PROJECTIVE-METRIC SPACES WITH QUADRATIC HYPERBOLAS

ÁRPÁD KURUSA AND JÓZSEF KOZMA

Abstract. A conjecture is formulated that a projective-metric space is classic if and only if it has a quadratic hyperbola. The conjecture is validated for Minkowski geometry and analytic Hilbert geometry.

1. Introduction

Let $(\mathcal{M}, d)$ be a metric space given in a set $\mathcal{M}$ with the metric d. If $\mathcal{M}$ is a projective space $\mathbb{P}^n$ or an affine space $\mathbb{R}^n \subset \mathbb{P}^n$ or a proper open convex subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$, and the metric d is complete, continuous with respect to the standard topology of $\mathbb{P}^n$, and the geodesic lines of d are exactly the non-empty intersection of $\mathcal{M}$ with the straight lines, then the metric d is called *projective*.

If $\mathcal{M} = \mathbb{P}^n$ and the geodesic lines of d are isometric with an Euclidean circle, or $\mathcal{M} \subseteq \mathbb{R}^n$ and the geodesic lines of d are isometric with an Euclidean straight line, then $(\mathcal{M}, d)$ is called a *projective-metric space* of *elliptic, parabolic or hyperbolic type*, respectively (see [1, p. 115] and [7, p. 188]).

In a projective-metric space $(\mathcal{M}, d)$ a set

(D_1) $\mathcal{H}^a_{d;F_1,F_2} := \{X : 2a = |d(F_1, X) - d(F_2, X)|\}$ is called *hyperbola* (*hyperboloid* in higher dimensions),

where $F_1, F_2 \in \mathcal{M}$ are different points, the *focuses*, and $a < \frac{1}{2}d(F_1, F_2)$ is a positive number.

A kind of folkloric result [6, Theorem 2.2][1] is that hyperboloids are quadratic in every classic projective-metric spaces. Following [6] we consider here the reverse of this asking if a projective-metric space is classic if it has a quadratic hyperbola.

We prove in Theorem 4.3 that a Minkowski geometry with *a* quadratic hyperboloid is Euclidean. This result can be understood as a hyperbolic version of Busemann's result [1, 25.4], for hyperbolas (hyperboloids).

Next we prove in Theorem 5.6 that an analytic Hilbert geometry with *a* quadratic hyperboloid is hyperbolic.

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[1] It is included in this article as Theorem 2.1 without proof for a complete treatment.

2. Preliminaries

To prove our results in arbitrary dimensions, we only need to work in the affine *plane*, because a convex body $\mathcal{K}$ in $\mathbb{R}^n$ $(n \geq 3)$ is an ellipsoid if and only if for any fixed $k \in \{2, \dots, n-1\}$ every k-plane through any point of $\mathcal{K}$ intersects $\mathcal{K}$ in a k-dimensional ellipsoid [2, (16.12), p. 91]. Therefore, in what follows, we restrict ourselves to the plane.

Points of $\mathbb{R}^2$ are denoted by $A, B, \dots$, vectors by $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots$. The open segment with endpoints A and B is denoted by $\overline{AB}$. The open ray starting from A and passing through B is denoted by $\overline{A}B$, and the line through A and B by AB.

Notations $\boldsymbol{u}_\varphi := (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}_\varphi^\perp := (\cos(\varphi + \pi/2), \sin(\varphi + \pi/2))$ are frequently used. It is worth to note that by these assumptions we have $\frac{\mathrm{d}}{\mathrm{d}\varphi}\boldsymbol{u}_\varphi = \boldsymbol{u}_\varphi^\perp$.

In most cases, we use *polar parameterization* for the boundary $\partial\mathcal{D}$ of a domain $\mathcal{D}$ in $\mathbb{R}^2$, starlike with respect to the origin $O \in \mathcal{D}$, by a function $\boldsymbol{r}\colon [-\pi, \pi) \to \mathbb{R}^2$ of the form $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, where r is the *radial function* of $\mathcal{D}$ with respect to O.

The *affine ratio* $(A, B; C)$ of the collinear points A, B and C is defined by $(A, B; C)\overrightarrow{BC} = \overrightarrow{AC}$. The *cross ratio* of the collinear points A, B and C, D is $(A, B; C, D) = (A, B; C)/(A, B; D)$ [1, page 243].

Following [6], we call a curve *analytic* if its coordinates depend on its arc length analytically. By [6, Lemma 2.1], the border of a convex domain is an analytic curve if and only if any one of its radial functions is analytic.

2.1. Basic properties of hyperbolas. Let a hyperboloid $\mathcal{H}^a_{d;F_1,F_2}$ be given. We define its *eccentricity* as $f := \frac{1}{2}d(F_1, F_2)$ and its *radius* as a.

Define the function $X \mapsto \Delta(X) := d(F_1, X) - d(X, F_2)$ on $\mathcal{M}$. It is clearly continuous by the continuity of d. Let the ordering '$\prec$' of line F_1F_2 be such that $F_1 \prec F_2$. Then, by the additivity of d, we have $\Delta(X) = -2f$ for every $X \preceq F_1$, $\Delta(X) = 2f$ for every $F_2 \preceq X$, and Δ is strictly monotonously increasing on $\overline{F_1F_2}$ with respect to '$\prec$'. As $f > a$, we deduce the existence of unique points $A, B \in \overline{F_1F_2}$ such that $F_1 \prec A \prec B \prec F_2$, $\Delta(A) = -2a$, $\Delta(B) = 2a$, and $d(A, B) = 2a$.

It follows that the *left branch* $\mathcal{H}^{a-}_{d;F_1,F_2} := \{X : \Delta(X) = -2a\}$ and the *right branch* $\mathcal{H}^{a+}_{d;F_1,F_2} := \{X : \Delta(X) = 2a\}$ of the hyperbola meets $\overline{F_1F_2}$ in A and B, respectively, and they are clearly disjoint sets.

The metric midpoint O of $\overline{F_1F_2}$ is called the *metric center* of $\mathcal{H}^a_{d;F_1,F_2}$. It is clearly in $\overline{AB}$, the *major axis* of length $2a$, and is therefore not on the hyperbola.

2.2. Classic geometries. A complete Riemannian manifold $\mathbb{M}^n$ of dimension n is called an abstract rotational manifold with *base point* $O \in \mathbb{M}^n$ if the induced linear action of the isotropy group of O on $T_O\mathbb{M}^n$ is equivalent to $O(n)$ [8].

