrm{th}\,q}{\mathrm{th}\,t}x\right)\left(\frac{\mathrm{th}^2 q}{\mathrm{th}^2 t}x^2-1\right)^{\frac{n-3}{2}}\mathrm{sh}^{2\lambda} t\right)x^{1-n}dx.$$

Let us recall that $C_m^\lambda(x)\left(1-x^2\right)^{\frac{n-3}{2}}$ is a polynomial of degree $m+n-3$ and that $\{C_m^\lambda(x)\}$ is an orthogonal system on [-1,1] with weight-function $\left(1-x^2\right)^{\frac{n-3}{2}}$ [6]. Now we prove that the polynomial $\mathcal{D}\left(C_m^\lambda(x\frac{\mathrm{th}\,q}{\mathrm{th}\,t})\left(\frac{\mathrm{th}^2 q}{\mathrm{th}^2 t}x^2-1\right)^{\frac{n-3}{2}}\mathrm{sh}^{2\lambda} t\right)$ is divisible by $x^{n-1}$ which implies $J=0$ and $I=0$ by the above facts. The coefficient of $x^k$, for $0\le k\le n-2$, vanishes if

$$\delta_1\delta_3\ldots\delta_{n-2}\left(\mathrm{sh}^{n-2} t\ \ \mathrm{cth}^k t\right)=0,$$

which can be verified by induction after establishing that

$$\delta_k(\mathrm{ch}^k t)=-k(k-1)\,\mathrm{ch}^{k-2} t\quad\text{and}\quad \delta_k(\mathrm{sh}^k t)=k(k-1)sh^{k-2}t.$$

Now equation $(*)$ gives just the expansion of the closed inversion formula stated for odd dimension.

Let us consider now the even-dimensional case, when $\lambda$ is integer. Let $\mathcal{D}$ denote the differential operator $\frac{d}{dt}\delta_2\delta_4\ldots\delta_{n-2}$. Theorem 2.5 says that
$(**)$

$$f_{l,m}(t)=-\mathcal{C}\mathcal{D}\int_t^\infty (Rf)_{l,m}(h)C_m^\lambda\left(\frac{\mathrm{th}\,h}{\mathrm{th}\,t}\right)\left(\frac{\mathrm{th}^2 h}{\mathrm{th}^2 t}-1\right)^{\frac{n-3}{2}}\frac{\mathrm{sh}^{n-2} t}{\mathrm{sh}\,h}\,\mathrm{cth}^{n-2} h\,dh.$$

For further use, we recall that $D_m^\lambda$ is the Gegenbauer function of the second kind and the polynomial $E_{m+2\lambda-1}^\lambda$ is given by (A.14) and (A.4) of [5] as

$$E_{m+2\lambda-1}^\lambda = \begin{cases} (-1)^{\lambda+1}2^{\lambda-1}\Gamma(\lambda)(1-x^2)^{\lambda-1/2}D_m^\lambda(x) & \text{if } 0<x<1 \\ 2^{\lambda-1}\Gamma(\lambda)(x^2-1)^{\lambda-1/2}(C_m^\lambda(x)-2D_m^\lambda(x)) & \text{if } 1<x. \end{cases}$$

The function $I_m^\lambda$ is defined by (22) of [5] as

$$I_m^\lambda(x) = \int_{-1}^1 C_m^\lambda(t)\left(1-t^2\right)^{\lambda-1/2}(x-t)^{-1}dt$$

and in (24) and (25) of [5] it is proved that

$$I_m^\lambda(x) = \begin{cases} \pi(1-x^2)^{\lambda-1/2}D_m^\lambda(x) & \text{if } 0<x<1 \\ 2\pi e^{-i\pi\lambda}(x^2-1)^{\lambda-1/2}D_m^\lambda(x) & \text{if } 1<x. \end{cases}$$

As in the odd-dimensional case, one can easily see that

$$0 = \mathcal{CD}\int_0^\infty (Rf)_{l,m}(h)E_{m+2\lambda-1}^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h\,dh,$$

because $E_{m+2\lambda-1}^\lambda$ is a polynomial of degree $m+n-3$ [5]. To proceed one has to rewrite this integral as $\int_0^t + \int_t^\infty$ and substitute the appropriate expressions of $E_{m+2\lambda-1}^\lambda$ into $\int_0^t$ and $\int_t^\infty$ respectively. Adding the result to equation (**) one obtains

$$f_{l,m}(t) = -\mathcal{CD}\Big[\int_t^\infty (Rf)_{l,m}(h)2D_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\left(\frac{\operatorname{th}^2 h}{\operatorname{th}^2 t}-1\right)^{\frac{n-3}{2}}\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h\,dh +$$
$$+\int_0^t (Rf)_{l,m}(h)(-1)^\lambda D_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\left(1-\frac{\operatorname{th}^2 h}{\operatorname{th}^2 t}\right)^{\frac{n-3}{2}}\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h\,dh\Big]$$

that can be written in the form

$$f_{l,m}(t) = -\frac{\mathcal{C}}{\pi}(-1)^\lambda\mathcal{D}\int_0^\infty (Rf)_{l,m}(h)I_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right)\frac{\operatorname{sh}^{n-2} t}{\operatorname{sh} h}\operatorname{cth}^{n-2} h\,dh,$$

where

$$I_m^\lambda\left(\frac{\operatorname{th} h}{\operatorname{th} t}\right) = \int_{-1}^1 C_m^\lambda(x)(1-x^2)^{\frac{n-3}{2}}\left(\frac{\operatorname{th} h}{\operatorname{th} t}-x\right)^{-1}dx.$$

Now the substitution $x = \operatorname{th} r/\operatorname{th} t$ and a change in the order of integrations result in

$$f_{l,m}(t) = \frac{-\mathcal{C}(-1)^\lambda}{\pi}\mathcal{D}\int_{-t}^t C_m^\lambda\left(\frac{\operatorname{th} r}{\operatorname{th} t}\right)\left(1-\frac{\operatorname{th}^2 r}{\operatorname{th}^2 t}\right)^{\frac{n-3}{2}}\times$$
$$\times\frac{\operatorname{sh}^{n-1} t}{\operatorname{sh} t}\frac{1}{\operatorname{ch}^2 r}\int_0^\infty (Rf)_{l,m}(h)\frac{\operatorname{cth}^{n-2} h/\operatorname{sh} h}{\operatorname{th} h-\operatorname{th} r}dh\,dr.$$

This equation is just the expansion of the closed inversion formula stated for even dimensions.

■

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