PROJECTIVE METRICS WITH QUADRATIC ELLIPSES

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Dedicated to Zoltán I. Szabó on his 70th birthday

Abstract. Considering projective-metric spaces having one or more quadratic ellipses, we prove variants of Buseman’s theorem for Minkowski geometries, and analogues for Hilbert geometries: A Minkowski geometry has a quadratic ellipse if and only if it is the Euclidean geometry; A Hilbert geometry has a quadratic ellipse if and only if it is a Cayley–Klein model of the hyperbolic geometry.

1. Introduction

Let \((\mathcal{M},d)\) be a metric space given by the set \(\mathcal{M}\) equipped with the metric \(d\). If \(\mathcal{M}\) is a projective space \(\mathbb{P}^n\), or an affine space \(\mathbb{R}^n \subset \mathbb{P}^n\), or a (not necessarily bounded) proper convex subset of an affine space \(\mathbb{R}^n \subset \mathbb{P}^n\), the metric is complete, continuous with respect to the usual topology of \(\mathbb{P}^n\), additive on the segments, and possesses the strict triangle inequality, then the pair \((\mathcal{M},d)\) is called projective-metric space of elliptic, parabolic or hyperbolic type, respectively, and the metric \(d\) is called projective.

For given points \(F_1, F_2 \in \mathcal{M}\), the focuses, and a given number \(a > d(F_1, F_2)/2\), (D1) an ellipse (ellipsoid in higher dimensions) is defined as the set
\[
\mathcal{E}_a^{d;F_1,F_2} := \{E : 2a = d(F_1, E) + d(E, F_2)\}.
\]
The distance \(2f := d(F_1, F_2)\) is called the (linear) eccentricity, and, obviously, the circles (spheres) are the ellipsoids of vanishing eccentricity.

As every isometry of a projective-metric space is a restriction of a projectivity \([9, (2.1)]\), and every projectivity keeps quadraticity, we deduce that every isomorphy of a projective-metric space maps ellipses to ellipses and also quadrics to quadrics. Thus, the extent to which ellipses in a projective-metric space are quadratic is an inner geometrical property of the projective-metric space. So the question arises

\[
\text{how much does the quadraticity of ellipses extend among projective-metric spaces.}
\]

There are only a few known answers. By a probably folkloric result Lemma 3.6 every ellipsoid is a quadric in constant curvature spaces. As a kind of reverse
statement, Busemann’s theorem [5, 25.4] states that a Minkowski geometry has a(!) quadratic sphere if and only if it is Euclidean.

The purpose of this paper is to provide several answers to question (1.1).

First, we prove in Theorem 4.1 that a Finslerian projective-metric space is Riemannian and hence of constant curvature by Beltrami’s theorem ([1]; see also [5, (29.3)]) if and only if every small sphere (circle) is quadratic. This extends Busemann’s theorem [5, 25.4].

Then we prove in Theorem 4.2 that a Hilbert geometry is a Cayley–Klein model of the hyperbolic geometry if and only if it has a(!) quadratic sphere (circle). For Hilbert geometries this is the analogue of Busemann’s theorem [5, 25.4].

Next, Theorem 5.3 extends Busemann’s theorem [5, 25.4] to ellipsoids (ellipses) with positive eccentricity by stating that a Minkowski geometry is the Euclidean geometry if and only if it has a(!) quadratic ellipsoid (ellipse).

Finally, Theorem 6.4 extends Theorem 4.2 to ellipsoids (ellipses) with positive eccentricity by stating that an analytic Hilbert geometry is a Cayley–Klein model of the hyperbolic geometry if and only if it has a(!) quadratic ellipsoid (ellipse). For Hilbert geometries this is the analogue of Theorem 5.3.

The last section discusses the analyticity of the Minkowski and Hilbert geometries, and formulates some conjectures.

2. Preliminaries

Points of \( \mathbb{R}^n \) \( (n \in \mathbb{N}) \) are denoted as \( A, B, \ldots \), vectors are \( \overrightarrow{AB} \) or \( a, b, \ldots \); however we use these latter notations also for points if the origin is fixed. The open segment with endpoints \( A \) and \( B \) is denoted by \( AB \). The open ray starting from \( A \) passing through \( B \) is \( AB \), and the line through \( A \) and \( B \) is denoted by \( AB \).

The affine ratio \((A, B; C)\) of the collinear points \( A, B \neq A \) and \( C \neq B \) satisfies \((A, B; C) \overrightarrow{BC} = \overrightarrow{AC}\). The cross ratio of the collinear points \( A, B \neq A \) and \( C \neq B, D \neq B, A \), is \((A, B; C, D) = (A, B; C)/(A, B; D) \) [5, page 243].

For a convex domain \( D \in \mathbb{R}^2 \) containing the origin \( O \in D \) we usually polar parameterize the boundary \( \partial D \) by \( r: [-\pi, \pi) \to \mathbb{R}^2 \) of the form \( r(\varphi) = r(\varphi)u_\varphi \), where \( r \) is the radial function of \( D \) with respect to \( O \), and \( u_\varphi = (\cos \varphi, \sin \varphi) \). We call a curve analytic if its coordinates depend on its arc length analytically.

Let \((\mathcal{M}, d)\) be a projective-metric space. Then \( \overrightarrow{\ell} := \mathcal{M} \cap \ell \) denotes its geodesic line on the projective straight line \( \ell \). The metric midpoint \( M \) of the focuses \( F_1, F_2 \) of an ellipse \((D_1)\) is called the metric center of the ellipse. The value \( a \) is the radius, and the segment of length \( 2a \) symmetric to \( M \) on line \( F_1F_2 \) id the major axis of the ellipse. We define the numerical eccentricity of an ellipse (ellipsoid) as

\[
\varepsilon = \begin{cases} 
\sin f/\sin a, & \text{if } (\mathcal{M}, d) \text{ is of elliptic type,} \\
f/a, & \text{if } (\mathcal{M}, d) \text{ is of parabolic type,} \\
\sinh f/\sinh a, & \text{if } (\mathcal{M}, d) \text{ is of hyperbolic type.}
\end{cases}
\]
Further, we have the notations $S_{d,O}^\varrho := E_{d,O,O}^\varrho$ for the sphere of radius $\varrho > 0$ with center $O$, $B_{d,O}^\varrho = \{ P : d(O,P) < \varrho \}$ for the open ball of radius $\varrho > 0$ with center $O$. We omit the metric from these notations if it is clear from the context.

2.1. CLASSICAL GEOMETRIES. A complete Riemannian manifold $\mathbb{M}^n$ of dimension $n \in \mathbb{N}$ is called an abstract rotational manifold with base point $O \in \mathbb{M}^n$ if the induced linear action of the isotropy group of $O$ on $\mathbb{T}_O \mathbb{M}^n$ is $O(n)$ [13].

The Riemannian metric on $\mathbb{M}^n$ is then completely described by its size function $\nu : [0, I_{\nu}) \to \mathbb{R}^+$ such that the geodesic sphere of radius $r$ and center $O$ in $\mathbb{M}^n$ is isometric to the Euclidean sphere of radius $\nu(r)$. This explains the notation $(\mathbb{M}^n, \nu)$. A complete abstract rotational manifold of real type is homogeneous if and only if it is of constant sectional curvature $\kappa$ [13]. In this case, a function $\mu : [0, I_{\nu}) \to [0, \infty)$, the projector function of $\mathbb{M}^n$ [8], exists such that the map $\bar{\mu} : \text{Exp}_O(pu) \to \mu(p)u$ from $\mathbb{M}^n$ into $\mathbb{R}^n$, where $u$ is a unit vector in the tangent space $T_O \mathbb{M}^n \cong \mathbb{R}^n$, takes geodesics into straight lines. From the quadratic model of the spaces of constant curvature one can easily read off the following:

<table>
<thead>
<tr>
<th>Classical geometries</th>
<th>$I_{\nu}$</th>
<th>$\kappa$</th>
<th>$\nu$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{H}^n$ (hyperbolic)</td>
<td>$\infty$</td>
<td>$-1$</td>
<td>$\sinh r$</td>
<td>$\tanh r$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$ (Euclidean)</td>
<td>$\infty$</td>
<td>$0$</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>$\mathbb{S}^n$ or $\mathbb{P}^n$ (elliptic)</td>
<td>$\pi/2$</td>
<td>$+1$</td>
<td>$\sin r$</td>
<td>$\tan r$</td>
</tr>
</tbody>
</table>

2.2. MINKOWSKI GEOMETRIES. Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^n$, symmetric to the origin $O$. The function $d_{\mathcal{I}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_{\mathcal{I}}(X,Y) = \inf \{ \lambda > 0 : \overrightarrow{XY}/\lambda \in \mathcal{I} \}$$

is a metric on $\mathbb{R}^n$ [5, IV.24], and is called Minkowski metric. Such a pair $(\mathbb{R}^n, d_{\mathcal{I}})$ is called Minkowski geometry (plane if $n = 2$), and $\mathcal{I}$ is called the indicatrix. We say that a Minkowski plane is analytic if $\partial \mathcal{I}$ is an analytic curve.

2.3. HILBERT GEOMETRIES. Let $\mathcal{I}$ be an open, strictly convex set in $\mathbb{R}^n$. The function $d_{\mathcal{I}} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ defined by $d_{\mathcal{I}}(X,X) = 0$, and by

$$d_{\mathcal{I}}(X,Y) = \frac{1}{2} | \ln(X,Y; A, B) |,$$

where $\overrightarrow{AB} = \mathcal{I} \cap XY$, if $X \neq Y$,

is a metric on $\mathcal{I}$ [5, page 297], and is called Hilbert metric. Such a pair $(\mathcal{I}, d_{\mathcal{I}})$ is called Hilbert geometry (plane if $n = 2$), and $\mathcal{I}$ is called the domain of the geometry. A Hilbert geometry bordered by an ellipsoid (ellipse) is a Cayley–Klein model of the hyperbolic geometry.

Hilbert geometries are isomorphic if and only if a projectivity map transforms their domain into each other. \hfill (2.1) \hfill (12, 15, 17)

We say that a Hilbert geometry is analytic if $\partial \mathcal{I}$ is an analytic curve.

\textsuperscript{2}Look for projective realization of constant curvature spaces in standard textbooks.
3. Utilities

The following technical lemmas are used in the next sections.

Lemma 3.1. For any collinear points $A, B, C, D$ satisfying $BC \subset AD$, and any point $D'$ out of the line $AD$, there is a unique perspectivity $\varpi$ such that $A = \varpi(A)$, $B' = \varpi(B)$, $C' = \varpi(C)$, $D' = \varpi(D)$ and $\overrightarrow{AB'} = \overrightarrow{C'D'}$.

Proof. Existence. Let $B' \in AD'$ be given by $(A, D'; B') = \sqrt{(A, D; B, C)}$ and let $P = DD' \cap BB'$. Then perspectivity $\varpi$ with center $P$ maps $AD$ to $AD'$ and maps $C$ into such a point $C'$ that satisfies

$$(A, D; B, C) = (A, D'; B', C') = \frac{(A, D'; B')}{(A, D'; C')},$$

hence $(A, D'; C') = 1/\sqrt{(A, D; B, C)} = 1/(A, D'; B')$.

