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Orbital integrals on the Lorentz space of curvature −1

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Abstract. We present a rotation symmetric model in the Euclidean space for the Lorentzian of curvature -1 in which the Lorentzian spheres around all the points of an a priori fixed spacelike totalgeodesic are straightlines. Investigating the mean value operators in this model yields to various representations of functions by means of their integrals over Lorentzian spheres.

1. Introduction

This article deals with the problem of recovering a function from its integrals over the spheres in pseudo Riemannian spaces. As Helgason pointed out in [4], and [5], formulas representing functions by their mean values play very important role in the theory of differential operators on higher rank symmetric spaces. For instance we refer the reader to Huygens' principle in the solution of the Cauchy problem.

To determine a function f from its spherical mean values $M^r f$ on Riemannian manifolds is simply done by $f = \lim_{r\to 0} M^r f / M^r 1$, where we adapted the commonly accepted notation $M^r f(x)$ for the integral of f on the sphere around x of radius $r > 0$. In Lorentzian spaces the situation changes considerable, because the Lorentzian spheres do not shrink to their center as their radius approaches zero. Other significant difference is that the Lorentzian spheres are neither compact nor connected.

In this paper we work on the Lorentz space $\mathcal{L}^n$ of signature $(1, n-1)$ with constant curvature -1. We represent the functions by means of their integrals

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over spheres in two essentially different way. The first is similar in spirit to the Riemannian formulation and the other resembles to the Radon transform.

The outline of the paper is as follows.

In the next section we present our model for $\mathcal{L}^n$, that is obtained as the orthogonal projection of S. Helgason's quadratic hypersurface model [5] along the straightline of the two ideal points. The Lorentzian spheres around points of the 'equator' become hyperplanes in this model. Further prerequisites are also given in this section.

In Section 3 we represent the function f with its spherical integrals $M^r f$ in a Riemannian-type formulation, i.e. $f(x)$ is calculated from mean values of f over Lorentzian spheres shrinking to x. The only result in this direction the author knows about is S. Helgason's formula for the timelike spheres (the part of the sphere in the 'retrograde' cone [5]). It works in even dimensions and involves radial differentiation of degree $n-2$ (1 for $n=2$). Our formulas contain radial differentiation of order $[n/2]$ in *any* dimension. For spacelike spheres we obtain formulas in even dimensions and show that there can not exist this type of representation in odd dimensions.

In Section 4 we turn to a Radon-type representation. We represent the functions by their integrals on a restricted set $\mathcal{M}^n$ of spheres centered to points on an a priori fixed spacelike totalgeodesic E, the equator. We prove inversion formulas for the corresponding integral transform M, that integrates functions over spheres in $\mathcal{M}^n$. This is the representation of Radon-type. We also prove support theorem, that ensures the representation's uniqueness despite of the restriction on the set of spheres.

2. The model

We start with Helgason's quadratic hypersurface model [4] for the Lorentz space of signature $(1, n-1)$ and of curvature -1, adapting also some of his notations. This model is defined in $\mathbb{R}^{n+1}$ by the bilinear form

$$B(x,y) = x_1 y_1 - \sum_{i=2}^{n+1} x_i y_i \tag{2.1}$$

on the hypersurface $\mathcal{Q}^n_{-1}$ that is defined by $B(x,x) = -1$. This implies immediately, that $\mathcal{Q}^n_{-1}$ with its Lorentz structure is axially symmetric around the x_1-axis, and is symmetric with respect to the origin. Helgason proved [4] that the geodesics are the intersections of $\mathcal{Q}^n_{-1}$ with the two dimensional subspaces of $\mathbb{R}^{n+1}$.

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Our model $\mathcal{L}^n$ of the Lorentzian is the orthogonal projection of $\mathcal{Q}^n_{-1}$ with its structure into the hyperplane $x_1 = 0$ along the axis x_1. Let $\mu\colon \mathcal{Q}^n_{-1} \to \mathcal{L}^n$ be this projection. Obviously $\mathcal{L}^n$ is rotation symmetric around the origin, because $\mathcal{Q}^n_{-1}$ is axially symmetric.

Let us take a point P on $\mathcal{Q}^n_{-1}$, and take the two dimensional subspace π of $\mathbb{R}^{n+1}$ containing P and the x_1-axis. Clearly, the intersection of π with $\mathcal{Q}^n_{-1}$ is a hyperbola (two sheeted). Say, the hyperbola through P intersects the subspace $x_1 = 0$ of $\mathbb{R}^{n+1}$ in the point O. Let r denote the Lorentzian distance from P to O. Then r is also the Lorentzian distance of P from the equator E, the intersection of $\mathcal{Q}^n_{-1}$ and the hyperplane $x_1 = 0$. Let the coordinates of P be $(p_1, p_2, \dots, p_{n+1})$ in $\mathbb{R}^{n+1}$. Relative to π we may use the coordinates $\rho_1 = p_1$ and $\rho_2 = \sqrt{\sum_{i=2}^{n+1} p_i^2}$. These coordinates are functions of r. For the arclength parameter r equation (2.1) and the definition of $\mathcal{Q}^n_{-1}$ imply

$$(2.2)\qquad \left(\frac{d\rho_1}{dr}\right)^2 - \left(\frac{d\rho_2}{dr}\right)^2 = +1 \quad \text{and} \quad \rho_1^2 - \rho_2^2 = -1$$

that gives $p_1 = \rho_1 = \sinh(r)$ and $\rho_2 = \cosh(r)$ by an easy calculation. Hence $|\mu(P)| = \cosh r$ and the projection $\mu(P)$ of P is in π. Figure 1 shows the situation on the plane π.

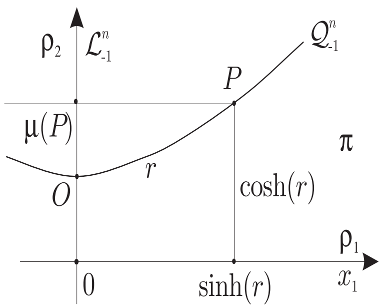

Figure 1.

We parameterize $\mathcal{L}^n$ so that (ω, r) means the point $\mu(P) = \omega \cosh r$, where $\omega \in S^{n-1}$. Identifying the points symmetric to the subspace $x_1 = 0$, the map μ becomes one-to-one. This can be done without loosing information, because all the objects under consideration will be symmetric to the hyperplane $x_1 = 0$.

Let $\langle ., . \rangle_{\mu(P)}$ be the Lorentzian inner product on $T_{\mu(P)}\mathcal{L}^n$ and let $\mu(P) = (\omega, \cosh r)$ in the polarsystem of the hyperplane $x_1 = 0$.