The Riemannian metric on $\mathbb{M}^n$ is then completely described by its *size function* $\nu\colon [0, I_\nu) \to \mathbb{R}_+$ such that the geodesic sphere of radius r and center O in $\mathbb{M}^n$ is isometric to the Euclidean sphere of radius $\nu(r)$. This explains the notation

$(\mathbb{M}^n, \nu)$. A complete abstract rotational manifold of real type is homogeneous if and only if it is of constant sectional curvature κ [8]. In this case, a function $\mu\colon [0, I_\nu) \to [0, \infty)$, the *projector function* of $\mathbb{M}^n$ [5], exists such that the map $\bar{\mu}\colon \exp_O(p\boldsymbol{u}) \mapsto \mu(p)\boldsymbol{u}$ from $\mathbb{M}^n$ into $\mathbb{R}^n$, where $\boldsymbol{u}$ is a unit vector in the tangent space $T_O\mathbb{M}^n \cong \mathbb{R}^n$, takes geodesics into straight lines. From the quadratic model of the spaces of constant curvature[2] one can easily read off the following:

$\mathbb{M}$	I_ν	κ	ν	μ
$\mathbb{H}^n$	∞	-1	$\sinh r$	$\tanh r$
$\mathbb{R}^n$	∞	0	r	r
$\mathbb{S}^n$ or $\mathbb{P}^n$	$\pi/2$	$+1$	$\sin r$	$\tan r$

Theorem 2.1 ([6, Theorem 2.2])**.** *Polar equation of every metric hyperbola $\mathcal{H}^a_{d;F_1,F_2}$ in any 2-dimensional manifold of constant curvature $\kappa \in \{-1, 0, 1\}$ is of the form*

$$\frac{1}{\nu^2(r(\omega))} = \frac{\cos^2\omega}{\nu^2(a)} + \frac{\sin^2\omega}{(\mu^2(a) - \mu^2(f))(1 - \kappa\nu^2(a))}, \tag{2.1}$$

where ν and μ are defined in the table above.

2.3. Minkowski geometry. Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^2$, symmetric to the origin O. The function $d_{\mathcal{I}}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_{\mathcal{I}}(X, Y) = \inf\left\{\lambda > 0 : \overrightarrow{XY}/\lambda \in \mathcal{I}\right\}$$

is a metric on $\mathbb{R}^2$ [1, IV.24], and is called *Minkowski metric*. We say that it is *analytic* if $\partial\mathcal{I}$ is an analytic curve. The pair $(\mathbb{R}^2, d_{\mathcal{I}})$ is called a *Minkowski plane*, $\mathcal{I}$ is its *indicatrix*. Note that Minkowski planes are isomorphic if and only if an affine map exists between their indicatrixes. If $\partial\mathcal{I}$ is an analytic curve, we speak of analytic Minkowski plane. The Euclidean plane is, in fact, a special analytic Minkowski plane $(\mathbb{R}^2, d_{\mathcal{E}})$ given by an ellipse $\mathcal{E}$ as indicatrix.

2.4. Hilbert geometry. Let $\mathcal{I}$ be an open, strictly convex set in $\mathbb{R}^2$ with boundary $\partial\mathcal{I}$. The function $d_{\mathcal{I}}\colon \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ defined by

$$d_{\mathcal{I}}(A, B) = \begin{cases} 0, & \text{if } A = B, \\ \frac{1}{2}\left|\ln(A, B; C, D)\right|, & \text{if } A \neq B, \text{ where } \overline{CD} = \mathcal{I} \cap AB, \end{cases}$$

is a metric on $\mathcal{I}$ [1, page 297], and is called the *Hilbert metric*. We say that it is *analytic* if $\partial\mathcal{I}$ is an analytic curve. The pair $(\mathcal{I}, d_{\mathcal{I}})$ is called the *Hilbert plane* given in $\mathcal{I}$. Note that two Hilbert planes are isomorphic if and only if a projectivity exists between their sets of points.

If $\partial\mathcal{I}$ is an analytic curve, we speak of an analytic Hilbert plane. Bolyai's hyperbolic plane is, in fact, a special analytic Hilbert plane $(\mathcal{E}, d_{\mathcal{E}})$ given by an ellipse $\mathcal{E}$.

[2] Look for projective realization of constant curvature spaces in standard textbooks.

3. Utilities

In this section the underlying plane is Euclidean. The technical lemmas obtained are used in the next sections, and are analogous to the similar technical lemmas in [6]. We give proofs here only because there are some disparateness in some details.

Lemma 3.1 ([6, Lemma 3.1]). *For any collinear points A, B, C, D satisfying $\overline{BC} \subsetneq \overline{AD}$, and a point D' out of the line AD, there is a unique perspectivity ϖ such that $A = \varpi(A)$, $B' = \varpi(B)$, $C' = \varpi(C)$, $D' = \varpi(D)$ and $\overrightarrow{AB'} = \overrightarrow{C'D'}$.*

Let $\mathcal{H}$ be a hyperbola in the plane, with center O, and foci F_1, F_2. Line $\ell = F_1F_2$ intersects $\mathcal{H}$ in points A and B such that $A \in \overline{F_1B}$, and $O \in \overline{AB} \subset \overline{F_1F_2}$. Let fix points V and W on F_1F_2 such that $F_1 \in \overline{VA}$ and $F_2 \in \overline{BW}$.

Straight lines ℓ_1 through F_1 and ℓ_2 through F_2 close angles α and β with ℓ, respectively. They intersect $\mathcal{H}$ in a common point $H(\varphi) = \ell_1 \cap \ell_2$, where φ is the angle $HOB\angle$. There is an angle $\Phi \in (0, \pi/2)$ such that for $\varphi \in (-\Phi, \Phi)$, points $H(\varphi)$ are on the 'right' branch $\mathcal{H}_r$ (containing B) of the hyperbola, while for $\varphi \in (\pi - \Phi, \pi + \Phi)$, points $H(\varphi)$ are on the 'left' branch $\mathcal{H}_l$ (containing A) of the hyperbola. It is clear, that angles α and β are functions of φ, and $\alpha \to 0$ and $\beta \to \pi$ when $\varphi \to 0$.

Starting from any $\varphi_0 \in (0, \Phi)$ we define sequences of points H_i and corresponding angles φ_i, $\alpha_i = \alpha(\varphi_i)$, $\beta_i = \beta(\varphi_i)$ recursively, as follows. (See Figure 3.1.)

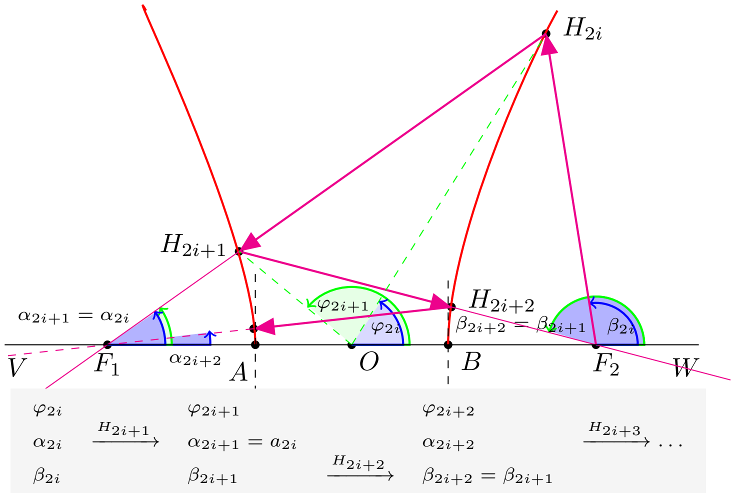


Figure 3.1. Sequence of angles

Let $H_0 = H(\varphi_0) \in \mathcal{H}_r$, $\alpha_0 = \angle WF_1H_0$ and $\beta_0 = \angle WF_2H_0$. Furthermore, $H_{2i+1} = \overline{F_1H_{2i}} \cap \mathcal{H}_l$, $\alpha_{2i+1} = \alpha_{2i}$, and $\beta_{2i+1} = \angle WF_2H_{2i+1}$. Finally, $H_{2i+2} = \overline{F_2H_{2i+1}} \cap \mathcal{H}_r$, $\alpha_{2i+2} = \angle H_{2i+2}F1W$, and $\beta_{2i+2} = \beta_{2i+1}$. It is easy to see that all these points and angles are well defined. Furthermore, we have $\varphi_{2i} \in (0, \Phi)$ and $\varphi_{2i+1} \in (\pi + \Phi, \pi - \Phi)$ for every $i \in \mathbb{N}$.