Uniqueness. The condition implies $(A, D'; B')(A, D'; C') = 1$ that gives

$$(A, D'; B', C') = \frac{(A, D'; B')}{(A, D'; C')} = (A, D'; B')^2, \quad \text{hence } (A, D'; B') = \sqrt{(A, D; B, C)}.$$ 

As perspectivity keeps cross ratio, this gives $(A, D'; B') = \sqrt{(A, D; B, C)}$ that determines the perspectivity uniquely. This proves the lemma.

Lemma 3.2. The border of a convex domain in the affine plane is an analytic curve if and only if any one of its radial function is analytic.

Proof. Let $D$ be an open convex domain in the Euclidean plane containing the origin $O$. Let $s \mapsto p(s)$ be an arc length parametrization of $\partial D$, where $s \geq 0$, and let $\varphi \mapsto r(\varphi) = r(\varphi)u_\varphi$ be a polar parametrization of $\partial D$ on $[-\pi, \pi)$ such that $p(0) = r(-\pi)$. Then

$$s(\xi) = \int_{-\pi}^{\xi} |r(\varphi)|d\varphi = \int_{-\pi}^{\xi} \sqrt{r^2(\varphi) + r^2(\varphi)}d\varphi,$$

hence the function $s: \xi \mapsto s(\xi)$ is strictly monotonously increasing, and therefore its inverse function $\sigma: s(\xi) \mapsto \xi$ exists and is strictly monotonously increasing.

Assume the analyticity of $r$. Then, as $r$ is bounded from below by a positive number, the integrand on the right-hand side of (3.1) is analytic, and therefore $s$ is analytic. As $s(\xi)$ is positive by (3.1), the analyticity of $\sigma$ follows from the analytic inverse function theorem [7, Theorem 4.2], that implies the analyticity of $p(s) = r(\sigma(s)) = r(\sigma(s))u_{\sigma(s)}$.

Conversely, assume that $p$ is analytic. As the derivatives of the cosine and sine functions do not vanish simultaneously, $u_{\sigma(s)} = p(s)/|p(s)|$ proves that $\sigma$ is analytic. As the derivative $\dot{s}(t) = 1/\dot{s}(\sigma(t))$ vanishes nowhere, analyticity of $s$ follows again by the analytic inverse function theorem [7, Theorem 4.2]. Then the analyticity of $r(\xi) = \langle p(s(\xi)), u_\xi \rangle$ follows.

The lemma is proved.
Notice that $r(\varphi) = p(s(\varphi))$, hence $r(\varphi) = \langle p(s(\varphi)), u_\varphi \rangle$. Differentiating this and then substituting the derivative of (3.1) leads to

$$
\dot{r}(\varphi) = \langle \dot{p}(s(\varphi)), u_\varphi \rangle \sqrt{\dot{r}^2(\xi) + r^2(\xi)}.
$$

(3.2)

Let $\mathcal{E}$ be a convex domain in the plane, and let line $\ell$ intersect its boundary $\partial \mathcal{E}$ in points $\{A, B\} = \partial \mathcal{E} \cap \ell$. We have different points $O, F_1, F_2 \in \ell$ such that $O \in F_1F_2 \subsetneq AB$ and $F_1 \in AF_2$. Let the straight lines $\ell_1$ through $F_1$ and $\ell_2$ through $F_2$ be such that the point $E(\varphi) = \ell_1 \cap \ell_2$ is in $\partial \mathcal{E}$, where $\varphi$ is the directed angle $EOB$. Let $\alpha = EF_1B \angle$ and $\beta = EF_2B \angle$ be directed angles. It is clear, that angles $\alpha$, $\beta$ and $\varphi$ are bijective functions of each others.

Starting from an arbitrary $\varphi_0 \in (0, \pi/2)$ we define the sequence $\varphi_i \in (0, \pi/2) \cup (\pi, 3\pi/2)$ recursively by $\alpha(\varphi_{i+1}) = \alpha(\varphi_i) + \pi$, and $\beta(\varphi_{i+2}) = \beta(\varphi_{i+1}) - \pi$. (See Figure 1.) To simplify notations we introduce $\alpha_i := \alpha(\varphi_i)$, $\beta_i := \beta(\varphi_i)$ and $E_i := E(\varphi_i)$ for each $i \in \mathbb{N}$.

![Figure 1. Sequence of angles](image-url)

**Lemma 3.3.** If $i \to \infty$, then $\alpha_{2i}$, $\beta_{2i}$ and $\varphi_{2i}$ tend to zero, $\alpha_{2i+1}$, $\beta_{2i+1}$ and $\varphi_{2i+1}$ tend to $\pi$, and $\alpha_{2i+2}/\alpha_{2i}$ tends to $(F_1, F_2; A, B)$.

**Proof.** Observe that $\varphi_{2i} \in (0, \pi/2)$, $\varphi_{2i+1} \in (\pi, 3\pi/2)$, and

$$
\beta_{2i+2} < \alpha_{2i} < \beta_{2i} \quad \text{and} \quad \beta_{2i+1} < \alpha_{2i+1} < \beta_{2i-1} \quad (i \in \mathbb{N}),
$$

hence the sequences $\beta_{2i}$, $\beta_{2i+1}$, $\varphi_{2i}$, $\varphi_{2i+1}$, $\alpha_{2i}$ and $\alpha_{2i+1}$ all strictly monotonously decrease, hence they are convergent.

Assume $\lim_i \beta_{2i} > 0$. Then $\lim_i \beta_{2i+1} > \pi$, $\lim_i \alpha_{2i} > 0$, $\lim_i \alpha_{2i+1} > \pi$, and therefore $\lim_i \frac{\beta_{2i+2}}{\beta_{2i}} = 1$, hence $\lim_i \frac{\alpha_{2i}}{\beta_{2i}} = 1$. By the sine law the last limit gives $\lim_i \frac{d(F_3, E_{2i})}{d(F_2, F_3)} = 1$, that, by the continuity of the functions involved, implies

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and, by the sine law, the quotient of these is

\[ \frac{\tan \alpha_i}{\tan \beta_{i+2}} = \frac{\alpha_i \cos \beta_{i+1}}{\beta_{i+2} \cos \alpha_{i+1}} \]

and, by the sine law,

\[ \frac{\tan \alpha_{i+2}}{\tan \alpha_i} = \frac{\alpha_{i+2} \cos \beta_{i+2}}{\alpha_i \cos \beta_{i+1}}. \]

The quotient of these is

\[ \frac{\tan \alpha_{i+2}}{\tan \alpha_i} = \frac{\alpha_{i+2} \cos \beta_{i+2}}{\alpha_i \cos \beta_{i+1}}. \]

which immediately implies the last statement of the lemma.

In the plane, let \( r_1 \) and \( r_2 \) be analytic arc length parametrization of curves, such that \( r_1(0) = r_2(0) \) and \( r_1(0) = r_2(0) \). Let \( \ell \) be the line through \( r_1(0) \) that is orthogonal to \( r_1(0) \), and let \( F_1, F_2 \) and \( B \) be different points on \( \ell \) such that

1. either \( F_1J \supseteq F_2J \supseteq BJ \), where \( J = r_1(0) \) and \( r_1(0) = e_{2}(0) \),
2. or \( IF_1 \subseteq IF_2 \subseteq IB \), where \( I = r_1(0) \) and \( r_1(0) = e_{2}(0) \).

Let \( e \) be an analytic arc length parametrization of a curve, such that \( B = e(0) \) and \( e(0) = u_{\pi/2} \). Every point \( E = e(s) \) determines two straight lines \( \ell_1 := F_1E \) and \( \ell_2 := F_2E \) closing small angles \( \alpha \) and \( \beta \) with \( \ell \), respectively. For sufficiently small \( s > 0 \) these lines intersect the curves \( r_1 \) and \( r_2 \) in points \( C_1 = r_1(s_{1,1}(\alpha)) \), \( D_1 = r_2(s_{1,1}(\alpha)) \) and \( C_2 = r_1(s_{1,2}(\beta)) \), \( D_2 = r_2(s_{2,2}(\beta)) \), respectively.

Let \( \delta_1 = \langle r_1(s_{1,1}(\alpha)) - r_2(s_{1,1}(\alpha)), u_\alpha \rangle \) and \( \delta_2 = \langle r_1(s_{1,2}(\beta)) - r_2(s_{2,2}(\beta)), u_\beta \rangle \).

**Lemma 3.4.** If \( E \) tends to \( B \) on the curve \( e \), \( K = r_1(0) \) and \( \delta_2(\beta) \neq 0 \), then

\[ \frac{\delta_1(\alpha)}{\delta_2(\beta)} \rightarrow (F_1, F_2; K, B)^k, \quad \text{where integer } k \geq 2. \]
Proof. If \( r_1^{(i)}(0) = r_2^{(i)}(0) \) for every \( i \in \mathbb{N} \), then, by the analyticity of \( r_1 \) and \( r_2 \), there does not exist any sequence \( \beta_j \) tending to 0 while \( \delta_2(\beta_j) \neq 0 \). Thus, \( k := \min\{i \in \mathbb{N} : r_1^{(i)}(0) \neq r_2^{(i)}(0)\} \) is well defined and is at least two.

If \( \lim_{s \to 0} \frac{\delta_1(\alpha)}{\delta_2(\beta)} \) exists, we may calculate by L’Hospital’s rule that results in

\[
\lim_{s \to 0} \frac{\delta_1(\alpha)}{\delta_2(\beta)} = \lim_{s \to 0} \frac{\delta_1(\alpha) \dot{\alpha}}{\delta_2(\beta) \dot{\beta}} = \lim_{s \to 0} \frac{\delta_1(\alpha)}{\delta_2(\beta)} \lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\beta}} = \cdots = \lim_{s \to 0} \frac{\delta_1^{(k)}(\alpha)}{\delta_2^{(k)}(\beta)} \left( \lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\beta}} \right)^k.
\]

In one hand, taking the orthogonal projection \( E^\perp \) of \( E \) onto \( \ell \), and using L’Hospital’s rule we obtain

\[
\frac{|F_2 - B|}{|F_1 - B|} = \lim_{s \to 0} \frac{|F_2 - E^\perp|}{|F_1 - E^\perp|} = \lim_{s \to 0} \frac{\tan \alpha}{\tan \beta} = \lim_{s \to 0} \frac{(1 + \tan^2 \alpha) \frac{d\alpha}{ds}}{(1 + \tan^2 \beta) \frac{d\beta}{ds}} = \lim_{s \to 0} \frac{\dot{\alpha}}{\dot{\beta}}. \tag{3.5} \tag{8, 13}
\]

In the other hand, the \( k \)-th derivative of \( \delta_j \) \((j = 1, 2)\) is of the form

\[
\delta_j^{(k)}(\xi) = \langle r_1^{(k)}(s_{1,j}(\xi)) s_{1,j}^k(\xi) - r_2^{(k)}(s_{2,j}(\xi)) s_{2,j}^k(\xi) + f(\xi), u_\xi \rangle + \langle g(\xi), u_{\xi + \pi/2} \rangle,
\]

where vectors \( f \) and \( g \) are composed from derivatives \( r_1^{(l)} \) multiplied by a sum of products of various derivatives of the form \( s_{i,j}^{(m)}(\xi) \) \((i = 1, 2, l < k \text{ and } m \leq k - l)\).