Obviously a vector v in $T_P\mathcal{Q}^n_{-1}$, the tangent space of $\mathcal{Q}^n_{-1}$ at the point P, orthogonal to π will be mapped by $\mu^*\colon T_P\mathcal{Q}^n_{-1} \to T_{\mu(P)}\mathcal{L}^n$, the induced map of μ,

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into a vector $\mu^*(v)$ having the same Euclidean direction. Clearly $|\mu^*(v)| = |v|$ in Euclidean meaning that, by (2.1), coincides with the Lorentzian meaning in these directions. If u is in $\pi \cap T_P \mathcal{Q}^n_{-1}$ then $\mu^*(u)$ is in $\pi \cap T_{\mu(P)}\mathcal{L}^n$ and is orthogonal to the x_1-axis. The Euclidean angle α of $\pi \cap T_P \mathcal{Q}^n_{-1}$ to the x_1-axis satisfies $\tan\alpha = \tanh r$, therefore $|\mu^*(u)| = |u|\sin\alpha$ gives $B(u,u) = |u|^2 \cdot \cos 2\alpha = |\mu^*(u)|^2 \cdot \sinh^{-2} r$. (A more formal way to prove this is to differentiate μ.) We can now conclude that

$$\langle (dr, d\omega), (dr', d\omega')\rangle_{\mu(P)} = dr dr' \sinh^{-2} r - \sum_{i=1}^{n-1} d\omega_i d\omega'_i,$$

where dr means the radial part of the vector $(dr, d\omega) \in T_{\mu(P)}\mathcal{L}^n$ and $d\omega$ means the part orthogonal, in Euclidean meaning, to the radius. We shall call $d\omega$ the spherical part.

The Lorentzian sphere falls into parts, determined by the timelike and spacelike vectors. We call these parts timelike and spacelike spheres, respectively. In dimension two there are two connected spacelike spheres and two connected timelike spheres. In higher dimensions there are still two connected timelike spheres, but 'only' one connected spacelike sphere.

As usual, we parameterize the set of hyperplanes in $\mathbb{R}^n$, so that $H(\omega, p)$ denotes the hyperplane perpendicular to $\omega \in S^{n-1}$ and going through $p \cdot \omega \in \mathbb{R}^n$, where $p \in \mathbb{R}_+$. The corresponding Lorentzian set of points in $\mathcal{L}^n$ will be denoted by $\hat{H}(\omega, p)$. This correspondence is not one-to-one in dimension 2 for $p < 1$, because the intersection of the corresponding plane with $\mathcal{Q}^2_{-1}$ falls into two parts.

Lemma 2.1. *For $p \neq 1$, the set $\hat{H}(\bar\omega, p)$ is the Lorentzian sphere around $(\bar\omega, 1) \in \mathcal{L}^n$ of radius*

$$r = \begin{cases} \operatorname{arccosh} p & \text{if } p > 1, \text{ timelike} \\ \arccos p & \text{if } p < 1, \text{ spacelike.} \end{cases}$$

$\hat{H}(\bar\omega, 1)$ is the lightlike totalgeodesic through $(\bar\omega, 1) \in \mathcal{L}^n$.

Proof. Because of the rotation symmetry the claim needs to be proved only for dimension two. Helgason proved in [5] that the geodesics in $\mathcal{Q}^2_{-1}$ are exactly the intersections of $\mathcal{Q}^2_{-1}$ with the planes through the origin. Let us take the plane π_α with normal vector $(-\sin\alpha, \cos\alpha, 0)$ in $\mathbb{R}^3$. $\mathbb{R}^3$ is parameterized by the orthogonal system (x, y, z). The quadratic model $\mathcal{Q}^2_{-1}$ is the surface $z^2 + y^2 = x^2 + 1$ and the intersection $\pi_\alpha \cap \mathcal{Q}^2_{-1}$ is the curve $c_\alpha(z)$ that we parameterize by arclength as follows:

$$c_\alpha(\cosh r) = \begin{cases} \left(\frac{\sinh r\cos\alpha}{\sqrt{\cos 2\alpha}}, \frac{\sinh r\sin\alpha}{\sqrt{\cos 2\alpha}}, \cosh r\right) & \text{if } -\pi/4 < \alpha < \pi/4, \text{ timelike,} \\ (r, r\tan\alpha, 1) & \text{if } \alpha = \pm\pi/4, \text{ lightlike, and} \\ \left(\frac{\sin r\cos\alpha}{\sqrt{-\cos 2\alpha}}, \frac{\sin r\sin\alpha}{\sqrt{-\cos 2\alpha}}, \cos r\right) & \text{if } {}^{-\pi/2\le\alpha<-\pi/4}_{\pi/4<\alpha\le\pi/2}, \text{ spacelike.} \end{cases}$$

Since the third coordinate does not depend on α, the points of $\{z = \cosh r\} \cap \mathcal{Q}^2_{-1}$ and $\{z = \cos r\} \cap \mathcal{Q}^2_{-1}$ has constant Lorentzian distance r to the point $c_\alpha(\cosh 0) = c_\alpha(1) = (0, 0, 1)$. This proves the statement.

■

By the above arguments we can easily determine the geodesics

$$g_\alpha(\cosh r) = \begin{cases} \left(0, \frac{\sinh r\sin\alpha}{\sqrt{\cos 2\alpha}}, \cosh r\right) & \text{if } -\pi/4 < \alpha < \pi/4, \text{ timelike and} \\ \left(0, \frac{\sin r\sin\alpha}{\sqrt{-\cos 2\alpha}}, \cos r\right) & \text{if } {}^{-\pi/2\le\alpha<-\pi/4}_{\pi/4<\alpha\le\pi/2}, \text{ spacelike,} \end{cases}$$

in $\mathcal{L}^n$. Introducing the natural parameter $p = |\tan\alpha|$ it turns out that these geodesics are the ellipses resp. hyperbolas of the equation $y^2(p^2-1)/p^2 + z^2 = 1$. Figure 2 shows a general picture about our results.

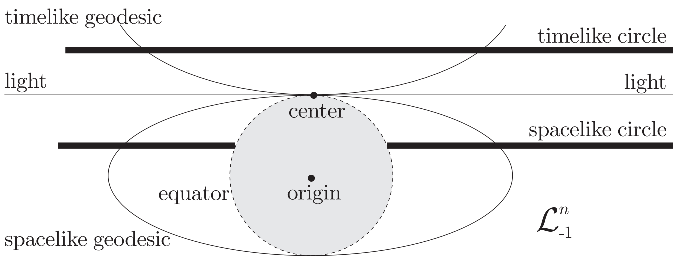

Figure 2.