Lemma 3.2. *If $i \to \infty$, then α_{2i} and φ_{2i} tend to zero, β_{2i}, β_{2i+1} and φ_{2i+1} tend to π, and $\alpha_{2i+2}/\alpha_{2i}$ tends to $(F_1, F_2; A, B)$.*

Proof. Simple consideration shows that $\varphi_{2i} < \Phi < \pi/2$ and $\varphi_{2i+1} > \pi - \Phi > \pi/2$, and therefore

$$\begin{aligned} \alpha_{2i} < \pi - \beta_{2i} \quad &\text{and} \quad \pi - \beta_{2i+1} < \alpha_{2i+1} \quad (\text{or} \quad \pi - \beta_{2i+2} < \alpha_{2i}), \\ \alpha_{2i+2} < \pi - \beta_{2i+2} \quad &\text{and} \quad \pi - \beta_{2i+1} < \alpha_{2i}, \end{aligned}$$

hence $\beta_{2i+2} > \beta_{2i}$, $\alpha_{2i+2} < \alpha_{2i}$, and $\pi - \beta_{2i+2} < \alpha_{2i} < \pi - \beta_{2i}$.

Thus, sequences β_{2i}, β_{2i+1} monotonously increase, while sequences α_{2i}, α_{2i+1} monotonously decrease. As these sequences are bounded, they are convergent.

Assuming $\lim_i \beta_{2i} < \pi$, i.e. $\lim_i(\pi - \beta_{2i}) > 0$, $\lim_i \frac{\pi-\beta_{2i+2}}{\pi-\beta_{2i}} = 1$, and $\lim_i \frac{\alpha_{2i}}{\pi-\beta_{2i}} = 1$ follow, hence the sinus law for triangle $\triangle F_1F_2H_{2i}$ implies

$$\lim_{i\to\infty} \frac{d(F_2, H_{2i})}{d(H_{2i}, F_1)} = \lim_{i\to\infty} \frac{\sin \alpha_{2i}}{\sin(\pi - \beta_{2i})} \cdot \lim_{i\to\infty} \frac{\pi - \beta_{2i}}{\alpha_{2i}} = 1,$$

which, by the continuity of d, gives $d(F_2, B) = d(B, F_1)$, a contradiction.

Thus $\lim_i \beta_{2i} = \pi$, hence also β_{2i+1} and φ_{2i+1} tend to π, and furthermore, sequences α_{2i}, α_{2i+1} and φ_{2i} tend to zero, and observing Figure 3.1, we see that

$$\begin{aligned} h_1(\alpha_{2i}) &:= d(F_1,H_{2i}) \to d(F_1, B), & h_1(\alpha_{2i+1}) &:= d(F_1,H_{2i+1}) \to d(F_1, A), \\ h_2(\beta_{2i}) &:= d(F_2,H_{2i}) \to d(F_2, B), & h_2(\beta_{2i+1}) &:= d(F_2,H_{2i+1}) \to d(F_2, A). \end{aligned} \tag{3.1}$$

The sine law in triangles $\triangle F_1F_2H_{2i}$ and $\triangle F_1F_2H_{2i+1}$ gives

$$\frac{h_2(\beta_{2i+1})}{h_1(\alpha_{2i+1})} = \frac{\sin \alpha_{2i+1}}{\sin(\pi - \beta_{2i+1})} \quad \text{and} \quad \frac{h_2(\beta_{2i+2})}{h_1(\alpha_{2i+2})} = \frac{\sin \alpha_{2i+2}}{\sin(\pi - \beta_{2i+2})},$$

respectively. Multiplying these by $\cos \beta_{2i+1}/\cos \alpha_{2i+1}$ and $\cos \beta_{2i+2}/\cos \alpha_{2i+2}$, respectively, and taking the ratio of the results give

$$\frac{\tan \alpha_{2i+2}}{\tan \alpha_{2i}} = \frac{h_2(\beta_{2i+2}) \cos \beta_{2i+2}}{h_1(\alpha_{2i+2}) \cos \alpha_{2i+2}} \frac{h_1(\alpha_{2i+1}) \cos \alpha_{2i+1}}{h_2(\beta_{2i+1}) \cos \beta_{2i+1}}.$$

By (3.1), the right-hand side of this equation tends to $(F_1, F_2; A, B)$, so the proof is complete. □

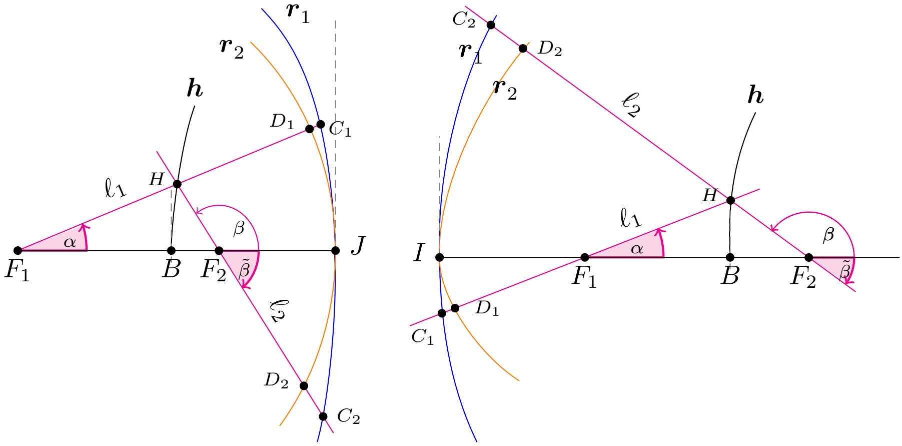
Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be curves in the plane with analytic arc length parametrization on $[-1, 1]$ such that at their common point $\boldsymbol{r}_1(0) = \boldsymbol{r}_2(0)$ they have common tangent $\dot{\boldsymbol{r}}_1(0) = \dot{\boldsymbol{r}}_2(0)$. Let line ℓ through $\boldsymbol{r}_1(0)$ be orthogonal to $\dot{\boldsymbol{r}}_1(0)$, and the analytic curve $\boldsymbol{h}$ parameterized on $[0, 1]$ by arc length intersects ℓ in $B = \boldsymbol{h}(0)$ orthogonally, and $\dot{\boldsymbol{r}}_1(0) = \dot{\boldsymbol{h}}(0)$.