As, clearly, \( r_1^{(l)}(0) = r_2^{(l)}(0) \) and \( s_{1,j}^{(m)}(0) = s_{2,j}^{(m)}(0) \) for every \( l < k \text{ and } m \leq k - l \), we get \( \delta_j^{(k)}(0) = \langle r_1^{(k)}(0) - r_2^{(k)}(0), u_0 \rangle s_{1,j}^k(0) \). Hence

\[
\lim_{s \to 0} \frac{\delta_1^{(k)}(\alpha)}{\delta_2^{(k)}(\beta)} = \left( \frac{s_{1,1}(0)}{s_{1,2}(0)} \right)^k = \left( \frac{|K - F_1|}{|K - F_2|} \right)^k.
\]

Notice that \( \tilde{\delta}_j(0) = \langle \tilde{r}_1(0) - \tilde{r}_2(0), u_0 \rangle s_{1,j}^2(0) = \pm (\kappa_1(0) - \kappa_2(0)) s_{1,j}^2(0) \), where \( \kappa_1 \) and \( \kappa_2 \) are the signed curvatures of the curves \( r_1 \) and \( r_2 \), respectively. Thus, the signed curvatures of the curves coincide if and only if \( k \geq 3 \).

Now supplement the previous configuration with two further straight lines \( \tilde{\ell}_1 \) and \( \tilde{\ell}_2 \) such that they pass through the midpoint \( O \) of the segment \( F_1 F_2 \), and close angles \( \alpha \) and \( \beta \) with \( \ell \), respectively.
Lemma 3.6. \textit{If }\delta(\alpha)/\delta(\beta)\textit{ is quadratic, but as no good reference was found, we prove it here for the case }\Gamma\textit{. This proves the lemma by (3.5).}

Proof. \textit{If }\mathbf{r}_1^{(i)}(0) = \mathbf{r}_2^{(i)}(0)\textit{ for every }i \in \mathbb{N},\textit{ then, by the analyticity of }\mathbf{r}_1\textit{ and }\mathbf{r}_2,\textit{ there does not exist any sequence }\beta_j\textit{ tending to }0\textit{ while }\delta(\beta_j) \neq 0.\textit{ Thus, }k := \min\{i \in \mathbb{N} : \mathbf{r}_1^{(i)}(0) \neq \mathbf{r}_2^{(i)}(0)\}\textit{ is well defined and is at least two.}

If \text{lim}_{s \to 0} \delta(\alpha)/\delta(\beta)\text{ exists, we may calculate by L'Hospital's rule that results in}

$$\lim_{s \to 0} \frac{\delta(\alpha)}{\delta(\beta)} = \lim_{s \to 0} \frac{\hat{\delta}(\alpha)\hat{\alpha}}{\hat{\delta}(\beta)\hat{\beta}} = \lim_{s \to 0} \frac{\hat{\delta}(\alpha)}{\hat{\delta}(\beta)} \lim_{s \to 0} \frac{\hat{\alpha}}{\hat{\beta}} = \cdots = \lim_{s \to 0} \frac{\delta^{(k)}(\alpha)}{\delta^{(k)}(\beta)} \lim_{s \to 0} \frac{\hat{\alpha}}{\hat{\beta}} = \left(\lim_{s \to 0} \frac{\hat{\alpha}}{\hat{\beta}}\right)^k. $$

This proves the lemma by (3.5). \hfill \Box

It is probably a folkloric fact that every ellipse in every space of constant curvature is quadratic, but as no good reference was found, we prove it here for the sake of completeness.

Lemma 3.6. \textit{Let }\mathcal{E}^\alpha_{d,F_1,F_2}\textit{ be an ellipse with metric center }O\textit{ in a }2\text{-dimensional Riemann manifold of constant curvature }\kappa \in \{-1,0,1\}.\textit{ The polar equation of }\mathcal{E}^\alpha_{d,F_1,F_2}\textit{ in the geodesic polar coordinatization with base point }O\textit{ is of the form}

$$\frac{1}{\nu^2(r(\omega))} = \frac{\cos^2 \omega}{\nu^2(a)} + \frac{\sin^2 \omega}{(\mu^2(a) - \mu^2(f))(1 - \kappa \nu^2(a))}. \quad (3.7) \quad (17)$$

Proof. \textit{Case of hyperbolic plane.} We use the quadratic model: on }\mathbb{H}_2 = \{(x, y, z) : x^2 + y^2 - z^2 = -1, z \geq 1\}, \textit{the upper sheet of the hyperboloid }z^2 - x^2 - y^2 = 1 \textit{of two sheets, the metric is }d(p, q) = \cosh^{-1}(p, q). \textit{Fix }f > 0\text{ and }a > f,\text{ and let }F_1 = (\sinh(-f), 0, \cosh f)\text{ and }F_2 = (\sinh f, 0, \cosh f).\textit{ For any point }P = (x, y, z) \in \mathcal{E}^\alpha_{d,F_1,F_2} \subset \mathbb{H}_2\textit{ there is a }t \in [0, f]\textit{ such that}

$$a + t = d(F_1, X) = \cosh^{-1}(F_1, X) = \cosh^{-1}(-x \sinh f + z \cosh f),$$

$$a - t = d(X, F_2) = \cosh^{-1}(X, F_2) = \cosh^{-1}(x \sinh f + z \cosh f).$$

Taking \cosh gives \cosh a \cosh t \pm \sinh a \sinh t = \cosh(a \pm t) = z \cosh f \mp x \sinh f,\textit{ hence }\cosh a \cosh t = z \cosh f \textit{ and }\sinh a \sinh t = -x \sinh f.\textit{ This implies}

$$\frac{x^2}{\cosh^2 a/\cosh^2 f} - \frac{z^2}{\cosh^2 a/\sinh^2 f} = \cosh^2 t - \sinh^2 t = 1,$$

hence

$$\frac{z^2 - x^2 \tanh^2 f}{\tanh^2 a} = \frac{\cosh^2 a}{\cosh^2 f} = \frac{1 - \tanh^2 f}{1 - \tanh^2 a}. $$

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The metric center $O$ of $E_{a,F_1,F_2}$ is $(0,0,1)$, therefore if the polar coordinates in $\mathbb{H}^2$ with respect to $O$ for $P$ is $(\omega, r)$, then $P = (\sinh r \cos \omega, \sinh r \sin \omega, \cosh r)$. Substituting this into the above equation results in

$$\frac{1}{\sinh^2 r} = \frac{\cos^2 \omega}{\sinh^2 a} + \frac{\sin^2 \omega}{(\tanh^2 a - \tanh^2 f) \cosh^2 a}.$$ 

The cases of the Euclidean and of the elliptic plane are so much analogous to the hyperbolic case, that we leave those proofs for the reader. □

Finally, we note that if $k \in \{2, \ldots, n - 1\}$ ($n \geq 3$), then, by [2, (16.12), p. 91],

the border $\partial K$ of a convex body $K \in \mathbb{R}^n$ is an ellipsoid if and only if every $k$-plane through an inner point of $K$ intersects $\partial K$ in a $k$-dimensional ellipsoid. (3.8) (9, 10, 14)

4. Projective-metric spaces with quadratic spheres (circles)

**Theorem 4.1.** A Finslerian projective-metric space is Riemannian and hence of constant curvature if and only if every small sphere (circle) is quadratic.

**Proof.** By Lemma 3.6 we only need to prove that if every small sphere (circle) is quadratic in a Finslerian projective-metric space $(M, d_F)$ of Finsler function $F$, then it is Riemannian and of constant curvature. For this, by Beltrami’s theorem it is enough to prove that the unit sphere $\{v \in T_P M : F(P, v) = 1\}$ in the tangent space $T_P M$ is quadratic for every point $P \in M$. According to (3.8), this should be done only in dimension two, i.e. $M \subseteq \mathbb{P}^2$.

All in one, we need to consider a Finslerian projective-metric $d_F$ given in a connected open bounded domain $\mathcal{N}$ of $\mathbb{R}^2$ with a Finsler function $F : TN \to \mathbb{R}_+$, where every circle $S_{d_F;O}^\rho$ with small enough radius $\rho > 0$ is quadratic.

Fix a point $O$. Then for every circle $S_{d_F;O}^\rho$ we have an affine ellipse with a center $C^\rho$. Let $A^\rho$ and $B^\rho$ be the intersections of $S_{d_F;O}^\rho$ with $OC^\rho$ such that $O \in C^\rho B^\rho$. Let $D^\rho$ be the point symmetric to $O$ in $C^\rho$. Finally, let $\ell$ be a straight line through $O$, and let $A^\rho_\ell$ and $B^\rho_\ell$ be the points, where $k$ intersects $S_{d_F;O}^\rho$. See Figure 2 for what we have.

![Figure 2. Ellipse of $S_{d_F;O}^\rho$](image-url)
Clearly, there are Euclidean metrics $d_{\rho,a}$ such that $S^\rho_{d_F;O} = \mathcal{E}^a_{d_{\rho,a};D^\rho,O}$. These metrics are multiples of each other by positive reals, and satisfy $d_{\rho,a}(C^\rho, B^\rho) = a$, and $d_{\rho,a}(C^\rho, O) = a(1 - (O, C^\rho; B^\rho)) =: f$. Then $\varepsilon_\rho = 1 - (O, C^\rho; B^\rho)$ is the numerical eccentricity of $\mathcal{E}^a_{d_{\rho,a};D^\rho,O}$ with respect to $d_{\rho,a}$, and we have
\[
\frac{d_F(A_\ell, O)}{d_{\rho,a}(A_\ell, O)} + \frac{d_F(B_\ell, O)}{d_{\rho,a}(B_\ell, O)} = \frac{\rho}{d_{\rho,a}(A_\ell, O)} + \frac{\rho}{d_{\rho,a}(B_\ell, O)}
= \rho \left( \frac{1}{d_{\rho,a}(A_\ell, O)} + \frac{1}{d_{\rho,a}(B_\ell, O)} \right)
= \frac{2a\rho}{a^2 - f^2} = \frac{2\rho/a}{1 - \varepsilon_\rho^2},
\]
where the second row follows from the equation of an ellipse in polar coordinates with a focus as origin.