Summing up, the Euclidean unit sphere in $\mathcal{L}^n$ is the equator $\mathcal{Q}^n_{-1} \cap \{x_1 = 0\}$. The Euclidean sphere around the origin of radius $\cosh r$ is the 'ring' $\mathcal{Q}^n_{-1} \cap \{x_1 = \sinh r\}$ of radius $\cosh r$. The lightlike geodesics are the straightlines touching the unit sphere. The spacelike resp. timelike geodesics are ellipses resp. hyperbolas. The spacelike resp. timelike spheres around the points of the equator are the straightlines meeting with resp. avoiding the unit sphere.

Note that we call a part of a sphere timelike sphere or spacelike sphere according to the type of geodesics it meets. We found this more natural than the naming convention in [5], based on the type of the tangent vectors.

Lemma 2.2. *The surface measure on the Lorentz sphere $\hat{H}(\bar{\omega}, p)$ at the point $X = (\omega, r) \in \mathcal{L}^n$ is*

$$dA = \frac{p^{n-1}\sqrt{|p^2-1|}}{\langle\omega,\bar{\omega}\rangle^{n-1}\sqrt{p^2-\langle\omega,\bar{\omega}\rangle^2}}d\omega,$$

where $\cosh r = p/\langle\omega,\bar{\omega}\rangle$, $0 < \langle\omega,\bar{\omega}\rangle < p$ *and* $\langle .,. \rangle$ *is the standard Euclidean inner product.*

Proof. Assume the dimension is two. If $d\omega$ is the infinitesimal element at ω on S^1, and $\overline{di}$ is the corresponding arclength element on $H(\bar{\omega}, p)$ at X, then $\overline{di} = \frac{p}{\cos^2\alpha}d\omega$, where $\cos\alpha = \langle\omega,\bar{\omega}\rangle$. The radial resp. spherical part of $\overline{di}$ are $dr = \overline{di}\sin\alpha$ resp. $ds = \overline{di}\cos\alpha$. Therefore

$$di^2 = \langle\overline{di}, \overline{di}\rangle_X = (dr^2\sinh^{-2} r - ds^2) = (\sin^2\alpha\sinh^{-2} r - \cos^2\alpha)\cdot\frac{p^2 d\omega^2}{\cos^4\alpha}.$$

Substituting $\sinh^{-2} r = (\cosh^2 r - 1)^{-1} = \cos^2\alpha/(p^2 - \cos^2\alpha)$ gives the statement for dimension two.

For higher dimension we use the rotation symmetry. The sphere $\mu(R)$ of radius $\cosh r$ centered to the origin in $\mathbb{R}^n$ corresponds to the 'ring' $R = \mathcal{Q}^n_{-1} \cap \{x_1 = \sinh r\}$ that has radius $\cosh r$ in $\mathbb{R}^{n+1}$. By (2.1) this means that the natural map $\nu\colon S^{n-1} \to \mu(R)$ induces a dilation ν^* of coefficient $\cosh r$ between the corresponding tangent spaces of S^{n-1} and $\mu(R)$. Since $\cosh r = p/\langle\omega,\bar{\omega}\rangle$ this proves the lemma. ■

Let X be a point in $\mathcal{L}^n$. We are looking for an angle measure in the tangents space $T_X\mathcal{L}^n$ that is invariant under the isotropy group. In other words we want to have a distance on the unit 'sphere' in $T_X\mathcal{L}^n$ according to the inner product $\langle .,. \rangle_X$. Let

$$d(\omega_1, \omega_2) = \frac{\langle\omega_1,\omega_2\rangle_X^2}{\langle\omega_1,\omega_1\rangle_X\langle\omega_2,\omega_2\rangle_X}. \tag{2.3}$$

Then d is a function on the pairs of elements of the unit sphere in $T_X\mathcal{L}^n$, and it is obviously invariant under the isotropy group. (We assume neither ω_1 nor ω_2 is lightlike.)

We define the angle of $\omega_1, \omega_2 \in T_X\mathcal{L}^n$ for timelike and spacelike vectors differently. Let $L_X \subset T_X\mathcal{L}^n$ be the cone of lightlike vectors, C_X be the set of timelike vectors and D_X be the set of spacelike vectors. Then we set the Lorentzian angle γ of ω_1 to ω_2 as

$$\begin{aligned} \cosh^2\gamma &= d(\omega_1,\omega_2) && \text{if} \quad \omega_1,\omega_2 \in C_X \\ \sinh^2\gamma &= -d(\omega_1,\omega_2) && \text{if} \quad \omega_1,\omega_2 \in D_X \end{aligned} \tag{2.4}$$

We denote the Lorentzian unit sphere at $X \in \mathcal{L}^n$ by Σ_X^{n-1}.

Lemma 2.3. *Let $X = (\bar{\omega}, r) \in \mathcal{L}^2$ and for every $\omega \in S^1$ let $\hat{\omega} \in \Sigma^1_X$ be tangent to $\hat{H}(\omega, \cosh r\langle\omega, \bar{\omega}\rangle)$. Then*

$$d\hat{\omega} = \frac{|\sinh r|}{\cosh^2 r\langle\omega,\bar{\omega}\rangle^2 - 1} d\omega \quad \text{on} \quad \Sigma^1_X \setminus L_X.$$

The same formula is also valid in higher dimensions.

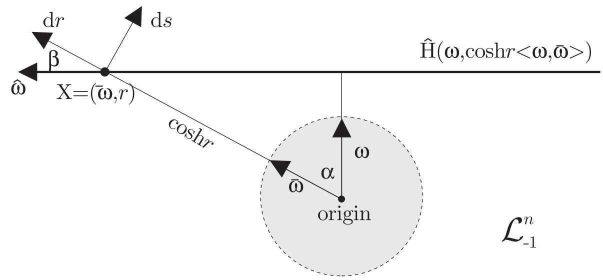

Figure 3.

Proof. Assuming $\hat{\omega} = dr\cos\beta + ds\sin\beta$ as on Figure 3, a simple calculation with (2.4) gives that

$$d(\hat{\omega}, dr) = \frac{\cos^2\beta}{1 - \cosh^2 r\sin^2\beta}. \tag{2.5}$$

From $\cos\alpha = \langle\omega, \bar{\omega}\rangle$ we see that $\alpha + \beta = \pi/2$. Let γ be the Lorentzian angle of $\hat{\omega}$ to dr as defined by (2.5). For timelike $\hat{\omega}$ we have $\cosh^2\gamma = d(\hat{\omega}, dr)$. Then substituting this into and differentiating (2.5) with respect to α we obtain

$$d\gamma = \frac{-|\sinh r|}{1 - \cosh^2 r\cos^2\alpha} d\alpha.$$

The formula can be proved for $\hat{\omega} \in D_X$ in the same way.

Since S^{n-1} and Σ^{n-1}_X are both axially symmetric with respect to the axis OX, the rotation symmetry of the model validates the formula in Lemma 2.3 for any dimension.