We are given different points F_1, F_2 on ℓ such that

(C1) either $\overline{F_2J} \subset \overline{BJ} \subset \overline{F_1J}$, where $J = \boldsymbol{r}_1(0)$, and $\dot{\boldsymbol{r}}_1(0) = \boldsymbol{u}_{\pi/2}$,

(C2) or $\overline{IF_1} \subset \overline{IB} \subset \overline{IF_2}$, where $I = \boldsymbol{r}_1(0)$, and $\dot{\boldsymbol{r}}_1(0) = \boldsymbol{u}_{-\pi/2}$.

For sufficiently small $s > 0$, points $H = \boldsymbol{h}(s)$ on the curve $\boldsymbol{h}$ define the straight lines $\ell_1 := F_1H$ and $\ell_2 := F_2H$, closing small angle α and $\tilde\beta$ with ℓ, respectively (where $\tilde\beta = \beta - \pi$).



The lines ℓ_1 and ℓ_2 intersect the curves $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in points $C_1 = \boldsymbol{r}_1(s_{1,1})$, $D_1 = \boldsymbol{r}_2(s_{2,1})$, and $C_2 = \boldsymbol{r}_1(s_{1,2})$, $D_2 = \boldsymbol{r}_2(s_{2,2})$, respectively, where $s_{i,j}$ is the arc length parameter of $\boldsymbol{r}_i$ at its intersection with ℓ_j $(i, j = 1, 2)$. Let $\delta_1 = \langle \boldsymbol{r}_1(s_{1,1}(\alpha)) - \boldsymbol{r}_2(s_{2,1}(\alpha)), \boldsymbol{u}_\alpha\rangle$ and $\delta_2 = \langle \boldsymbol{r}_1(s_{1,2}(\tilde\beta)) - \boldsymbol{r}_2(s_{2,2}(\tilde\beta)), \boldsymbol{u}_{\tilde\beta}\rangle$.

Lemma 3.3. *If H tends to B on the curve $\boldsymbol{h}$, $K = \boldsymbol{r}_1(0)$ and $\delta_2(\tilde\beta) \neq 0$, then*

$$\frac{\delta_1(\alpha)}{\delta_2(\tilde\beta)} \to (F_1, F_2; K, B)^k, \quad \textit{where } k \geq 2. \tag{3.2}$$

Proof. If there is $\tilde\beta$ in every neighborhood of zero such that $\delta_2(\tilde\beta) \neq 0$, then, by the analyticity of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, integer $k := \min\{i \in \mathbb{N} : \frac{d^i \boldsymbol{r}_1}{ds}(0) \neq \frac{d^i \boldsymbol{r}_2}{ds}(0)\}$ is well defined, and $k \geq 2$.

If $\lim_{s\to0} \frac{\delta_1(\alpha)}{\delta_2(\tilde\beta)}$ exists, then we can apply L'Hospital's rule, which results in

$$\lim_{s\to0} \frac{\delta_1(\alpha)}{\delta_2(\tilde\beta)} = \lim_{s\to0} \frac{\frac{d\delta_1}{d\alpha}\frac{d\alpha}{ds}}{\frac{d\delta_2}{d\tilde\beta}\frac{d\tilde\beta}{ds}} = \lim_{s\to0} \frac{\frac{d\delta_1}{d\alpha}}{\frac{d\delta_2}{d\tilde\beta}} \lim_{s\to0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}} = \lim_{s\to0} \frac{\frac{d^2\delta_1}{d\alpha^2}}{\frac{d^2\delta_2}{d\tilde\beta^2}} \left(\lim_{s\to0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}\right)^2$$

$$= \cdots = \lim_{s\to0} \frac{\frac{d^k\delta_1}{d\alpha^k}}{\frac{d^k\delta_2}{d\tilde\beta^k}} \left(\lim_{s\to0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}\right)^k. \tag{3.3}$$

For the second limit in (3.3), take the orthogonal projection $H^\perp$ of H onto ℓ, and use L'Hospital's rule to get

$$\frac{|F_2 - B|}{|F_1 - B|} = \lim_{s\to0} \frac{|F_2 - H^\perp|}{|F_1 - H^\perp|} = \lim_{s\to0} \frac{\tan\alpha}{-\tan\tilde\beta} = -\lim_{s\to0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}. \tag{3.4}$$

For the first limit in (3.3), we first observe that

$$\frac{d^k\delta_j}{d\xi^k}(\xi) = \Big\langle \frac{d^k\boldsymbol{r}_1}{ds_{1,j}^k}(s_{1,j}(\xi))\Big(\frac{ds_{1,j}}{d\xi}(\xi)\Big)^k - \frac{d^k\boldsymbol{r}_2}{ds_{2,j}^k}(s_{2,j}(\xi))\Big(\frac{ds_{2,j}}{d\xi}(\xi)\Big)^k, \boldsymbol{u}_\xi\Big\rangle + \Delta,$$

where $\xi=\alpha$ for $j=1$, $\xi=\tilde\beta$ for $j=2$, and $\Delta=\langle \boldsymbol{f}(\xi),\boldsymbol{u}_\xi\rangle+\langle \boldsymbol{g}(\xi),\boldsymbol{u}_{\xi+\pi/2}\rangle$, where vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ are composed of lower order derivatives $d^\ell\boldsymbol{r}_i/ds_{i,j}^\ell(s_{i,j}(\xi))$ $(\ell<k)$ multiplied by a sum of products of various lower order derivatives $(m\le k-\ell)$ of the form $d^m s_{i,j}/d\xi^m(\xi)$ $(i=1,2)$. As for every $0<m\le k-\ell$, $\frac{d^\ell\boldsymbol{r}_1}{ds_{1,j}^\ell}(0)=\frac{d^\ell\boldsymbol{r}_2}{ds_{2,j}^\ell}(0)$ and $\frac{d^m s_{1,j}}{d\xi^m}(0)=\frac{d^m s_{2,j}}{d\xi^m}(0)$, we obtain $\frac{d^k\delta_j}{d\xi^k}(0)=\big\langle\frac{d^k\boldsymbol{r}_1}{ds_{1,j}^k}(0)-\frac{d^k\boldsymbol{r}_2}{ds_{2,j}^k}(0),\boldsymbol{u}_0\big\rangle\big(\frac{ds_{1,j}}{d\xi}(0)\big)^k$. Substituting this, (3.4), and the evident equations $\frac{ds_{1,1}}{d\alpha}(0)=\frac{ds_{2,1}}{d\alpha}(0)=|F_1-K|$ and $\frac{ds_{1,2}}{d\tilde\beta}(0)=\frac{ds_{2,2}}{d\tilde\beta}(0)=-|F_2-K|$ into (3.3) we arrive at