Let $a_\rho = \rho/(1 - \varepsilon_\rho^2)$. By the Busemann–Mayer theorem [6] we have
\[
\lim_{\rho \to 0} \frac{d_F(A_\ell, O)}{d_{\rho,a}(A_\ell, O)} = \lim_{\rho \to 0} \frac{d_F(B_\ell, O)}{d_{\rho,a}(B_\ell, O)} = \lim_{\rho \to 0} F\left( O, \frac{A_\ell - O}{d_{\rho,a}(A_\ell, O)} \right),
\]
hence
\[
1 = \lim_{\rho \to 0} \frac{\rho/a_\rho}{1 - \varepsilon_\rho^2} = \lim_{\rho \to 0} \frac{F\left( O, \frac{A_\ell - O}{d_{\rho,a}(A_\ell, O)} \right) + F\left( O, \frac{O - B_\ell}{d_{\rho,a}(B_\ell, O)} \right)}{2}
\]
\[
= \lim_{\rho \to 0} \left( F\left( O, \frac{A_\ell - O}{d_{\rho,a}(A_\ell, O)} \right) \frac{1}{2} + \frac{d_{\rho,a}(A_\ell, O)}{d_{\rho,a}(A_\ell, O)} \frac{O - B_\ell}{d_{\rho,a}(B_\ell, O)} \frac{1}{2} \right)
\]
\[
= \lim_{\rho \to 0} F\left( O, \frac{A_\ell - O}{d_{\rho,a}(A_\ell, O)} \right) = F\left( O, \lim_{\rho \to 0} \frac{A_\ell - O}{d_{\rho,a}(A_\ell, O)} \right).
\]
Thus, the unit circle of $F(O, \cdot)$ is the limit of the unit circles of $d_{\rho,a}$. This means that the closed quadratic curves given by the unit circles of $d_{\rho,a}$ converge to the strictly convex closed curve given by the unit circle of $F(O, \cdot)$, hence the unit circle of $F(O, \cdot)$ is a quadratic closed curve, i.e. an ellipse.

Thus, $(\mathcal{M}, d_F)$ is a Riemannian projective metric-space, and therefore it is of constant curvature by Beltrami’s theorem.

**Theorem 4.2.** If a Hilbert geometry has a quadratic sphere (circle), then it is a Cayley–Klein model of the hyperbolic geometry.

**Proof.** By (3.8), we need to consider only a Hilbert plane $(\mathcal{I}, d_\mathcal{I})$, where $\mathcal{I} \subset \mathbb{R}^2$ is a strictly convex domain.
Let the circle $\mathcal{E}_{d_1;F,F}^a$ be quadratic with affine center $O$. Let $\ell$ be a straight line through $O$ and $F$. Let $A, B$ and $I, J$ be the intersections of $\ell$ with $\mathcal{E}_{d_1;F,F}^a$ and $\partial \mathcal{I}$, respectively, and for having clear notations, assume $A \in IT$. Finally, let $t_A$ and $t_B$ be the tangents of $\mathcal{E}_{d_1;F,F}^a$ at $A$ and $B$, respectively, and let $t_I$ and $t_J$ be the tangents of $\mathcal{I}$ at $I$ and $J$, respectively.

Choose the Euclidean metric $d_e$ such that the quadric $\mathcal{E}_{d_1;F,F}^a$ is a unit circle with respect to $d_e$. Then $O$ is the Euclidean center and $t_A \perp \ell \perp t_B$.

Let $E(\varphi)$ be the point of $\mathcal{E}_{d_1;F,F}^a$, where $\varphi = \angle(E(\varphi)FB)$. As point $E(\varphi)$ moves along $\mathcal{E}_{d_1;F,F}^a$, we have the functions $e(\varphi) = d_e(F, E(\varphi))$ and $r(\varphi) = d_e(F, V_0(\varphi))$, where $V_0(\varphi) = FE(\varphi) \cap \partial \mathcal{I}$. Taking $U_0(\varphi) = E(\varphi)F \cap \partial \mathcal{I}$, we have

$$a = d_\mathcal{I}(F, E(\varphi))$$

$$= \frac{1}{2} \ln|U_0(\varphi), V_0(\varphi); F, E(\varphi)| = \frac{1}{2} \ln \left( \frac{r(\varphi + \pi)}{r(\varphi)} \right) \left( \frac{r(\varphi + \pi) + e(\varphi)}{r(\varphi) - e(\varphi)} \right)$$

$$= -\frac{1}{2} \ln \left( \frac{r(\varphi + \pi)}{r(\varphi)} \frac{r(\varphi) - e(\varphi)}{r(\varphi + \pi) + e(\varphi)} \right).$$

As the positive values in the brackets are clearly smaller than 1, it follows that

$$e^{2a}r(\varphi + \pi)(r(\varphi) - e(\varphi)) = (r(\varphi + \pi) + e(\varphi))r(\varphi).$$

With some rearrangements this becomes

$$\frac{e^{2a} - 1}{e(\varphi)} = r(\varphi + \pi) + e^{2a}r(\varphi) \quad \text{and} \quad \frac{e^{2a} - 1}{e(\varphi + \pi)} = r(\varphi) + e^{2a}r(\varphi + \pi), \quad (4.1)$$

FIGURE 3. Metric circle in the Hilbert plane
where \( \bar{r}(\varphi) := 1/r(\varphi) \). Multiplying the first equation by \( e^{2a} \), then subtracting it from the second one, and dividing the result by \( 1 - e^{2a} \) lead to
\[
\frac{e^{2a}}{e(\varphi)} - \frac{1}{e(\varphi + \pi)} = \bar{r}(\varphi)(1 + e^{2a}).
\]

As \( \dot{e}(0) = \dot{e}(\pi) = 0 \), the last equation proves \( \dot{r}(0) = 0 \) and \( \dot{r}(\pi) = 0 \), that means \( t_I \parallel t_J \perp t_A \parallel t_B \).

By Lemma 3.1, we have a perspectivity \( \varpi \) such that \( I = \varpi(I), \ A' = \varpi(A), \ B' = \varpi(B), \ J' = \varpi(J) \) and \( \bar{IA}' = \bar{B'}J' \), meanwhile \( \varpi(t_I) = t_I \parallel t_{A'} \parallel t_{B'} \parallel t_{J'} \) holds true as well. As perspectivity keeps cross ratio, (2.1) allows us to assume \( \bar{IA} = \bar{BJ} \) without loss of generality.

Having \( \bar{IA} = \bar{BJ} \), it follows immediately, that \( O \equiv F \), and therefore, by (4.2), for every straight line \( \ell \) through \( F \) the tangents \( t_A, \ t_B, \ t_I \) and \( t_J \) are parallel. As every straight line \( \ell \) through \( F \) intersects \( \mathcal{E}^a_{d_\mathcal{I};F,F} = \mathcal{E}^{1a}_{d_\mathcal{I};F,F} \) orthogonally, this implies that every \( \ell \) intersects \( \mathcal{I} \) orthogonally too. This means that \( \mathcal{I} \) is a circle with respect to \( d_e \), which proves the theorem in this case. \( \square \)

5. MINKOWSKI GEOMETRIES WITH A QUADRATIC ELLIPSOID (ELLIPSE)

We start by considering the Minkowski plane \( (\mathbb{R}^2, d_\mathcal{I}) \) with indicatrix \( \mathcal{I} \), and ellipse \( \mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \) with different focuses.

![Figure 4. Ellipse \( \mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \) in a Minkowski plane](image)

Let \( A, B \) be the intersections of line \( F_1F_2 \) with \( \mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \) such that \( F_1 \in \overline{AF_2} \) and \( F_2 \in \overline{F_1B} \). Let \( \mathcal{I}_O \) be the indicatrix centered at the midpoint \( O \) of \( F_1F_2 \), and denote the intersections of line \( F_1F_2 \) with \( \partial \mathcal{I}_O \) by \( I, J \), so that \( I \in \overline{OF_1} \) and \( J \in \overline{OF_2} \). Further, let \( t_A, t_B \) and \( t_I, t_J \), respectively, denote the tangents of the appropriate curve \( \mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \) or \( \partial \mathcal{I}_O \) at \( A, B \) and \( I, J \), respectively.

For any fixed Euclidean metric \( d_e \) we let \( r \) be the radial function of \( \mathcal{I}_O \) with respect to \( O \), \( \alpha = \angle(EF_1O), \ \beta = \angle(EF_2B) \) and \( \varphi = \angle(EOB) \). We consider these angles as functions \( \alpha(\varphi), \ \beta(\varphi) \), and also define the functions \( e_1(\alpha) = d_e(F_1,E), \)
Lemma 5.1. If the ellipse $E_{d_{\mathcal{I}}:F_1:F_2}$ in the Minkowski plane $(\mathbb{R}^2,d_{\mathcal{I}})$ is quadratic, then the tangents $t_A$, $t_B$, $t_I$ and $t_J$ are all parallel.

Proof. The symmetry of $\mathcal{I}$ entails that $t_I \parallel t_J$, and it also follows that the affine center of the quadric $E_{d_{\mathcal{I}}:F_1:F_2}$ coincides with its metric center $O$, hence $t_A \parallel t_B$.

Choose the Euclidean metric $d_e$ so that $E_{d_{\mathcal{I}}:F_1:F_2}$ is a circle of radius $a$. With this metric, we have $t_A \perp F_1 F_2 \perp t_B$.

Differentiation of (5.1) with respect to $\varphi$ gives

$$0 = \frac{\dot{e}_1(\alpha)r(\alpha) - e_1(\alpha)r(\alpha)}{r(\alpha)} \dot{\alpha} + \frac{\dot{e}_2(\beta)r(\beta) - e_2(\beta)r(\beta)}{r(\beta)} \dot{\beta}.$$ 

As $\dot{e}_1(0) = \dot{e}_2(0) = 0 = \dot{e}_1(\pi) = \dot{e}_2(\pi)$, this gives at $\varphi = 0$ and at $\varphi = \pi$ that

$$0 = \dot{r}(0)(e_1(0)\dot{\alpha}(0) + e_2(0)\dot{\beta}(0)),$$
$$0 = \dot{r}(\pi)(e_1(\pi)\dot{\alpha}(\pi) + e_2(\pi)\dot{\beta}(\pi)),$$

respectively. Applying (3.5) for the present configuration, we obtain

$$e_1(0)\dot{\alpha}(0) = e_2(0)\dot{\beta}(0) \neq 0 \quad \text{and} \quad e_1(\pi)\dot{\alpha}(\pi) = e_2(\pi)\dot{\beta}(\pi) \neq 0,$$

which prove $\dot{r}(0) = \dot{r}(\pi) = 0$ in (5.2), hence the lemma. \hfill \Box

Lemma 5.2. If the ellipse $E_{d_{\mathcal{I}}:F_1:F_2}$ in the Minkowski plane $(\mathbb{R}^2,d_{\mathcal{I}})$ is an analytic curve, then the curve $\partial \mathcal{I}_O$ is analytic in a neighborhood of $I$ and $J$.

Proof. By the proof of Lemma 3.2 the functions $\alpha \mapsto e_1$, $\beta \mapsto e_2$, $s \mapsto \alpha$ and $s \mapsto \beta$, and the inverses $\alpha \mapsto s$ and $\beta \mapsto s$ are analytic. Specifically these imply that $\dot{\beta}(\alpha)$ is also an analytic function.