■

Note that both Lemma 2.2 and 2.3 calculate the surface measure on Lorentzian spheres, but in basically different coordinate systems.

3. Spherical means on concentric spheres

In this section we represent the functions with their orbital integrals on concentric spheres similarly to the Riemannian case. We call a function on the Lorentzian space even resp. odd, if its representating function f on $\mathcal{Q}^{n-1}_{-1}$ satisfies $f(x_1,\dots,x_n)=f(-x_1,\dots,x_n)$ resp. $f(x_1,\dots,x_n)=-f(-x_1,\dots,x_n)$. The orbital integral of odd functions vanish on the spacelike spheres centered to points on the equator. The timelike spheres centered to points on the equator never meet with the equator, hence the formulas representing the functions by their orbital integrals over timelike spheres can not depend on the parity. To make the mapping μ essentially one-to-one, we restrict the considerations onto the set of the even functions. Keep in mind that the formulas on timelike spheres are all valid for any type of functions.

First we introduce notations for the orbital integrals, that fit to our situation. Let $Mf(\omega,p)$ denote the integral of the integrable function f over the sphere $\hat{H}(\omega,p)$ with respect to the Lorentzian surface measure determined in Lemma 2.2. Although more generality could be allowed, we point our attention to the set $C^\infty_c(\mathcal{L}^n)$ of infinitely differentiable functions of compact support. Then by Lemma 2.2 we have

$$(3.1)\qquad Mf(\bar\omega,p)=\int_{S^{n-1}_{\bar\omega,p}} f\Big(\omega,\operatorname{arccosh}\Big(\frac{p}{\langle\omega,\bar\omega\rangle}\Big)\Big)\frac{p^{n-1}\sqrt{|p^2-1|}}{\langle\omega,\bar\omega\rangle^{n-1}(p^2-\langle\omega,\bar\omega\rangle^2)^{1/2}}d\omega,$$

where $S^{n-1}_{\bar\omega,p}=\{\omega\in S^{n-1}:0<\langle\omega,\bar\omega\rangle<p\}$, and we used the parameterization of $\mathcal{L}^n$ for parameterizing f. Note that both arguments of $Mf(\omega,p)$ are Euclidean objects.

We shall frequently use the radially acting differential operators $\mathcal{D}_t$ resp. $\mathcal{D}_s$ that is defined on the functions $f\in C^\infty_c(\mathcal{L}^n)$ as

$$(3.2)\qquad \mathcal{D}_tf=\frac{d}{dr}\Big(\frac{f(\omega,r)}{\sinh r\cosh r}\Big)\quad\text{and}\quad \mathcal{D}_sf=\frac{d}{dr}\Big(\frac{f(\omega,r)}{\sin r\cos r}\Big)$$

Theorem 3.1. *Let $f\in C^\infty_c(\mathcal{L}^2)$. For timelike spheres*

$$f(\omega,0)=\lim_{r\to0}\frac{-\sinh r}{2}\mathcal{D}_t\Big(\cosh r Mf(\omega,\cosh r)\Big)$$

and for spacelike spheres

$$f(\omega,0)=\lim_{r\to0}-\sin r\mathcal{D}_s\Big(\cos r Mf(\omega,\cos r)\Big).$$

Before the proof note that $\sinh r$ as well as $\sin r$ could be replaced by r in which case Theorem 3.1 reproduces Helgason's formula for time like spheres in [5].

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Proof. In dimension two the formula (3.1) takes the form

$$Mf(\alpha,p)=\int_I f\Big(\alpha+\omega,\operatorname{arccosh}\Big(\frac{p}{\cos\omega}\Big)\Big)\frac{p\sqrt{|p^2-1|}}{\cos\omega(p^2-\cos^2\omega)^{1/2}}d\omega, \tag{3.3}$$

where $I=[-\pi/2,\pi/2]$ for $p>1$, $I=[-\pi/2,0]\cap\{\cos\omega<p\}$ or $I=[0,\pi/2]\cap\{\cos\omega<p\}$ for $p<1$ and we used angles instead of vectors.

First we deal with the timelike case, i.e. $p>1$. The timelike sphere $\hat{H}(\alpha,p)$ is symmetric to its point closest to the equator, therefore it is enough to prove for symmetric functions. With this in mind, substituting $\cos\omega=\frac{\cosh r}{\cosh z}$ yields

$$Mf(\alpha,\cosh r)=2\int_r^\infty \bar{f}(z)\frac{\cosh z\sinh r}{\sqrt{\cosh^2 z-\cosh^2 r}}dz,$$

where $\bar{f}(z)=(f(\alpha+\omega,z)+f(\alpha-\omega,z))/2$ with $\omega=\arccos(p/\cosh z)$. Since $\cosh^2 z-\cosh^2 r=\sinh^2 z-\sinh^2 r$, partial integration with $\bar{f}(z)/\sinh z$ gives

$$Mf(\alpha,\cosh r)=-2\sinh r\int_r^\infty\left(\frac{\bar{f}(z)}{\sinh z}\right)'\sqrt{\sinh^2 z-\sinh^2 r}dz.$$

Differentiating this with respect to r yields

$$\begin{aligned}\frac{-\sinh r}{2}\mathcal{D}_t\Big(\cosh r Mf(\alpha,\cosh r)\Big)&\\ =\sinh^2 r\cosh r\int_r^\infty&\frac{\bar{f}(z)\cosh z-\bar{f}'(z)\sinh z}{\sinh^2 z\sqrt{\sinh^2 z-\sinh^2 r}}dz.\end{aligned}\tag{3.4}$$

The estimate

$$\left|\sinh^2 r\int_r^\infty\frac{\bar{f}'(z)\sinh z}{\sinh^2 z\sqrt{\sinh^2 z-\sinh^2 r}}dz\right|\le\frac{\pi}{2}\sup|\bar{f}'(z)|\sinh r.$$

is easy to prove by substituting the variable $t=\sinh r/\sinh z$. Since the right hand side clearly goes to zero as $r\to 0$ the only part in (3.4) not vanishing as $r\to 0$ is

$$\sinh^2 r\cosh r\int_r^\infty\frac{\bar{f}(z)\cosh z}{\sinh^2 z\sqrt{\sinh^2 z-\sinh^2 r}}dz.$$

Substituting again the variable $t=\sinh r/\sinh z$, this turns out to be $\bar{f}(0)$ that proves the first part of the statement.