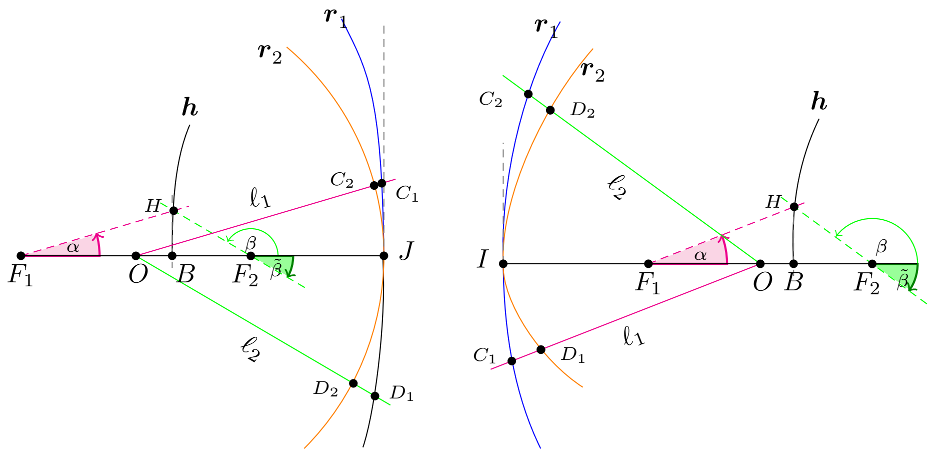
$$\lim_{s\to 0}\frac{\delta_1(\alpha)}{\delta_2(\tilde\beta)} = \Big(\frac{|F_1-K|}{|F_2-K|}\Big)^k\Big(\frac{|F_1-B|}{|F_2-B|}\Big)^k = (F_1,F_2;K,B)^k. \qquad \square$$

Notice that

$$\frac{d^2\delta_j}{d\xi^2}(0) = \Big\langle\frac{d^2\boldsymbol{r}_1}{ds_{1,j}^2}(0)-\frac{d^2\boldsymbol{r}_2}{ds_{2,j}^2}(0),\boldsymbol{u}_0\Big\rangle\Big(\frac{ds_{1,j}}{d\xi}(0)\Big)^2 = \pm(\kappa_1(0)-\kappa_2(0))\Big(\frac{ds_{1,j}}{d\xi}(0)\Big)^2,$$

where κ_1 and κ_2 are the signed curvatures of the curves $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. Thus, the signed curvatures of the curves coincide if and only if $k\ge 3$.

Now modify the previous configuration by changing the position and role of the lines ℓ_1 and ℓ_2.



Let they pass through the midpoint O of the segment $\overline{F_1F_2}$, and close angles α and β with ℓ, respectively. Denote the intersections of ℓ_1 and ℓ_2 with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by $\bar C_1$, $\bar D_1$ and $\bar C_2$, $\bar D_2$, respectively. Finally, let s_i be the arc length parameter of

$\boldsymbol{r}_i$ $(i = 1, 2)$, and introduce $\delta(\alpha) = \langle C_1 - D_1, \boldsymbol{u}_\alpha\rangle$ and $\delta(\tilde\beta) = \langle C_2 - D_2, \boldsymbol{u}_{\tilde\beta}\rangle$ where $\tilde\beta = \beta - \pi$.

Lemma 3.4. *If H tends to B on the curve $\boldsymbol{h}$, $K = \boldsymbol{r}_0(0)$ and $\delta(\tilde\beta) \neq 0$, then*

$$\frac{\delta(\alpha)}{\delta(\tilde\beta)} \to (F_2, F_1; B)^k, \quad \textit{where } k \geq 2. \tag{3.5}$$

Proof. If there is $\tilde\beta$ in every neighborhood of zero such that $\delta(\tilde\beta) \neq 0$, then, by the analyticity of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, integer $k := \min\{i \in \mathbb{N} : \frac{d^i\boldsymbol{r}_1}{ds}(0) \neq \frac{d^i\boldsymbol{r}_2}{ds}(0)\}$ is well defined and $k \geq 2$.

If $\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\tilde\beta)}$ exists, then L'Hospital's rule can be used, which results in

$$\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\tilde\beta)} = \lim_{s\to 0} \frac{\frac{d\delta(\alpha)}{d\alpha}\frac{d\alpha}{ds}}{\frac{d\delta(\tilde\beta)}{d\tilde\beta}\frac{d\tilde\beta}{ds}} = \lim_{s\to 0} \frac{\frac{d\delta(\alpha)}{d\alpha}}{\frac{d\delta(\tilde\beta)}{d\tilde\beta}} \lim_{s\to 0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}} = \lim_{s\to 0} \frac{\frac{d^2\delta(\alpha)}{d\alpha^2}}{\frac{d^2\delta(\tilde\beta)}{d\tilde\beta^2}} \left(\lim_{s\to 0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}\right)^2$$
$$= \cdots = \lim_{s\to 0} \frac{\frac{d^k\delta(\alpha)}{d\alpha^k}}{\frac{d^k\delta(\tilde\beta)}{d\tilde\beta^k}} \left(\lim_{s\to 0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}\right)^k = \left(\lim_{s\to 0} \frac{\frac{d\alpha}{ds}}{\frac{d\tilde\beta}{ds}}\right)^k.$$

By (3.4), this proves the lemma. □

Notice again that the signed curvatures of the curves coincide if and only if $k \geq 3$.

4. Minkowski planes with a quadratic hyperbola

We consider the quadratic hyperbola $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ with eccentricity $2f = d_\mathcal{I}(F_1, F_2)$ in the Minkowski plane $(\mathbb{R}^2, d_\mathcal{I})$ with indicatrix $\mathcal{I}$. By [4, (ii) of Theorem 3] every straight line parallel to F_1F_2 intersects $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ in exactly two points, hence $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is a hyperbolic quadric.

According to Subsection 2.1, the left-branch and right-branch of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ intersect $\overline{F_1F_2}$ in the points A and B. Let t_A, t_B be the tangents of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ at A, B, respectively. Then, the obvious symmetry in the midpoint O of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ entails $t_A \parallel t_B$.

Let $\mathcal{I}_O$ be the translate of $\mathcal{I}$ centered at O, and denote its intersections with line F_1F_2 by I, J so that I is on the ray $\overline{O}F_1$ and J is on $\overline{O}F_2$. Denote the tangents of $\mathcal{I}$ at I, J by t_I, t_J, respectively. Then $t_I \parallel t_J$ by the symmetry of $\mathcal{I}$.

As a hyperbolic quadric, $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ has two asymptotes ℓ_+ and ℓ_- through O. These intersect the straight lines ℓ_1 and ℓ_2 through F_1 and F_2, respectively, in points $P_1^\pm = \ell_\pm \cap \ell_1$ and $P_2^\pm = \ell_\pm \cap \ell_2$.

Introduce now an affine coordinate system such as $O = (0, 0)$, $J = (1, 0)$, and $P_2^+ = (f, \sqrt{f^2 - a^2})$. Choose the Euclidean metric d_e so that $\{(1, 0), (0, 1)\}$ is an orthonormal basis. Then $F_1 = (-f, 0)$, $F_2 = (f, 0)$, $A = (-a, 0)$, and $B = (a, 0)$.