As $x \mapsto 1/x$ is analytic in a neighborhood of $r(0) > 0$, $r(\beta)$ is analytic in a neighborhood of the zero too, so we only need to prove that $\ddot{r}(\beta) := 1/r(\beta)$ is analytic in a neighborhood of the zero. With this in mind we reformulate (5.1) as

$$\ddot{r}(\beta) = \frac{e_1(\alpha(\beta))}{-e_2(\beta)} \ddot{r}(\alpha(\beta)) + \frac{2a}{e_2(\beta)}.$$ 

Using the notations in formula [7, (4.20)], we have $\phi(\beta) := \ddot{r}(\beta)$, $f(\beta) := \alpha(\beta)$, $g(\beta) := \frac{e_1(\alpha(\beta))}{-e_2(\beta)}$, and $h(\beta) := \frac{2a}{e_2(\beta)}$; and these functions are analytic in a neighborhood of 0, $f(0) = 0$, $\dot{f}(0) = \frac{e_2(0)}{e_1(0)} < 1$, $g(0) = \frac{-e_2(0)}{e_1(0)} \in (-1,0)$, $h(0) = 2a/e_2(0)$, and $\phi(0) = 1/r(0)$. By [7, Theorem 4.6], we deduce that $\phi$ is unique and analytic in a neighborhood of 0, because equation [7, (4.20)] has only one solution for the sequence $c_k$ starting with $c_0 = h(0)/(1-g(0)) = \frac{2ae_1(0)/e_2(0)}{e_1(0)+e_2(0)}$. \hfill \Box
Theorem 5.3. A Minkowski geometry has a quadratic ellipsoid if and only if it is Euclidean.

Proof. As every ellipsoid in the Euclidean geometry is a quadric, we only have to prove that a Minkowski geometry with a quadratic ellipsoid is Euclidean.

By (3.8), we only need to consider a Minkowski plane \((\mathbb{R}^2, d_I)\), so we can use all our notations and lemmas above.

Assume that \(E_{d_I}^a: F_1, F_2\) is a quadric.

If \(F_1 = F_2\), then \(E_{d_I}^a: F_1, F_2\) is a homothetic image of \(\partial I\), hence \(\partial I\) is quadratic, and therefore \(d_I\) is Euclidean by [5, 25.4].

From now on we assume that \(F_1 \neq F_2\).

We have \(t_A \parallel t_I \parallel t_J \parallel t_B\) by Lemma 5.1, and, as every (plane) quadric is an analytical curve, the border \(\partial I_O\) is analytic in a neighborhood of \(I\) and \(J\) by Lemma 5.2, where \(O\) is the midpoint of \(F_1F_2\). Further, by the central symmetry of \(I_O\) and the definition of \(E_{d_I}^a: F_1, F_2\), we have \(f = d_I(F_1, O), AF_1^3 = F_2B\) and \(\overrightarrow{IA} = \overrightarrow{BJ}\), so \(O\) is the (affine) midpoint of both \(IJ\) and \(AB\). Additionally, we have \(a \cdot d_I(O, J) = d_I(O, B)\), because the definition of \(E_{d_I}^a: F_1, F_2\) implies

\[
2d_I(O, B) = 2d_I(O, F_2) + 2d_I(F_2, B) = d_I(F_1, O) + d_I(O, F_2) + d_I(F_2, B) + 2a - d_I(F_1, B) = 2a.
\]

Denote by \(C_1\) and \(C_2\) the points where the line through \(O\) parallel to \(t_f\) intersects \(E_{d_I}^a: F_1, F_2\). Fix the affine coordinate system, where \(O = (0, 0), J = (1, 0)\) and \(C_1 = (0, \sqrt{a^2 - f^2})\), and choose the Euclidean metric \(d_e\) such that \(\{(1, 0), (0, 1)\}\) is an orthonormal basis. Let \(C\) denote the unit circle of \(d_e\).

Then both \(E_{d_e}^a: F_1, F_2\) and \(E_{d_I}^a: F_1, F_2\) are quadratic curves, and they have four common points \(A, B\) and \(C_1, C_2\), and two common tangents \(t_A\) and \(t_B\), hence they coincide.

Figure 5. Coinciding ellipse \(E_{d_I}^a: F_1, F_2 \equiv E_{d_e}^a: F_1, F_2\).
By the definition of $\mathcal{E}_{d_x: F_1, F_2}^a$ we have $e_1(\alpha) + e_2(\beta) = 2a$. Using this fact and introducing $\delta: \varphi \mapsto r(\varphi) - 1$ into (5.1) lead to
\[ \delta(\alpha) = -\delta(\beta) \frac{e_2(\beta)}{e_1(\alpha) + 2a\delta(\beta)}. \] (5.4)Taking the limit of this as $\varphi \to 0$, we obtain
\[ \lim_{\varphi \to 0} \frac{\delta(\alpha)}{\delta(\beta)} = -\frac{a - f}{a + f} = -(F_2, F_1; B). \]
Comparing this to (3.6), we obtain $(F_2, F_1; B) = -1$, which contradicts $F_2 \in F_1B$. This contradiction implies that $\delta(\beta) = 0$ in a neighborhood of $J$.

In the other hand, (5.4) implies that if $\delta(\beta_0) \neq 0$ for some $\beta_0$, then $\delta(\beta_2i)$ never vanishes in the process described in Lemma 3.3, meanwhile $\beta_2i$ tends to zero by Lemma 3.3. This is a contradiction as $\delta$ vanishes in a neighborhood of $J$, hence $\delta \equiv 0$, so the theorem is proved.

6. Hilbert geometries with a quadratic ellipsoid (ellipse)

We consider a Hilbert plane $(\mathcal{I}, d_\mathcal{I})$ in the strictly convex bounded domain $\mathcal{I} \subset \mathbb{R}^2$, with an ellipse $\mathcal{E}_{d_x: F_1, F_2}^a$ of different focuses.

Denote the unique straight line through the focuses by $\ell = F_1F_2$, and let $f = d_\mathcal{I}(F_1, F_2)/2$. Further, let $A, B$ and $I, J$ be the intersections of $\ell$ with $\mathcal{E}_{d_x: F_1, F_2}^a$ and $\partial \mathcal{I}$, respectively, such that $A \in TB$.

Lemma 6.1. If $\mathcal{E}_{d_x: F_1, F_2}^a$ is quadratic, then the tangents $t_A$ and $t_B$ of $\mathcal{E}_{d_x: F_1, F_2}^a$ at $A$ and $B$ and the tangents $t_I$ and $t_J$ of $\mathcal{I}$ at $I$ and $J$ are concurrent.

Proof. If $t_I$ intersects $t_J$, then let $i$ be a straight line through their intersection that avoids $\mathcal{I}$, and choose a projectivity $\varpi$ of the plane that sends line $i$ to the ideal line. As, by the definition of the Hilbert metric $d_\mathcal{I}$, we have $d_\mathcal{I}(P, Q) = d_{\varpi(\mathcal{I})}(\varpi(P), \varpi(Q))$, we obtain $\varpi(\mathcal{E}_{d_x: F_1, F_2}^a) = \mathcal{E}_{d_{\varpi(\mathcal{I})}: \varpi(F_1), \varpi(F_2)}^a$ and we also see that $\varpi(\mathcal{E}_{d_x: F_1, F_2}^a)$ is a quadric. Further, $\varpi(\mathcal{I})$ is convex and bounded, and the tangents $t_{\varpi(I)} = \varpi(t_I)$ and $t_{\varpi(J)} = \varpi(t_J)$ are parallel. Thus, based on (2.1), we assume $t_I \parallel t_J$ from now on without loss of generality.

By Lemma 3.1, we have a perspectivity that maps $I, F_1, F_2, J$ into $I, F'_1, F'_2, J'$ such that $IF'_1 = F'_2J'$, meanwhile $t_I \parallel t_J$. Thus, by similar reasoning as above, we assume from now on that $IF'_1 = F'_2J$, that immediately implies $AF'_1 = F_2B$ by the definition of $\mathcal{E}_{d_x: F_1, F_2}^a$.

Choose a Euclidean metric $d_e$ such that $t_I \perp \ell \perp t_J$.

Let $O$ be the metric center of $\mathcal{E}_{d_x: F_1, F_2}^a$, and let $E$ be a moving point on $\mathcal{E}_{d_x: F_1, F_2}^a$. Define $\alpha = \angle(EF_1B), \beta = \angle(EF_2B)$ and $\varphi = \angle(EOB)$. As point $E$ moves on $\mathcal{E}_{d_x: F_1, F_2}^a$, we get the functions $\alpha(\varphi)$ and $\beta(\varphi)$, and these define the functions $r_1(\alpha) = d_e(F_1, V_1), r_2(\beta) = d_e(F_2, V_2)$, and $e_1(\alpha) = d_e(F_1, E), e_2(\beta) = d_e(F_2, E)$, where $V_1 = F_1E \cap \partial \mathcal{I}, V_2 = F_2E \cap \partial \mathcal{I}$. (See Figure 6.)
Figure 6. Ellipse $\mathcal{E}_{d_{2}:F_{1},F_{2}}^{a}$ in the Hilbert plane $(\mathcal{I}, d_{2})$

Letting $U_{1} = \overline{EF_{1}} \cap \partial \mathcal{I}$, $U_{2} = \overline{EF_{2}} \cap \partial \mathcal{I}$, we can calculate the Hilbert-distances

\[
\begin{align*}
    d_{\mathcal{I}}(F_{1}, E) &= -\frac{1}{2} \ln \left( \frac{r_{1}(\alpha + \pi)}{r_{1}(\alpha)} \right) \left( \frac{r_{1}(\alpha + \pi) + e_{1}(\alpha)}{r_{1}(\alpha) - e_{1}(\alpha)} \right), \\
    d_{\mathcal{I}}(E, F_{2}) &= -\frac{1}{2} \ln \left( \frac{r_{2}(\beta + \pi)}{r_{2}(\beta)} \right) \left( \frac{r_{2}(\beta + \pi) + e_{2}(\beta)}{r_{2}(\beta) - e_{2}(\beta)} \right).
\end{align*}
\]  

(6.1) \hspace{1cm} (16)

As $2a = d_{\mathcal{I}}(F_{1}, E) + d_{\mathcal{I}}(E, F_{2})$, it follows that

\[
2a = -\frac{1}{2} \ln \left[ \left( \frac{r_{1}(\alpha + \pi)}{r_{1}(\alpha)} \right) \frac{r_{1}(\alpha + \pi) + e_{1}(\alpha)}{r_{1}(\alpha) - e_{1}(\alpha)} \right] \left( \frac{r_{2}(\beta + \pi)}{r_{2}(\beta)} \frac{r_{2}(\beta + \pi) + e_{2}(\beta)}{r_{2}(\beta) - e_{2}(\beta)} \right),
\]

that is

\[
e^{-4a} \left( 1 + \frac{e_{1}(\alpha)}{r_{1}(\alpha + \pi)} \right) \left( 1 + \frac{e_{2}(\beta)}{r_{2}(\beta + \pi)} \right) = \left( 1 - \frac{e_{1}(\alpha)}{r_{1}(\alpha)} \right) \left( 1 - \frac{e_{2}(\beta)}{r_{2}(\beta)} \right). \hspace{1cm} (6.2) \hspace{1cm} (16, 19)
\]

Let $2t = d_{\mathcal{I}}(F_{1}, E) - d_{\mathcal{I}}(E, F_{2})$. It follows from (6.1) that

\[
2t = -\frac{1}{2} \ln \left[ \left( \frac{r_{1}(\alpha + \pi)}{r_{1}(\alpha)} \right) \frac{r_{1}(\alpha + \pi) + e_{1}(\alpha)}{r_{1}(\alpha) - e_{1}(\alpha)} \right] \left( \frac{r_{2}(\beta + \pi)}{r_{2}(\beta)} \frac{r_{2}(\beta + \pi) + e_{2}(\beta)}{r_{2}(\beta) - e_{2}(\beta)} \right),
\]

that is

\[
e^{-4t} \left( 1 + \frac{e_{1}(\alpha)}{r_{1}(\alpha + \pi)} \right) \left( 1 - \frac{e_{2}(\beta)}{r_{2}(\beta)} \right) = \left( 1 - \frac{e_{1}(\alpha)}{r_{1}(\alpha)} \right) \left( 1 + \frac{e_{2}(\beta)}{r_{2}(\beta + \pi)} \right). \hspace{1cm} (6.3) \hspace{1cm} (16)
\]