We turn to the spacelike case, i.e. $p < 1$. Choosing any of the intervals for the integration in (3.3), the substitution $\cos\omega = \cos r/\cosh z$ yields

$$Mf(\alpha, \cos r) = \int_0^\infty \bar{f}(z)\frac{\cosh z \sin r}{\sqrt{\cosh^2 z - \cos^2 r}}dz.$$

Since $\cosh^2 z - \cos^2 r = \sinh^2 z + \sin^2 r$ the differentiation with respect to r leads to

$$-\sin r\mathcal{D}_s\Big(\cos rMf(\alpha, \cos r)\Big) = \sin^2 r \cos r \int_0^\infty \frac{\bar{f}(z)\cosh z}{(\sinh^2 z + \sin^2 r)^{3/2}}dz. \tag{3.5}$$

Let $\bar{f}(z) = \bar{f}(0) + g(z)\sinh z$, where $g \in C_c^\infty(\mathcal{L}^2)$. Then by partial integration we obtain

$$\begin{aligned}\left|\sin^2 r \int_0^\infty \frac{g(z)\sinh z\cosh z}{(\sinh^2 z + \sin^2 r)^{3/2}}dz\right| &\le |g(0)\sin r| + \sin^2 r\int_0^\infty \frac{|g'(z)|}{\sqrt{\sinh^2 z + \sin^2 r}}dz \\ &\le \sin r\Big(|g(0)| + \int_0^\infty |g'(z)|dz\Big).\end{aligned}$$

The right hand side again goes to zero as $r \to 0$, therefore the only not trivial part in (3.5) is

$$\sin^2 r \int_0^\infty \frac{\bar{f}(0)\cosh z}{(\sinh^2 z + \sin^2 r)^{3/2}}dz = f(0)\int_0^\infty (y^2+1)^{-3/2}dy = f(0)$$

that completes the proof.

■

The following theorem generalizes Helgason's similar formula [4], [5] in the sense that it works also in odd dimensions. However there are differences: our differential operator is of half the degree than Helgason's one, and Helgason's mean value is normalized, i.e. it is defined to be $\sinh^{1-n} rMf(\omega, \cosh r)$.

Theorem 3.2. *Let $f \in C_c^\infty(\mathcal{L}^n)$. For timelike spheres in even dimensions*

$$f(\omega, 0) = \lim_{r\to 0}\frac{(n-3)!!\sinh r}{|S^{n-2}|(-1)^{[n/2]}}\mathcal{D}_t^{[n/2]}\Big(\cosh rMf(\omega, \cosh r)\Big)$$

and in odd dimensions

$$f(\omega, 0) = \lim_{r\to 0}\frac{(n-3)!!}{|S^{n-2}|(-1)^{[n/2]}}\mathcal{D}_t^{[n/2]}\Big(\cosh rMf(\omega, \cosh r)\Big).$$

Proof. First we observe, that the timelike spheres are rotation symmetric and the operator M commutes with the rotations around ω. This allows us to work only with functions rotation symmetric around ω that simplifies formula (3.1) into

$$Mf(\omega, \cosh r) = |S^{n-2}| \int_0^1 f\Big(\omega, \operatorname{arccosh}\Big(\frac{\cosh r}{t}\Big)\Big) \frac{\cosh^{n-1} r \sinh r (1-t^2)^\delta}{t^{n-1}(\cosh^2 r - t^2)^{1/2}} dt,$$

where $\delta = (n-3)/2$. We substitute $t = \cosh r / \cosh z$ and obtain

$$Mf(\omega, \cosh r) = |S^{n-2}| \sinh r \int_r^\infty f(\omega, z)(\sinh^2 z - \sinh^2 r)^\delta \cosh z dz. \tag{3.6}$$

Let

$$F_\delta(r) = \frac{1}{|S^{n-2}| \sinh r} Mf(\omega, \cosh r) = \int_r^\infty f(\omega, z)(\sinh^2 z - \sinh^2 r)^\delta \cosh z dz.$$

A simple differentiation gives

$$F'_\delta(r) = \begin{cases} -\cosh r f(\omega, r) & \text{if } \delta = 0 \\ -2\delta \cosh r \sinh r F_{\delta-1}(r) & \text{if } \delta > 0. \end{cases} \tag{3.7}$$

Recursively using this on (3.6), $[n/2]-1$ steps will give either $\delta = 0$ for n odd or $\delta = -1/2$ for n even. This proves the statement for n odd. For even n Theorem 3.1 completes the proof.

■

In odd dimensions there can not exist this type of representation for spacelike spheres. We shall show this in the proof of the following theorem.

Theorem 3.3. *Let $f \in C_c^\infty(\mathcal{L}^n)$. For spacelike spheres in even dimensions $n \geq 4$*

$$f(\omega, 0) = \lim_{r \to 0} \frac{(n-3)!! \sin r}{|S^{n-2}|} \mathcal{D}_s^{[n/2]}\Big(\cos r M f(\omega, \cos r)\Big).$$

Note that in dimension $n = 2$ an inversion formula is already proved in Theorem 3.1.

Proof of Theorem 3.3. Just like in the proof of Theorem 3.2, we need to work only with functions rotation symmetric around ω. Therefore formula (3.1) simplifies into

$$Mf(\omega, \cos r) = |S^{n-2}| \int_0^{\cos r} f\Big(\omega, \operatorname{arccosh}\Big(\frac{\cos r}{t}\Big)\Big) \frac{\cos^{n-1} r \sin r (1-t^2)^\delta}{t^{n-1}(\cos^2 r - t^2)^{1/2}} dt,$$

where $\delta = (n-3)/2$. By substituting $t = \cos r / \cosh z$ we obtain

$$Mf(\omega, \cos r) = |S^{n-2}| \sin r \int_0^\infty f(\omega, z)(\sinh^2 z + \sin^2 r)^\delta \cosh z dz. \tag{3.8}$$

Let

$$F_\delta(r) = \frac{1}{|S^{n-2}| \sin r} Mf(\omega, \cos r) = \int_0^\infty f(\omega, z)(\sinh^2 z + \sin^2 r)^\delta \cosh z dz.$$

A simple differentiation gives

$$F'_\delta(r) = \begin{cases} 0 & \text{if } \delta = 0 \\ 2\delta \cos r \sin r F_{\delta-1}(r) & \text{if } \delta > 0. \end{cases} \tag{3.9}$$

Using this on (3.8) recursively, $[n/2] - 1$ steps will give either $\delta = 0$ for n odd or $\delta = -1/2$ for n even. From this latter case Theorem 3.1 gives the result. ∎

As (3.9) clearly shows that no representation of the above kind can exist in odd dimensions, because Mf contains only some moments of f. The idea to look for other representations arises from this insufficiency.

4. Spherical means on a restricted set of spheres

In this section we consider representation of functions by their spherical integrals on spheres centered to points of the equator. (Note that any spacelike totalgeodesic can be chosen as equator by the homogenity of the space $\mathcal{L}^n$.) These spheres are the hyperplanes in our model. As in the previous section, we restrict the considerations onto the even functions, to make the mapping μ essentially one-to-one, but keep in mind that the timelike formulas again remain valid for functions of any type.