Given the Euclidean metric d_e, we can define r as the radial function of $\partial\mathcal{I}_O$ with respect to O, the angles $\alpha = \angle(HF_1O)$, $\tilde{\beta} = \angle(HF_2B)$ ($\beta := \pi - \tilde{\beta}$) and $\varphi = \angle(HOB)$ for the points H on the B-branch (that contains B) of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$. Finally, we define the lengths $h_1(\alpha) := d_e(F_1, H)$, $h_2(\beta) := d_e(F_2, H)$, and $h(\varphi) := d_e(O, H)$. Then $d_\mathcal{I}(F_1, H) = h_1(\alpha)/r(\alpha)$, and $d_\mathcal{I}(F_2, H) = h_2(\beta)/r(\beta)$, so we have

$$2a = \frac{h_1(\alpha)}{r(\alpha)} - \frac{h_2(\beta)}{r(\beta)}. \tag{4.1}$$

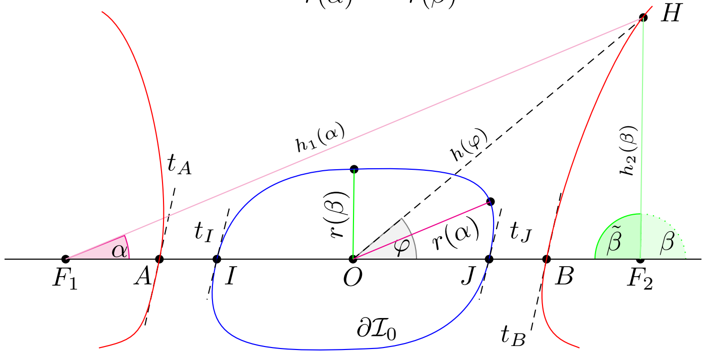


FIGURE 4.1. A hyperbola in a Minkowski plane

Lemma 4.1. *Tangents t_A, t_B, t_I and t_J are all parallel.*

Proof. Due to the quadraticity, φ and H are bijectively related, hence the functions $\alpha(\varphi)$, $\beta(\varphi)$ are also well defined. Differentiating (4.1) with respect to φ leads to

$$0 = \frac{\frac{dh_1(\alpha)}{d\alpha} r(\alpha) - h_1(\alpha)\frac{dr(\alpha)}{d\alpha}}{r^2(\alpha)} \frac{d\alpha}{d\varphi} - \frac{\frac{dh_2(\beta)}{d\beta} r(\beta) - h_2(\beta)\frac{dr(\beta)}{d\beta}}{r^2(\beta)} \frac{d\beta}{d\varphi}. \tag{4.2}$$

As $\varphi = 0$ implies $\alpha = 0$, $\beta = \pi$, $r(0) = r(\pi) = 1$, and $\frac{dh_1}{d\alpha}(0) = \frac{dh_2}{d\beta}(\pi) = 0$ follows from $t_B \perp_{d_e} F_1F_2$, (4.2) gives at $\varphi = 0$ that

$$r'(0)\Big[-h_1(0)\frac{d\alpha}{d\varphi}(0) + h_2(\pi)\frac{d\beta}{d\varphi}(0)\Big] = 0.$$

According to (3.4), $h_1(0) = |F_1 - B|$, and $h_2(\pi) = |F_2 - B|$, the second factor in the left-hand side is positive, hence $r'(0) = 0$ follows that is $t_B \perp_{d_e} F_1F_2$, which proves the lemma. □

Lemma 4.2. *The curve $\partial\mathcal{I}_O$ is analytic in a neighborhood of I and J.*

Proof. The radial functions h_1, h_2, the angles $\alpha(s)$, $\beta(s)$, and the inverses of the angles, where s is the arc length parameter, are clearly analytic, hence we deduce that $\beta(\alpha)$ and $\alpha(\beta)$ are also analytic functions.

As $x \mapsto 1/x$ is analytic in a neighborhood of 1, in order to prove that $r(\alpha)$ is analytic in a neighborhood of 0, it is enough to prove that $\bar{r}(\alpha) := 1/r(\alpha)$ is analytic in a neighborhood of 0.

Bearing this in mind, we reformulate (4.1) as

$$\bar{r}(\alpha) = \frac{h_2(\tilde{\beta}(\alpha))}{h_1(\alpha)}\bar{r}(\tilde{\beta}(\alpha)) + \frac{2a}{h_1(\alpha)}. \tag{4.3}$$

Let us now introduce the functions $f(\alpha) := \tilde{\beta}(\alpha)$, $g(\alpha) := \frac{h_2(\tilde{\beta}(\alpha))}{h_1(\alpha)}$, and $h(\alpha) := \frac{2a}{h_1(\alpha)}$. Then $\phi(\alpha) := \bar{r}(\alpha)$ is a solution of the functional equation

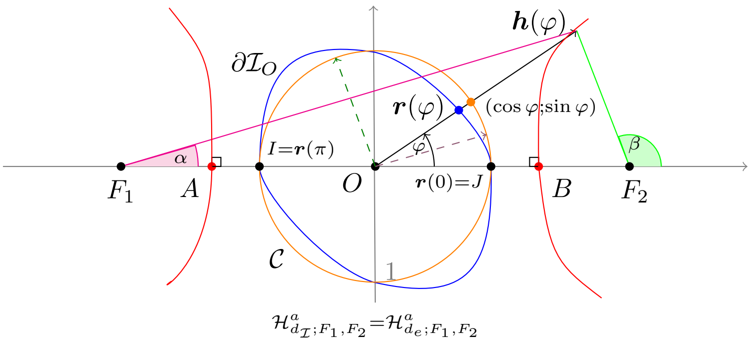
$$\phi(\alpha) = g(\alpha)\phi(f(\alpha)) + h(\alpha),$$

in which functions f, g and h are analytic in a neighborhood of 0, $\frac{df}{d\alpha}(0) = \frac{h_2(0)}{h_1(0)} < 1$, $g(0) = \frac{h_2(0)}{h_1(0)} < 1$, and $h(0) = \frac{2a}{h_1(0)} = \frac{2a}{|F1-B|} < 1$. By [3, Theorem 4.6], such a functional equation has a unique solution for ϕ, which additionally is analytic in a neighborhood of 0. Consequently, $r(\alpha)$ is the reciprocal of that unique analytic solution, so $\partial\mathcal{I}_O$ is analytic around J, and, by its symmetry, around I too. □

Theorem 4.3. *A Minkowski-plane that has a quadratic hyperbola is Euclidean.*

Proof. We compare $\partial\mathcal{I}_O$, analytic by Lemma 4.2, with the unit circle $\mathcal{C}$ of d_e.

Observe that hyperbolas $\mathcal{H}^a_{d_e;F_1,F_2}$ and $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ have two common tangents t_A and t_B, two common asymptotes, and two common points A and B, hence, due to their quadraticity, they coincide.



By the definition of $\mathcal{H}^a_{d_e;F_1,F_2}$ we have $h_1(\alpha) - h_2(\beta) = 2a$, which together with (4.1) implies

$$\delta(\alpha) = \delta(\beta)\frac{h_2(\beta)}{h_1(\alpha) + 2a\delta(\beta)}, \tag{4.4}$$

where $\delta(\alpha) = 1 - r(\alpha)$ is the radial difference of $\mathcal{C}$ and $\partial\mathcal{I}_O$.