The square root of the product of (6.2) and (6.3) is

\[1 + \frac{e_{1}(\alpha)}{r_{1}(\alpha + \pi)} = e^{2a+2t} \left( 1 - \frac{e_{1}(\alpha)}{r_{1}(\alpha)} \right),
\]

hence

\[
e_{1}(\alpha) = \frac{(e^{2a+2t} - 1)r_{1}(\alpha)(\alpha + \pi)}{e^{2a+2t}r_{1}(\alpha + \pi) - r_{1}(\alpha)}. \hspace{1cm} (6.4) \hspace{1cm} (17)
\]
By the triangle inequality with respect to $d_{\mathcal{I}}$, we clearly have $2t \leq d_{\mathcal{I}}(F_2, F_1)$, where the inequality becomes equality if and only if $E \in \ell$, i.e. $t$ has extreme values at $A$ and $B$. As $r_1$ has also extreme values at $A$ and $B$, an easy differentiation by $\varphi$ shows that the derivative of (6.4) vanishes at $\varphi = 0$, because $\dot{t} = \dot{r}_1 = 0$. This proves that $t_A \perp \ell \perp t_B$. \hfill $\square$

Suppose that there exists a quadratic ellipse $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$.

By Lemma 6.1, we have the (maybe ideal) point $T = t_A \cap t_B \cap t_I \cap t_J$. Choose a straight line $t$ through $T$ such that $t \cap \mathcal{I} = \emptyset$. As a projectivity that takes $t$ to the ideal line maps lines $t_I, t_A, t_B, t_J$ into parallel lines, (2.1) means that we can assume without loss of generality that the quadratic ellipse $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$ is such that $t_I \parallel t_A \parallel t_B \parallel t_J$. By Lemma 3.1, we have a perspectivity $\varpi$ such that $I = \varpi(I), A' = \varpi(A), B' = \varpi(B), J' = \varpi(J)$, and $\mathcal{I}' = \mathcal{I} = \mathcal{I}$. Meanwhile, $\varpi(t_I) = t_I \parallel t_A \parallel t_B \parallel t_J$, too. Again by (2.1), we can assume without loss of generality that the quadratic ellipse $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$ is additionally such that $\mathcal{I}' = \mathcal{I}$.

We also have $(I, J; A, F_1)(I, J; A, F_2) = e^{2a} = (I, J; F_1, B)(I, J; F_2, B)$ by the definition of $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$, hence $(I, J; A)(I, J; B) = (I, J; F_1)(I, J; F_2)$ follows. This gives $\mathcal{I}' = \mathcal{I}$, because $(I, J; A)(I, J; B) = 1$, hence the midpoints of the segments $\mathcal{I}J, AB$, and $F_1F_2$ coincide with $O$, the affine center of $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$.

Take the straight line $t$ through $O$ that is parallel to $t_J$. Let $C$ be one of the two intersections of $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$ and $t$. Choose an affine coordinate system such that $O = (0, 0), J = (1, 0)$ and $C = (0, c)$, where $\sinh^2 c = (\tanh^2 a - \tanh^2 f) \cosh^2 a$. Let $d_e$ be the Euclidean metric such that $\{(1, 0), (0, 1)\}$ is an orthonormal basis.

Then the unit circle $C$ of $d_e$ is such that $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$ coincide with $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$. For, we see that both curves are quadratic$^3$, they have common tangents $t_A$ and $t_B$ at their common points $A$ and $B$, and they intersect each other at the points $C$ and $-C$ by (3.7).

**Proposition 6.2.** If $\partial \mathcal{I}$ is analytic around the points $I, J$, then either $\mathcal{I}$ coincides with $C$ in a neighborhood of the points $I, J$, or

\[
1 = \frac{1 - \tanh a}{1 + \tanh a} \frac{(\tanh a - \tanh f)^{k-1}}{(\tanh a + \tanh f)^{k-1}} \frac{(1 - \tanh f)^{k-1}}{1 + \tanh f} + 1
\]

(6.5) \hfill (20, 22, 23)

for some integer $k \geq 2$.

**Proof.** For any point $E \in E_{d_{\mathcal{I}}}^{a} : F_1, F_2$ we define the angles $\varphi = \angle E O J, \alpha = \angle E F_1 J$ and $\beta = \angle E F_2 J$ and notice that these are bijective functions of each other. We define the points $E(\varphi), C(\varphi)$ and $R(\varphi)$, as the intersections of $\partial E$ with $E_{d_{\mathcal{I}}}^{a} : F_1, F_2$, $C$ and $\mathcal{I}$, respectively. We define the points $C_j(\varphi)$ and $R_j(\varphi)$, as the intersections

$^3$The first one because of Lemma 3.6, the second one by condition.
of $F_jE$ with $C$ and $I$, respectively, where $j \in \{1, 2\}$. Let $\overrightarrow{OE}(\varphi) = e(\varphi)u_\varphi$, $\overrightarrow{OC}(\varphi) = c(\varphi)u_\varphi$, and $\overrightarrow{OR}(\varphi) = r(\varphi)u_\varphi$. Further, for let $\overrightarrow{F_jE}(\varphi) = e_j(\xi)u_\xi$, $\overrightarrow{F_jR_j}(\xi) = r_j(\xi)u_\xi$, and $\overrightarrow{F_jR_j}(\xi) = r_j(\xi)u_\xi$, where $j \in \{1, 2\}$ and $\xi$ is $\alpha$ if $j = 1$, and $\beta$ if $j = 2$.

Further, we introduce
\[ \delta_1(\alpha) := r_1(\alpha) - c_1(\alpha) \quad \text{and} \quad \delta_2(\beta) := r_2(\beta) - c_2(\beta). \tag{6.6} \tag{19} \]

For non-vanishing denominators we define
\[ \sigma_1(\alpha) := \frac{\delta_1(\alpha)}{\delta_1(\alpha + \pi)} \quad \text{and} \quad \sigma_2(\beta) := \frac{\delta_2(\beta)}{\delta_2(\beta + \pi)} \tag{6.7} \tag{19} \]

as well as
\[ \tau(\alpha) := \frac{\delta_1(\alpha + \pi)}{\delta_2(\beta + \pi)} \quad \text{and} \quad \rho(\alpha) := \frac{\sigma_1(\alpha)}{\sigma_2(\beta)}. \tag{6.8} \tag{19, 20} \]

Notice that $\varphi \to 0$ implies
\[ c_1(\alpha), r_1(\alpha) \to 1 + \tanh f, \quad c_1(\alpha + \pi), r_1(\alpha + \pi) \to 1 - \tanh f, \]
\[ c_2(\beta), r_2(\beta) \to 1 - \tanh f, \quad c_2(\beta + \pi), r_2(\beta + \pi) \to 1 + \tanh f, \]
and, by Lemma 3.4, for non-vanishing denominators we have

\[
\tau(\alpha) \to (F_1, F_2; I, B)^k, \\
g(\alpha) = \frac{\delta_1(\alpha) \delta_2(\beta + \pi)}{\delta_2(\beta) \delta_1(\alpha + \pi)} \to \frac{(F_1, F_2; J, B)^k}{(F_1, F_2; I, B)^k} = (F_1, F_2; J, I)^k, \tag{6.9} \tag{20}
\]

where integer \( k \) is at least two.

Following (6.2) for \( I \) as well as for \( C \), we have

\[
\left(1 + \frac{e_1(\alpha)}{c_1(\alpha + \pi)}\right) \left(1 + \frac{e_2(\beta)}{r_1(\alpha + \pi)}\right) = e^{4a} \left(1 - \frac{e_1(\alpha)}{c_1(\alpha)}\right) \left(1 - \frac{e_2(\beta)}{c_2(\beta)}\right), \tag{6.10} \tag{19}
\]

\[
\left(1 + \frac{e_1(\alpha)}{r_1(\alpha + \pi)}\right) \left(1 + \frac{e_2(\beta)}{c_2(\beta + \pi)}\right) = e^{4a} \left(1 - \frac{e_1(\alpha)}{c_1(\alpha)}\right) \left(1 - \frac{e_2(\beta)}{c_2(\beta)}\right). \tag{6.11} \tag{19, 21}
\]

Substitution of (6.6) into (6.10) gives

\[
\left(1 + \frac{e_1(\alpha)}{c_1(\alpha + \pi)} - \frac{e_1(\alpha)\delta_1(\alpha + \pi)}{c_1(\alpha + \pi)(c_1(\alpha + \pi) + \delta_1(\alpha + \pi))}\right) \times
\left(1 + \frac{e_2(\beta)}{c_2(\beta + \pi)} - \frac{e_2(\beta)\delta_2(\beta + \pi)}{c_2(\beta + \pi)(c_2(\beta + \pi) + \delta_2(\beta + \pi))}\right) = e^{4a} \left(1 - \frac{e_1(\alpha)}{c_1(\alpha)} + \frac{e_1(\alpha)\delta_1(\alpha)}{c_1(\alpha)(c_1(\alpha) + \delta_1(\alpha))}\right) \times
\left(1 - \frac{e_2(\beta)}{c_2(\beta)} + \frac{e_2(\beta)\delta_2(\beta)}{c_2(\beta)(c_2(\beta) + \delta_2(\beta))}\right). \tag{6.12} \tag{21}
\]