We shall use now all the spherical integrals $Mf(\omega, p)$, and from now on we think $Mf(\omega, p)$ as a transform of the function f into a function on the set of spheres

$\hat{H}(\omega, p)$. Then the dual transform M^t transforms a function F, integrable on the set of spheres $\hat{H}(\omega, p)$ as

$$M^t F(X) = \int_{X\in\hat{H};\hat{\omega}\in\Sigma_X^{n-1}} F(\hat{H}) d\hat{\omega}, \tag{4.1}$$

where $d\hat{\omega}$ is the Lorentzian measure on Σ_X^{n-1} and $\hat{\omega}$ is the normal of $\hat{H}$ at $X \in \mathcal{L}^n$, that is $d(\hat{\omega}, \eta) = 0$ for any $\eta \in T_X\hat{H}$. Let $\mathcal{H}^n$ be the manifold of the spheres $\hat{H}(\omega, p)$.

The following proposition follows easily from Lemma 2.3 and from the note after it.

Proposition 4.1. *The dual spherical integral transform of $F \in C^\infty(\mathcal{H}^n)$ is*

$$M^t F(\bar{\omega}, r) = \int_{S^{n-1},\langle\omega,\bar{\omega}\rangle>0} F(\hat{H}(\omega, \cosh r\langle\omega,\bar{\omega}\rangle)) \frac{|\sinh r|}{|\cosh^2 r\langle\omega,\bar{\omega}\rangle^2 - 1|} d\omega.$$

Call attention that F is parameterized according to the correspondence between $H(\bar{\omega}, p)$ and $\hat{H}(\bar{\omega}, p)$ while the function $M^t F$ is parameterized by the natural polar coordinates on $\mathcal{L}^n$.

The next result connects the spherical integral transform with the Radon transform on the Euclidean space. We define $\mathbb{E}^n = \mathbb{R}^n \setminus B^n$, where B^n is the open unit ball.

Theorem 4.2. *Let $g: \mathbb{E}^n \to \mathbb{R}$ be defined by $g(\omega, \cosh r) = f(\omega, r)/|\sinh r|$, $f \in C(\mathcal{L}^n)$. Then*

$$\bar{R}g(\bar{\omega}, p) = Mf(\bar{\omega}, p) \cdot \frac{1/2}{\sqrt{|p^2 - 1|}} \qquad \begin{cases} p \neq 1 & \text{for } \ n > 2 \\ p > 1 & \text{for } \ n = 2 \end{cases},$$

where $\bar{R}$ denotes the Euclidean Radon transform on $\mathbb{R}^n$.

Proof. First, f is well defined as an even function on $\mathcal{L}^n$. It is well known [5] that

$$2 \cdot \bar{R}g(\bar{\omega}, p) = \int_{S^{n-1}} g\Big(\omega \cdot \frac{p}{\langle\omega,\bar{\omega}\rangle}\Big) \frac{p^{n-1}}{|\langle\omega,\bar{\omega}\rangle|^n} d\omega.$$

On the other hand, by formula (3.1) we have

$$\begin{aligned}\frac{Mf(\bar{\omega}, p)}{\sqrt{|p^2-1|}} &= \int_{S^{n-1}_{\bar{\omega},p}} f\Big(\omega, \operatorname{arccosh}\Big(\frac{p}{\langle\omega,\bar{\omega}\rangle}\Big)\Big) \frac{p^{n-1}}{\langle\omega,\bar{\omega}\rangle^{n-1}(p^2 - \langle\omega,\bar{\omega}\rangle^2)^{1/2}} d\omega \\ &= \int_{S^{n-1}_{\bar{\omega},p}} g\Big(\omega \cdot \frac{p}{\langle\omega,\bar{\omega}\rangle}\Big)\Big(\frac{p^2}{\langle\omega,\bar{\omega}\rangle^2} - 1\Big)^{1/2} \frac{p^{n-1}\langle\omega,\bar{\omega}\rangle^{1-n}}{(p^2 - \langle\omega,\bar{\omega}\rangle^2)^{1/2}} d\omega\end{aligned}$$

that gives the statement immediately. ■

The above argument shows that $\bar{R}g(\bar{\omega},p)2\sqrt{|p^2-1|}$ is the sum of the two spacelike spherical integrals of f if $p<1$ and $n=2$.

Theorem 4.2 can be used to transfer most of the results known about the Radon transform on the Euclidean space to the spherical integral transform on the Lorentzian space. We make the transfer only for the support theorem and the inversion formula.

Theorem 4.3. *Let $f \in C(\mathcal{L}^n)$ (not necessarily even!), $f(\omega,r)\cosh^k r$ is bounded for all $k \geq 0$. If there exists an $A>1$ such that the spherical integrals $Mf(\omega,p)$ vanish for $p \geq A$ then f vanishes for $r \geq \operatorname{arccosh} A$.*

Proof. We prove this result by tracing it back to Helgason's support theorem [5] via our Theorem 4.2.

Let g be defined as in Theorem 4.2. Then $\bar{R}g(\bar{\omega},p) = Mf(\bar{\omega},p)\frac{1/2}{\sqrt{|p^2-1|}} = 0$ for $p>A$. Therefore we only have to show that g satisfies the conditions of Helgason's support theorem. One can obviously alter g near the unit sphere so that it becomes continuous on $\mathbb{R}^n$ hence we need only to show $\lim_{x\to\infty} g(x)|x|^m = 0$ for all $m \in \mathbb{N}$. This can be proved by the following sequence of equations.

$$\lim_{x\to\infty} g(x)|x|^m = \lim_{r\to\infty} g(\bar{\omega}\cosh r)\cosh^m r = \lim_{r\to\infty} f(\bar{\omega},r)\cosh^m r/\sinh r < \infty. \qquad ■$$

It is interesting to observe, that the support theorem provided by Theorem 4.3 needs vanishing of Mf only on timelike spheres!

To formulate the transferred inversion formula we need to define the operator Λ as

$$\Lambda\Phi(\omega,p) = \begin{cases} \dfrac{\partial^{n-1}}{\partial p^{n-1}}\Phi(\omega,p) & \text{if } n \text{ odd} \\ \dfrac{i}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\partial^{n-1}}{\partial t^{n-1}}\Phi(\omega,p)\dfrac{dt}{t-p} & \text{if } n \text{ even} \end{cases}$$

for Φ in the Schwartz space of the set of hyperplanes in $\mathbb{R}^n$.