If in every neighborhood of I curves $\mathcal{C}$ and $\partial\mathcal{I}_O$ differ, then (4.4) implies

$$\lim_{\varphi\to 0}\frac{\delta(\alpha)}{\delta(\beta)} = \frac{f-a}{f+a} = (F_2, F_1; B),$$

which contradicts (3.5). It follows that in a neighborhood of I curves $\mathcal{C}$ and $\partial\mathcal{I}_O$ coincide.

However, if $\delta(\beta_0) \neq 0$ for any β_0, then no value of the 0-convergent sequence β_{2i} constructed in Lemma 3.2 can vanish by (4.4), therefore no β_0 can exist for which $\delta(\beta_0) \neq 0$.

By the symmetry of the configuration, we deduce also, that no α can exist for which $\delta(\alpha) \neq 0$, hence $\mathcal{C}$ and $\partial\mathcal{I}_O$ coincide. $\square$

5. Hilbert planes with a quadratic hyperbola

Let $\mathcal{I} \subset \mathbb{R}^2 \subset \mathbb{P}^2$ be a bounded, strictly convex open domain, and consider the Hilbert plane $(\mathcal{I}, d_{\mathcal{I}})$. Let $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ be a quadratic hyperbola in $(\mathcal{I}, d_{\mathcal{I}})$.

Let the intersections of line $\ell = F_1F_2$ with $\partial\mathcal{I}$ be denoted by I and J so that $F_1 \in \overline{IF_2}$, and denote the tangents of $\mathcal{I}$ at I and J by t_I and t_J, respectively.

Take the point $T_{\mathcal{I}} = t_I \cap t_J$ in $\mathbb{P}^2$ and a straight line $\ell \subset \mathbb{P}^2$ through $T_{\mathcal{I}}$ that avoids $\mathcal{I}$. From now on, we consider the configuration in the affine plane $\mathbb{R}^2$ in which $T_{\mathcal{I}}$ is on the ideal line, hence $t_I \parallel t_J$.

By Lemma 3.1, there is a perspectivity such that the respective image points I, F_1', F_2', J' of I, F_1, F_2, J satisfy $\overrightarrow{IF_1'} = \overrightarrow{F_2'J'}$, meanwhile $t_I \parallel t_{J'}$. Thus, considering the configuration in the image plane allows us to assume from now on that $\overrightarrow{IF_1} = \overrightarrow{F_2J}$.

Denote the intersections of line ℓ with $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ by A and B so that $A \in \overline{F_1B}$. By the definition of $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ we have

$$(I,J;A,F_1)(I,J;A,F_2) = e^{-4a} = (I,J;F_1,B)(I,J;F_2,B),$$

which implies $(I,J;A)(I,J;B) = (I,J;F_1)(I,J;F_2)$. As $(I,J;F_1)(I,J;F_2) = 1$ follows from $\overrightarrow{IF_1} = \overrightarrow{F_2J}$, and $(I,J;A)(I,J;B) = 1$ gives $\overrightarrow{IA} = \overrightarrow{BJ}$, we found that the affine and metric midpoints of the segments $\overline{IJ}$, $\overline{AB}$ and $\overline{F_1F_2}$ coincide. Let this point be denoted by O.

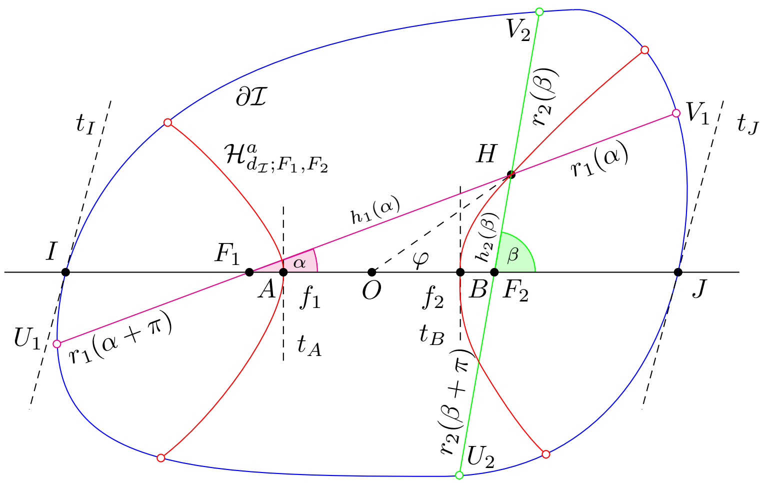


Figure 5.1. Metric hyperbola in the Hilbert plane

Take the straight line ℓ_O through O that is parallel to t_I. Fix an affine coordinate system so that $O=(0,0)$, $J=(1,0)$ and a point $Y\in\ell_O\setminus\{O\}$ is $(0,1)$. Let d_e be the Euclidean metric such that $\{(0,1),(1,0)\}$ is an orthonormal bases.

Let H be a moving point on $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ that defines angles $\alpha=\angle(HF_1J)$, $\beta=\angle(HF_2J)$ and $\varphi=\angle(HOJ)$; points $V_1=\overline{F_1}H\cap\partial\mathcal{I}$, $V_2=\overline{F_2}H\cap\partial\mathcal{I}$, $U_1=\overline{H}F_1\cap\partial\mathcal{I}$, and $U_2=\overline{H}F_2\cap\partial\mathcal{I}$; and distances $r_1(\alpha)=d_e(F_1,V_1)$, $r_2(\beta)=d_e(F_2,V_2)$, $h_1(\alpha)=d_e(F_1,H)$, and $h_2(\beta)=d_e(F_2,H)$ (see Figure 5.1).

Lemma 5.1. *The respective tangents t_A and t_B of $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ at A and B, respectively, are parallel with t_I and t_J.*

Proof. We start with the Hilbert distances

$$\begin{aligned}d_{\mathcal{I}}(F_1,H)&=-\frac{1}{2}\ln\Big(\frac{r_1(\alpha+\pi)}{r_1(\alpha)}\Big/\frac{r_1(\alpha+\pi)+h_1(\alpha)}{r_1(\alpha)-h_1(\alpha)}\Big),\\ d_{\mathcal{I}}(H,F_2)&=-\frac{1}{2}\ln\Big(\frac{r_2(\beta+\pi)}{r_2(\beta)}\Big/\frac{r_2(\beta+\pi)+h_2(\beta)}{r_2(\beta)-h_2(\beta)}\Big).\end{aligned}\tag{5.1}$$