Carrying out the multiplications, subtracting (6.11) and using (6.7) and (6.8) we arrive at

\[
e^{-4a} \frac{e_1(\alpha)\delta_1(\alpha + \pi)}{c_1(\alpha + \pi)(c_1(\alpha + \pi) + \delta_1(\alpha + \pi))} \frac{e_2(\beta)}{c_2(\beta + \pi)(c_2(\beta + \pi) + \delta_2(\beta + \pi))} -
- e^{-4a} \frac{e_1(\alpha)\tau(\alpha)}{c_1(\alpha + \pi)(c_1(\alpha + \pi) + \delta_1(\alpha + \pi))} \left(1 + \frac{e_2(\beta)}{c_2(\beta + \pi)}\right) -
- e^{-4a} \left(1 + \frac{e_1(\alpha)}{c_1(\alpha + \pi)}\right) \frac{e_2(\beta)}{c_2(\beta + \pi)(c_2(\beta + \pi) + \delta_2(\beta + \pi))} =
\frac{e_1(\alpha)\delta_1(\alpha)}{c_1(\alpha)(c_1(\alpha) + \sigma_1(\alpha)\delta_1(\alpha + \pi))} \frac{e_2(\beta)}{c_2(\beta)(c_2(\beta) + \sigma_2(\beta)\delta_2(\beta + \pi))} +
+ \frac{e_1(\alpha)\delta_1(\alpha)}{c_1(\alpha)(c_1(\alpha) + \sigma_1(\alpha)\delta_1(\alpha + \pi))} \left(1 - \frac{e_2(\beta)}{c_2(\beta)}\right) +
+ \left(1 - \frac{e_1(\alpha)}{c_1(\alpha)}\right) \frac{e_2(\beta)}{c_2(\beta)(c_2(\beta) + \sigma_2(\beta)\delta_2(\beta + \pi))}. \tag{6.13} \tag{20}
\]
For any point $\hat{\sigma}_2$ of accumulation of $\sigma_2$ in $\mathbb{R} \cup \{-\infty, \infty\}$, (6.13) gives with the help of (3.3) and of (6.9) that
\[
-e^{-4a} \frac{(\tanh a + \tanh f)(F_1, F_2; I, B)^k}{(1 - \tanh f)^2} \left(1 + \frac{\tanh a - \tanh f}{1 + \tanh f}\right) - e^{-4a} \left(1 + \frac{\tanh a + \tanh f}{1 - \tanh f}\right) \frac{\tanh a - \tanh f}{(1 + \tanh f)^2} = \hat{\sigma}_2 \left[\frac{(\tanh a + \tanh f)(F_1, F_2; J, B)^k}{(1 + \tanh f)^2} \left(1 - \frac{\tanh a - \tanh f}{1 - \tanh f}\right)\right. \\
left. + \left(1 - \frac{\tanh a + \tanh f}{1 + \tanh f}\right) \frac{\tanh a - \tanh f}{(1 - \tanh f)^2}\right].
\]
As $1 + \frac{\tanh a + \tanh f}{1 + \tanh f} = \frac{1 + \tanh a}{1 + \tanh f}$, $1 - \frac{\tanh a + \tanh f}{1 + \tanh f} = \frac{1 - \tanh a}{1 + \tanh f}$, and $e^{2a} = \frac{1 + \tanh a}{1 - \tanh a}$, some simplification of the above equation leads to
\[
\hat{\sigma}_2 = e^{-2a} \frac{(\tanh a + \tanh f)(F_1, F_2; I, B)^k}{(1 - \tanh f)^2} - \frac{\tanh a - \tanh f}{1 + \tanh f} \frac{(\tanh a + \tanh f)(F_1, F_2; J, B)^k}{(1 + \tanh f)^2} + \frac{\tanh a - \tanh f}{1 - \tanh f}.
\]
Using $(F_1, F_2; I, B) = \frac{1 - \tanh f}{1 + \tanh f} \frac{\tanh a - \tanh f}{\tanh a + \tanh f}$ and $(F_1, F_2; J, B) = \frac{1 + \tanh f}{1 + \tanh f} \frac{\tanh a - \tanh f}{\tanh a + \tanh f}$, and making some simplifications give
\[
\hat{\sigma}_2 = -e^{-2a} \frac{(F_1, F_2; I, B)^k - 1 - \tanh f}{(F_1, F_2; J, B)^k + 1 + \tanh f} = \frac{(J, I; B)}{(J, F; J, B)^{k-1} + 1 + \tanh f}.
\]
Comparing (6.8), (6.9) and $(F_1, F_2; J)(F_1, F_2; I) = 1$ we arrive at
\[
\frac{\hat{\sigma}_1}{(F_1, F_2; J)^k} = \frac{\hat{\sigma}_2}{(F_1, F_2; I)^k} = \frac{\left((J, I; B)\right)}{(F_1, F_2; I)^{k-1} + 1}.
\]
Reflection in $O$ shows that $1 = \hat{\sigma}_1 \hat{\sigma}_2$, hence the last equation with a little rearrangement gives
\[
1 = \sqrt{\hat{\sigma}_1 \hat{\sigma}_2} = \frac{|(J, I; B)|}{(F_1, F_2; I)^{k-1}/(F_1, F_2; B)^{k-1} + 1}.
\]
This becomes (6.5) after substituting $(J, I; B) = \frac{1 - \tanh a}{1 + \tanh f}$, $(F_1, F_2; I) = \frac{1 - \tanh a}{1 - \tanh f}$, $(F_1, F_2; B) = \frac{\tanh a + \tanh f}{\tanh a - \tanh f}$, and $(F_1, F_2; J) = \frac{1 + \tanh f}{1 + \tanh f}$.

Thus, (6.5) is valid if (6.9) fulfills, i.e. there is a non-vanishing sequence $\alpha_i \to 0$, such that $\delta_1(\alpha_i + \pi) \neq 0 \neq \delta_2(\beta_i + \pi)$ and $\delta_2(\beta_i) \neq 0$.

If, in contrast, $\delta_1(\alpha_i + \pi) \delta_2(\beta_i + \pi) \delta_2(\beta_i) = 0$ for every non-vanishing sequence $\alpha_i \to 0$, then choose one such non-vanishing sequence $\alpha_i \to 0$. The, by the pigeonhole principle, one of the factors in the product vanishes infinitely many times, hence, by the analyticity of $\partial L$, there is an $\varepsilon > 0$ such that either $\delta_1(\alpha) = 0 = \delta_2(\beta)$ or $\delta_1(\alpha + \pi) = 0 = \delta_2(\beta + \pi)$ for $|\alpha| < \varepsilon$.

Now we prove that $\delta_1(\alpha) = 0 = \delta_2(\beta)$ if and only if $\delta_1(\alpha + \pi) = 0 = \delta_2(\beta + \pi)$ for $|\alpha| < \varepsilon$. For symmetry reasons, we only have to prove either “if” or only only if.
Say, we have \( \delta_1(\alpha + \pi) = 0 = \delta_2(\beta + \pi) \) for \(|\alpha| < \varepsilon \). For such small values of \( \alpha \), the difference of (6.12) and (6.11) gives

\[
0 = \left(1 - \frac{e_1(\alpha)}{c_1(\alpha)}\right) \frac{e_2(\beta)\delta_2(\beta)}{c_2(\beta)(c_2(\beta) + \delta_2(\beta))} + \frac{e_1(\alpha)\delta_1(\alpha)}{c_1(\alpha)(c_1(\alpha) + \delta_1(\alpha))} \left(1 - \frac{e_2(\beta)}{c_2(\beta)}\right)
\]

(6.14) (21)

If there is a non-vanishing sequence \( \alpha_i \to 0 \) such that \( \delta_2(\beta_i) \neq 0 \) fulfilled for all index, then dividing (6.14) by \( \delta_2(\beta_i) \) and applying Lemma 3.4, we arrive at

\[
0 = \left(1 - \frac{e_1(0)}{c_1(0)}\right) \frac{e_2(0)}{c_2(0)} + \frac{e_1(0)(F_1, F_2; J, B)^k}{c_1^2(0)} \left(1 - \frac{e_2(0)}{c_2(0)}\right),
\]

where integer \( k \) is at least two. This is a contradiction, because both summands on the right are positive. Thus, for every non-vanishing sequence \( \alpha_i \to 0 \), \( \delta_2(\beta_i) \neq 0 \) can only be fulfilled for finitely many index \( i \), hence \( \delta_2 \) vanishes in a neighborhood of the zero.

The proof is complete. \( \square \)

**Proposition 6.3.** If \( I \) coincides with \( C \) in a neighborhood of \( I \) and \( J \), then \( I \equiv C \).

**Proof.** The condition of the proposition means that there is an \( \varepsilon > 0 \) such that

\[
\delta_1(\alpha(\varphi)) = \delta_2(\beta(\varphi)) = \delta_1(\alpha(\varphi) + \pi) = \delta_2(\beta(\varphi) + \pi) = 0 \quad \text{if} \quad \varphi \in (-\varepsilon, \varepsilon).
\]

Define the positive values

\[
\beta_{J^+} = \inf\{\xi \in (0, \pi) : \delta_2(\xi) \neq 0\}, \quad \alpha_{J^+} = \inf\{\xi \in (0, \pi) : \delta_1(\xi) \neq 0\},
\]

\[
\beta_{J^-} = \inf\{\xi \in (0, \pi) : \delta_2(-\xi) \neq 0\}, \quad \alpha_{J^-} = \inf\{\xi \in (0, \pi) : \delta_1(-\xi) \neq 0\},
\]

\[
\beta_{I^+} = \inf\{\xi \in (0, \pi) : \delta_2(\pi-\xi) \neq 0\}, \quad \alpha_{I^+} = \inf\{\xi \in (0, \pi) : \delta_1(\pi-\xi) \neq 0\},
\]

\[
\beta_{I^-} = \inf\{\xi \in (0, \pi) : \delta_2(\xi+\pi) \neq 0\}, \quad \alpha_{I^-} = \inf\{\xi \in (0, \pi) : \delta_1(\xi+\pi) \neq 0\}.
\]

Let \( I^\pm = R_1(\pi \mp \alpha_{I^\pm}) = R_2(\pi \mp \beta_{I^\pm}) \) and \( J^\pm = R_1(\alpha_{J^\pm}) = R_2(\beta_{J^\pm}) \). By these, we clearly have

\[
\alpha_{J^+} < \beta_{J^+}, \quad \beta_{I^-} < \alpha_{I^-}, \quad \alpha_{I^+} > \beta_{I^+}, \quad \text{and} \quad \beta_{I^-} > \alpha_{J^-}.
\]

(6.15) (22)

Assume that \( \beta_{I^-} < \beta_{J^+} \). Let \( \hat{\varphi} \) be such that \( \beta(\hat{\varphi}) = \beta_{I^-} \). Then \( \alpha(\hat{\varphi}) < \beta(\hat{\varphi}) = \beta_{J^+} \), therefore not only \( \delta_2(\beta(\hat{\varphi})) = 0 \), but also \( \delta_1(\alpha(\hat{\varphi})) = 0 \). Further, \( \alpha(\hat{\varphi}) < \beta(\hat{\varphi}) = \beta_{I^-} < \alpha_{I^-} \), so \( \delta_1(\alpha(\hat{\varphi}) + \pi) = 0 \). By continuity, there is an \( \varepsilon > 0 \) such that \( \alpha(\varphi) < \beta(\varphi) \) and \( \beta(\varphi) < \beta_{J^+} \) for every \( \varphi \in (\hat{\varphi}, \hat{\varphi} + \varepsilon) \). For such \( \varphi \) we have also \( \alpha(\varphi) < \beta(\varphi) < \alpha_{J^-} \), therefore \( \delta_1(\alpha(\varphi)) = 0, \delta_2(\beta(\varphi)) = 0, \) and \( \delta_1(\alpha(\varphi) + \pi) = 0 \). In this case the difference of (6.12) and (6.11) is

\[
\left(1 + \frac{e_1(\alpha)}{c_1(\alpha + \pi)}\right) \frac{e_2(\beta)\delta_2(\beta + \pi)}{c_2(\beta + \pi)(c_2(\beta + \pi) + \delta_2(\beta + \pi))} = 0
\]

which implies \( \delta_2(\beta(\varphi) + \pi) = 0 \). This contradicts the definition of \( \beta_{I^-} \), hence \( \beta_{I^-} \geq \beta_{J^+} \).
Similar reasoning shows that $\beta_I + \geq \beta_J -$, $\alpha_J + \geq \alpha_I -$, and $\alpha_J - \geq \alpha_I +$ fulfills too. From these and (6.16) the sequences of inequalities

$$\alpha_J + \leq \beta_J + \leq \alpha_I - \leq \alpha_J + \text{ and } \alpha_J + \geq \beta_J - \geq \alpha_J - \geq \alpha_I +$$

follows. Both sequences of inequalities are contradictory, therefore the values in (6.15) cause contradiction, hence at least one of the defining sets is empty, that completes the proof of the proposition. □

**Theorem 6.4.** Assume that

1. there are two quadratic ellipses, or
2. there is a quadratic ellipse of numerical eccentricity bigger then $1/\sqrt{3}$, with focuses $F_1$ and $F_2$ in a Hilbert plane $(\mathcal{I}, d_\mathcal{I})$, such that $\partial \mathcal{I}$ is analytic in a neighborhood of $F_1 F_2 \cap \partial \mathcal{I}$. Then the Hilbert plane is Bolyai’s hyperbolic plane.