Theorem 4.4. *If all the derivatives of $f \in S(\mathcal{L}^n)$ vanish at the equator, then*

$$cf = M^t\left(|1-p^2|\Lambda\left(\frac{Mf(\omega,p)}{2\sqrt{|1-p^2|}}\right)\right),$$

where $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

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Proof. Using Proposition 4.1 it is easy to verify that

$$M^t F(\omega, r) = |\sinh r| \bar{R}^* G(\omega, \cosh r)$$

for $G(\omega, p) = F(\hat{H}(\omega, p))/|1-p^2|$, where $\bar{R}^*$ is the Euclidean dual Radon transform. Substitute this and Theorem 4.2 into Helgason's Theorem 3.4 in [5]. ■

This result shows that the inversion is local in odd dimensions contrary the even dimensions, where the reconstruction needs Mf on all the spheres. However it hides the important aspect of the spherical integral transform, namely, that the spherical integral transform is invertible even if only the timelike spheres are taken into account.

To prove this, we need some facts about the spherical harmonics. Briefly, the spherical harmonics, $Y_{\ell,m}$ constitute a complete polynomial orthonormal system in the Hilbert space $L^2(S^{n-1})$. If $f \in C^\infty(S^{n-1} \times \mathbb{R}_+)$ and $f_{\ell,m}(p)$ is the corresponding coefficient of $Y_{\ell,m}(\omega)$ in the expansion of $f(\omega, p)$, ie. $f_{\ell,m}(p) = \int_{S^{n-1}} f(\omega, p)\overline{Y_{\ell,m}(\omega)} d\omega$, then the series $\sum_{\ell,m}^{\infty} f_{\ell,m}(p) Y_{\ell,m}(\omega)$ converges uniformly absolutely on compact subsets of $S^{n-1} \times \mathbb{R}$ to $f(\omega, p)$. For further references, including the Funk-Hecke theorem,

$$Y_{\ell,m}(\bar{\omega}) \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_{-1}^{1} C_m^\lambda(x)(1-x^2)^{\lambda - \frac{1}{2}} f(x) dx = \int_{S^{n-1}} f(\langle \omega, \bar{\omega} \rangle) Y_{\ell,m}(\omega) d\omega,$$

where C_m^λ is the Gegenbauer polynomial of degree m and $\lambda = (n-2)/2$, refer to [9]. Below we shall use the expansions

$$g(\varphi, p) = \sum_{m=-\infty}^{\infty} g_m(p) \exp(im\varphi) \quad \text{and} \quad g(\omega, p) = \sum_{\ell,m}^{\infty} g_{\ell,m}(p) Y_{\ell,m}(\omega) \tag{4.2}$$

in dimension two and in higher dimensions, respectively. In dimension two, φ will mean the angle of the respective unit vector to a fixed direction.

Now we give the spherical harmonic expansion of the spherical integral transform.

Proposition 4.5. (i) *If $f(\varphi, p) \in S(\mathcal{L}^2)$, then*

$$(Mf)_m(p) = 2 \int_{\operatorname{arccosh} p}^{\infty} f_m(q) \frac{\cos\left(m \arccos\left(\frac{p}{\cosh q}\right)\right)\sqrt{p^2-1}}{\sqrt{1-p^2/\cosh^2 q}} dq$$

for $p > 1$ and

$$(Mf)_m(p) = 2\int_0^\infty f_m(q)\frac{\exp\left(im\arccos\left(\frac{p}{\cosh q}\right)\right)\sqrt{1-p^2}}{\sqrt{1-p^2/\cosh^2 q}}dq$$

for $p < 1$.

(ii) *If* $f(\omega, p) \in S(\mathcal{L}^n)$ *and* $n > 2$ *then*

$$(Mf)_{l,m}(p) = \frac{|S^{n-2}|}{C_m^\lambda(1)}\int_{\max}^\infty f_{\ell,m}(q)\sqrt{|p^2-1|}\cosh^{n-2} q\times \\ \times\left(1-\frac{p^2}{\cosh^2 q}\right)^{\frac{n-3}{2}} C_m^\lambda\left(\frac{p}{\cosh q}\right)dq,$$

where $\max = \operatorname{arccosh}\max(1, p)$.

Proof. For (i), formula (3.1) can be read as

$$Mf(\alpha, p) = \int_I f\left(\alpha+\omega, \operatorname{arccosh}\left(\frac{p}{\cos\omega}\right)\right)\frac{p\sqrt{|p^2-1|}}{\cos\omega(p^2-\cos^2\omega)^{1/2}}d\omega, \tag{4.3}$$

where we used angles instead of vectors and $I = [-\pi/2, +\pi/2]$ if $p > 1$ and $I = [-\pi/2, 0] \cap \{\cos\omega < p\}$ or $I = [0, \pi/2] \cap \{\cos\omega < p\}$ for $p < 1$. Substituting the (4.2)-type expansions of Mf and f into this formula we obtain

$$(Mf)_m(p) = \int_I f_m\left(\operatorname{arccosh}\left(\frac{p}{\cos\varphi}\right)\right)\frac{\exp(im\varphi)p\sqrt{|p^2-1|}}{\cos\varphi(p^2-\cos^2\varphi)^{1/2}}d\varphi.$$

For $p > 1$, the domain of this integration is symmetric to the origin, therefore the imaginary part of $\exp(im\varphi)$ makes zero in the integral. For $p < 1$ the factor $\exp(im\varphi)$ remains and the substitution $\cos\varphi = p/\cosh q$ gives the desired formulas.

For (ii), the first to observe is that the domain of the integration is rotation symmetric around $\bar{\omega}$. Substituting (4.2) type spherical harmonic expansions of Mf and f into formula (3.1) and using the Funk-Hecke formula yield immediately to

$$(Mf)_{\ell,m}(p) = \frac{|S^{n-2}|}{C_m^\lambda(1)}\int_0^{\min(p,1)} f_{\ell,m}\left(\operatorname{arccosh}\left(\frac{p}{t}\right)\right)\frac{p^{n-1}\sqrt{|p^2-1|}}{t^{n-1}(p^2-t^2)^{1/2}}\times \\ \times(1-t^2)^{\lambda-\frac{1}{2}}C_m^\lambda(t)dt.$$

Substituting now $t = p/\cosh q$ we are lead to the desired formula.