**Proof.** For variables $0 < x < y < 1$ and integer $k \geq 2$ define

$$g_k(x, y) := \frac{1 - y}{1 + y} \left( \frac{1 - x}{1 + x} \right)^{k-1} + \frac{1 - (\frac{1-x}{1+x})^{2k-2}}{(1 - \frac{2x}{y+x})^{k-1} + (\frac{1-x}{1+x})^{k-1}}. \quad (6.17)$$

It is clear, that $g_k(x, \cdot)$ is a strictly decreasing function on $(x, 1)$, hence the equation $y_k : (0, 1) \to (0, 1)$ is uniquely defined by the equation $g_k(x, y_k(x)) = 1$ that implies

$$y_k(x) = \frac{1 - (\frac{1-x}{1+x})^{k-1} 1 - (1 - \frac{2x}{y_k(x)+x})^{k-1}}{1 + (\frac{1-x}{1+x})^{k-1} 1 + (1 - \frac{2x}{y_k(x)+x})^{k-1}}. \quad (6.18)$$

As the right-hand side of this is clearly strictly monotone increasing in $k$ and strictly monotone decreasing in $y_k$, we deduce that $y_k(x)$ is strictly monotone increasing in $k$. Finally, observe, that some easy rearrangements give that

$$(6.5) \text{ is equivalent with } 1 = g_k(\tanh f, \tanh a). \quad (\ast) \quad (22)$$

By the strict decrease of $g_k(x, \cdot)$, (\ast) says that (6.5) can not be satisfied for two ellipses with common focuses, hence Proposition 6.2 implies that $\mathcal{I}$ coincides with $\mathcal{C}$ in some neighborhoods of $I$ and $J$. Thus, if condition (1) applies, then our Theorem follows by Proposition 6.3.

From now on we assume that condition (2) applies.

By the strict increase of $y_\bullet(x)$, (\ast) says that that the numerical eccentricity $\sinh f / \sinh a$ with $f$ and $a$ satisfying (6.5) can be maximal only for the smallest $k$ for which (6.18) has solution. If $k = 2$, then there is no solution for (6.18), but if $k = 3$, then (6.18) takes the form

$$y = \frac{1 - (\frac{1-x}{1+x})^2 1 - (\frac{y-x}{y+x})^2}{1 + (\frac{1-x}{1+x})^2 1 + (\frac{y-x}{y+x})^2} = \frac{2x}{1+x^2} \frac{2xy}{x^2 + y^2}, \text{ hence } y^2 = x^2 \frac{3 - x^2}{1 + x^2}.$$
Substitution of $x = \tanh f$ and $y = \tanh a$ gives 
\[
\tanh^2 a = \tanh^2 f \frac{3 - \tanh f}{1 + \tanh^2 f} = 1 - \frac{1}{1 + 2 \sinh^2 f} \quad \text{and therefore} \quad (1 + \sinh^2 f)(1 + 2 \sinh^2 f) = 1 + \sinh^2 a \quad \text{hence}
\]
\[
\frac{\sinh^2 f}{\sinh^2 a} = \frac{1}{3 + 2 \sinh^2 f}.
\]
This takes its maximum at $f = 0$, where the numerical eccentricity is $1/\sqrt{3}$.

This proves that in case (2) equation (6.5) can not be satisfied, hence, by Proposition 6.2, $I$ coincides with $C$ in some neighborhoods of $I$ and $J$.

Thus, if condition (2) applies, (6.5) does not fulfills, hence the statement of our Theorem follows again by Proposition 6.3.

The proof is completed. \hfill \Box

Notice that condition (2) could be changed to a duller but more precise criterion that excludes the ellipses with eccentricity $f$ if $a = \arctanh (y_k (\tanh f))$ for every integer $k \geq 2$.

7. Discussion

About the analyticity of the Minkowski geometries, we have the following.

**Theorem 7.1.** A Minkowski plane is analytic if and only if it has an analytic ellipse.

**Proof.** First assume that the Minkowski plane $(\mathbb{R}^2, d_I)$ is analytic. Then the circles are also analytic, because they are homothetic to the boundary of the indicatrix, so we only need to prove the analyticity of ellipses $E_{d \mathcal{I}; F_1, F_2}$.

We use the notations given in Section 5.

Fix an arbitrary point $E_0 \in \mathcal{E}_{d \mathcal{I}; F_1, F_2}$, and let the point $R_i \in \mathcal{I}$ ($i = 1, 2$) be such that $O \overline{R_i} \parallel F_i \overline{E_0}$. Let the straight line $t_i$ ($i = 1, 2$) tangent to $\mathcal{I}$ at $R_i$. Let $d_e$ be the Euclidean metric which satisfies $t_2 \perp OR_2$, $d_e(O, R_1) = d_e(O, R_2)$, and $d_e(O, J) = 1$. Then we have

\[
e_2^2(\beta) = e_1^2(\alpha) + 4 f^2 - 4 e_1(\alpha) f \cos \alpha, \quad \text{and} \quad \beta = \arcsin \frac{e_1(\alpha) \sin \alpha}{e_2^3(\beta)}.
\]

Substituting this into (5.1) results in the analytic equation

\[
F(\alpha, e_1(\alpha)) := \left(2a - \frac{e_1(\alpha)}{r(\alpha)} \right)^2 - \frac{e_1^2(\alpha) + 4 f^2 - 4 e_1(\alpha) f \cos \alpha}{r^2 \left( \frac{\arcsin \frac{e_1(\alpha) \sin \alpha}{\sqrt{e_1^2(\alpha) + 4 f^2 - 4 e_1(\alpha) f \cos \alpha}}} \right) } = 0.
\]

As

\[
\partial_2 F(\alpha, e_1(\alpha)) = 2 \frac{e_2(\beta)}{r(\beta)} \frac{-1}{r(\alpha)} - 2 \frac{e_1(\alpha) - 4 f \cos \alpha}{r^2(\beta)} + \frac{2 \frac{e_2^2(\beta)}{r^3(\beta)} \frac{\sin \alpha}{e_2(\beta)} \frac{-1}{2} \frac{e_1(\alpha) \sin \alpha (2 e_1(\alpha) - 4 f \cos \alpha)}{e_2^3(\beta)},
\]

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\( \partial_2 F(\alpha, e_1(\alpha)) \) vanishes if and only if

\[
e_2(\beta) + \frac{e_1(\alpha) - 2f \cos \alpha}{r(\beta)} = \frac{e_2(\beta) \sin \alpha \dot{r}(\beta)}{r(\beta) \cos \beta r(\beta)} \left( 1 - \frac{e_1(\alpha)(e_1(\alpha) - 2f \cos \alpha)}{e_2^2(\beta)} \right).
\]

By (3.2), we have \( \frac{\dot{r}(\beta)}{r(\beta)} = \cot \theta \), where \( \theta \) is the angle between \( F_1 E \) and the tangent vector at \( E \) of \( \mathcal{E}_d^{a}_{F_1, F_2} \). Further, it can be easily seen that \( e_1(\alpha) - 2f \cos \alpha = e_2(\beta) \cos(\beta - \alpha) \). Thus, the above equation is equivalent to

\[
\frac{e_2(\beta)}{r(\alpha)} + \frac{e_1(\alpha) - 2f \cos \alpha}{r(\beta)} = \frac{\cot \theta}{r(\beta) \cos \beta} \left( e_2(\beta) \sin \alpha - e_1(\alpha) \cos(\beta - \alpha) \sin \alpha \right).
\]

Since \( e_2(\beta) \sin \beta = e_1(\alpha) \sin \alpha \), this equation simplifies to

\[
\frac{1}{r(\alpha)} + \frac{\cos(\beta - \alpha)}{r(\beta)} = \frac{\cot \theta \sin \alpha - \sin \beta \cos(\beta - \alpha)}{\cos \beta} = -\sin(\beta - \alpha) \cot \theta.
\]

In sum, \( \partial_2 F(\alpha, e_1(\alpha)) \) vanishes if and only if

\[
\frac{r(\beta)}{r(\alpha)} + \sin(\beta - \alpha)(\cot \theta + \cot(\beta - \alpha)) = 0.
\]

(7.1) (24)

At \( E_0 \) we have \( \theta = \pi/2 \), and \( r(\beta) = r(\alpha) \), therefore (7.1) requests \( 1 + \cos(\beta - \alpha) = 0 \), that gives \( \beta = \pi + \alpha \), a contradiction. Thus \( \partial_2 F(\alpha, e_1(\alpha)) \neq 0 \) at \( E_0 \), hence the analytic implicit function theorem [7, Theorem 4.1] implies the analyticity of \( e_1 \) in a neighborhood of \( \alpha \). As the point \( E_0 \) was chosen arbitrarily on \( \mathcal{E}_d^{a}_{F_1, F_2} \), this proves that \( \mathcal{E}_d^{a}_{F_1, F_2} \) is analytic.

Assuming now that the ellipse is analytic, Lemma 5.2 proves the analyticity of the border of the indicatrix, where the major axis of the ellipse intersects it. By (5.3) we have

\[
\dot{r}(\beta(\alpha)) = \frac{e_1(\alpha)}{-e_2(\beta(\alpha))} \dot{r}(\alpha) + \frac{2a}{e_2(\beta(\alpha))}.
\]

This shows that if \( \dot{r} \) is analytic in an interval \((-\varepsilon, \varepsilon)\), then it is also analytic in the interval \((-\beta(\varepsilon), \beta(\varepsilon))\). According to Lemma 3.3 this means that the border of the indicatrix is analytic.

About the analyticity of the Hilbert geometries, we only have a strong conjecture.

**Conjecture 7.2.** A Hilbert geometry is analytic if and only if it has an analytic ellipse.

If this conjecture holds, then the analyticity condition in Theorem 6.4 becomes unnecessary, and the following strengthened version of Theorem 6.4 should be valid.

**Conjecture 7.3.** If a Hilbert geometry has a quadratic ellipsoid (ellipse), then it is Bolyai’s hyperbolic geometry.
Finally we formulate a conjecture that, if proves valid, would strengthen Theorem 4.1 considerably.

**Conjecture 7.4.** *A continuously differentiable regular projective-metric space is Riemannian and hence of constant curvature if and only if every point is the center of a quadratic sphere (circle).*

If the requirement of the regularity dropped, then one probably needs to require that every sphere (circle) is quadratic.

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**References**


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