■

To prove the inversion formulas by means of these spherical harmonic expansions we need the formula
(4.4)

$$\mathcal{M}\left(\frac{\cosh q-\cosh s}{\cosh q\cosh s}\right)^{n-2}=\int_q^s C_m^\lambda\left(\frac{\cosh r}{\cosh q}\right)\left(1-\frac{\cosh^2 r}{\cosh^2 q}\right)^{\frac{n-3}{2}}\times$$
$$\times C_m^\lambda\left(\frac{\cosh r}{\cosh s}\right)\left(\frac{\cosh^2 r}{\cosh^2 s}-1\right)^{\frac{n-3}{2}}\frac{\sinh r}{\cosh^{n-1} r}dr,$$

where

$$\mathcal{M}=\frac{\pi 2^{3-n}}{\Gamma(n-1)}\left(\frac{\Gamma(m+n-2)}{\Gamma(m+1)\Gamma(\lambda)}\right)^2,$$

$m\in\mathbb{Z}$, $n>2$, $\lambda=\frac{n-2}{2}$. This can be proved from the corresponding formula of [1] by a simple substitution.

The following theorem represents functions by their integrals on timelike spheres centered to points on the equator. Here we use differentiation of degree $n-1$. Compared to Theorem 3.2 here we pay higher degree differentiation for using less spheres for reconstructing f.

Theorem 4.6. *For $f\in S(\mathcal{L}^n)$ and the timelike spherical integrals we have* (i) *for $n\geq 3$*

$$f_{\ell,m}(s)=(-1)^{n-1}\frac{\Gamma(m+1)\Gamma(\lambda)}{\pi^{n/2}\Gamma(m+n-2)}\mathcal{D}_r^{n-1}F_{\ell,m}(s),$$

where $\mathcal{D}_r=\frac{d}{ds}\left(\frac{\cdot}{\sinh s}\right)$, $\lambda=(n-2)/2$ and

$$F_{\ell,m}(s)=-\int_s^\infty (Mf)_{\ell,m}(\cosh r)C_m^\lambda\left(\frac{\cosh r}{\cosh s}\right)\left(\frac{\cosh^2 r}{\cosh^2 s}-1\right)^{\frac{n-3}{2}}\times$$
$$\times\frac{\sinh s\cosh^{n-2} s}{\cosh^{n-1} r}dr.$$

(ii) *for $n=2$*

$$f_m(s)=\frac{-1}{\pi}\frac{d}{ds}\int_s^\infty (Mf)_m(\cosh r)\frac{\cosh(m\operatorname{arccosh}(\cosh r/\cosh s))}{\cosh r\sqrt{\cosh^2 r/\cosh^2 s-1}}dp.$$

Proof. (i) First we multiply $(Mf)_{l,m}(\cosh r)$, using the appropriate formula of Proposition 4.5 with

$$-C_m^\lambda\left(\frac{\cosh r}{\cosh s}\right)\left(\frac{\cosh^2 r}{\cosh^2 s}-1\right)^{\frac{n-3}{2}}\frac{\cosh^{n-2} s}{\cosh^{n-1} r}\sinh s$$

and integrate from s to ∞ by r. Denoting the result by $F_{\ell,m}$ we get

$$\begin{aligned}F_{\ell,m}(s) = -\int_s^\infty C_m^\lambda\left(\frac{\cosh r}{\cosh s}\right)\left(\frac{\cosh^2 r}{\cosh^2 s}-1\right)^{\frac{n-3}{2}}\frac{\cosh^{n-2} s}{\cosh^{n-1} r}\sinh s\times\\ \times\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_r^\infty f_{\ell,m}(q)C_m^\lambda\left(\frac{\cosh r}{\cosh q}\right)\left(1-\frac{\cosh^2 r}{\cosh^2 q}\right)^{\frac{n-3}{2}}\times\\ \times\cosh^{n-2} q\sinh r\,dq\,dr.\end{aligned}$$

Changing the order of the integrations we see

$$\begin{aligned}F_{\ell,m}(s) = \frac{-|S^{n-2}|}{C_m^\lambda(1)}\int_s^\infty f_{\ell,m}(q)(\cosh s\cosh q)^{n-2}\sinh s\times\\ \times\int_s^q C_m^\lambda\left(\frac{\cosh r}{\cosh s}\right)C_m^\lambda\left(\frac{\cosh r}{\cosh q}\right)\left(\frac{\cosh^2 r}{\cosh^2 s}-1\right)^{\frac{n-3}{2}}\times\\ \times\left(1-\frac{\cosh^2 r}{\cosh^2 q}\right)^{\frac{n-3}{2}}\frac{\sinh r}{\cosh^{n-1} r}dr\,dq.\end{aligned}$$

According to formula (4.4) we obtain

$$F_{\ell,m}(s) = \frac{\mathcal{M}|S^{n-2}|}{C_m^\lambda(1)}\int_s^\infty f_{\ell,m}(q)(\cosh q-\cosh s)^{n-2}\sinh s\,dq$$

that implies the formula by the observation

$$\mathcal{D}_r\Phi_k(s) = \begin{cases} f(s) & \text{if } k=0\\ -k\Phi_{k-1} & \text{if } k>0,\end{cases}$$

where

$$\Phi_k(s) = \int_s^\infty f_{\ell,m}(q)(\cosh q-\cosh s)^k\sinh s\,dq.$$

For (ii), we multiply the first formula of Proposition 4.5 with

$$\frac{\cosh(m\operatorname{arccosh}(p/\cosh s))}{p\sqrt{p^2-1}\sqrt{p^2/\cosh^2 s-1}}$$

and integrate from $\cosh s$ to ∞ by p. The result is

$$F_m(s) = \int_{\cosh s}^{\infty} \frac{\cosh(m \operatorname{arccosh}(p/\cosh s))}{p\sqrt{p^2/\cosh^2 s - 1}} \times \\ \times 2 \int_{\operatorname{arccosh} p}^{\infty} f_m(q) \frac{\cos(m \arccos(p/\cosh q))}{\sqrt{1 - p^2/\cosh^2 q}} dq dp.$$

Substituting $p = \cosh r$ and changing the order of the integrations lead to

$$F_m(s) = 2\int_s^{\infty} f_m(q) \int_s^q \frac{\cos(m \arccos(\cosh r/\cosh q))}{\sqrt{1 - \cosh^2 r/\cosh^2 q}} \times \\ \times \frac{\cosh(m \operatorname{arccosh}(\cosh r/\cosh s))}{\sqrt{\cosh^2 r/\cosh^2 s - 1}} \tanh r dr dq.$$

The inner integral is known to be $\pi/2$ [1] and so

$$F_m(s) = \pi \int_s^{\infty} f_m(q) dq$$

that gives the statement.

■

The connection between the timelike spherical integral transform and the exterior Radon transform [8] is striking. A singular value decomposition is possible from Quinto's result via Theorem 4.4.

After Theorem 4.6 the question of invertibility arises naturely for the spacelike spherical integral transform. The formula in (ii) of Proposition 4.5 in odd dimension shows that $(MF)_{\ell,m}$ for spacelike spheres contains only some finite moments of $f_{\ell,m}$, therefore the inversion is impossible in odd dimensions.

In even dimensions, the situation is much more complicated.

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