As we have $d_{\mathcal{I}}(F_1,B)-d_{\mathcal{I}}(B,F_2)=2a$, and $d_{\mathcal{I}}$ is continuous, there is a neighborhood $\mathcal{B}$ of B in which $d_{\mathcal{I}}(F_1,H)>d_{\mathcal{I}}(H,F_2)$ for every point H. Thus, the equation

$$2a=d_{\mathcal{I}}(F_1,H)-d_{\mathcal{I}}(H,F_2)=-\frac{1}{2}\ln\frac{\frac{r_1(\alpha+\pi)}{r_1(\alpha)}\Big/\frac{r_1(\alpha+\pi)+h_1(\alpha)}{r_1(\alpha)-h_1(\alpha)}}{\frac{r_2(\beta+\pi)}{r_2(\beta)}\Big/\frac{r_2(\beta+\pi)+h_2(\beta)}{r_2(\beta)-h_2(\beta)}}$$

describes $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ in $\mathcal{B}$. After some rearrangement this gives

$$e^{-4a}\Big(1+\frac{h_1(\alpha)}{r_1(\alpha+\pi)}\Big)\Big(1-\frac{h_2(\beta)}{r_2(\beta)}\Big)=\Big(1-\frac{h_1(\alpha)}{r_1(\alpha)}\Big)\Big(1+\frac{h_2(\beta)}{r_2(\beta+\pi)}\Big).\tag{5.2}$$

By (5.1), the sum $2t(\alpha)=d_{\mathcal{I}}(F_1,H)+d_{\mathcal{I}}(H,F_2)$ is

$$2t(\alpha)=-\frac{1}{2}\ln\Big[\Big(\frac{r_1(\alpha+\pi)}{r_1(\alpha)}\Big/\frac{r_1(\alpha+\pi)+h_1(\alpha)}{r_1(\alpha)-h_1(\alpha)}\Big)\Big(\frac{r_2(\beta+\pi)}{r_2(\beta)}\Big/\frac{r_2(\beta+\pi)+h_2(\beta)}{r_2(\beta)-h_2(\beta)}\Big)\Big],$$

which, after some rearrangements, results in

$$e^{-4t(\alpha)}\Big(1+\frac{h_1(\alpha)}{r_1(\alpha+\pi)}\Big)\Big(1+\frac{h_2(\beta)}{r_2(\beta+\pi)}\Big)=\Big(1-\frac{h_1(\alpha)}{r_1(\alpha)}\Big)\Big(1-\frac{h_2(\beta)}{r_2(\beta)}\Big).$$

Multiplying (5.2) with this and taking square root of the product yield

$$e^{2a+2t(\alpha)}\Big(1-\frac{h_1(\alpha)}{r_1(\alpha)}\Big)=1+\frac{h_1(\alpha)}{r_1(\alpha+\pi)}.$$

Expressing $h_1(\alpha)$ gives

$$h_1(\alpha)=\frac{(e^{2a+2t(\alpha)}-1)r_1(\alpha)r_1(\alpha+\pi)}{e^{2a+2t(\alpha)}r_1(\alpha+\pi)+r_1(\alpha)}.$$

The derivative of this vanishes at 0, because the derivative of r_1 vanishes at 0 and at π as $t_I \perp \ell \perp t_J$, and the derivative of t also vanishes at 0 as $2t(\alpha) \geq d_{\mathcal{I}}(F_2, F_1)$ by the triangle inequality of $d_{\mathcal{I}}$, and equality holds if and only if $H \in \overline{F_1F_2}$, i.e. when $H = B$, due to the strictness of the triangle inequality. Thus, $t_B \perp \ell$.

The very same reasoning for point A leads to the deduction of $t_A \perp \ell$, so the lemma is proved. □

Lemma 5.2. *If $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ is the intersection of a hyperbolic quadric $\mathcal{H}$ with $\mathcal{I}$, then point $Y \in \ell_O$ can be chosen so that for the open unit disc $\mathcal{D}$ of d_e the hyperbola $\mathcal{H}^a_{d_{\mathcal{D}};F_1,F_2}$ in the hyperbolic plane $(\mathcal{D}, d_{\mathcal{D}})$ coincides with $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ in $\mathcal{I} \cap \mathcal{D}$.*

Proof. The touching points of two parallel tangents of $\mathcal{H}$ are symmetric in the center of $\mathcal{H}$, hence O is the center of $\mathcal{H}$. Therefore, the asymptotes ℓ_+ and ℓ_- of $\mathcal{H}$ intersect each other in O. Let C be the opposite vertice of A in the parallelogram defined by edges ℓ and t_A, vertice O, and diagonal ℓ_-. (See Figure 5.2.)

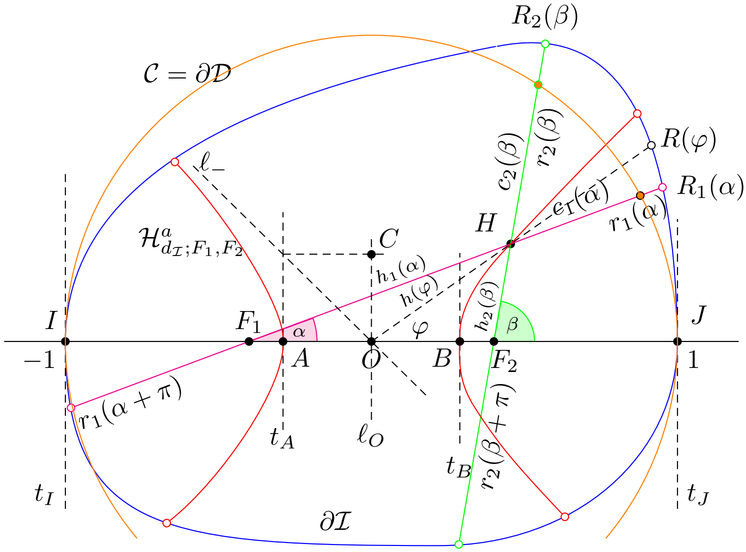


FIGURE 5.2. Common hyperbola of Hilbert planes $(\mathcal{I}, d_{\mathcal{I}})$ and $(\mathcal{C}, d_{\mathcal{C}})$.

By (2.1), hyperbola $\mathcal{H}^a_{d_{\mathcal{D}};F_1,F_2}$ is the intersection of $\mathcal{D}$ and a hyperbolic quadric $\mathcal{H}'$, and, by symmetry, the asymptotes ℓ'_+ and ℓ'_- of $\mathcal{H}'$ intersect each other in O. Let C' be the opposite edge of A in the parallelogram defined by vertices ℓ and t_A, edge O, and diagonal ℓ'_-.

Points A, B and tangents t_A, t_B are common of $\mathcal{H}$ and $\mathcal{H}'$ by Lemma 5.1.

Equation (2.1) gives the polar-equation

$$\frac{1}{\sinh^2(r(\omega))} = \frac{\cos^2 \omega}{\sinh^2(a)} + \frac{\sin^2 \omega}{(\tanh^2(a) - \tanh^2(f))\cosh^2(a)},$$

for $\mathcal{H}^a_{d_{\mathcal{D}};F_1,F_2}$, that shows $C' = \big(0, \sqrt{\tanh^2 f - \tanh^2 a}\big)$.

Thus, choosing $Y \in \ell_O$ so that the coordinates of C be $\big(0, \sqrt{\tanh^2 f - \tanh^2 a}\big)$ in the affine coordinate system given by $O = (0,0)$, $J = (1,0)$ and $Y = (0,1)$, results in $\mathcal{H}'$ such that $C = C'$. In this case $\mathcal{H}$ and $\mathcal{H}'$ have A, B, t_A, t_B and $\ell_\pm$ in common, therefore, as they are quadrics, they coincidence, hence the statement of the lemma. □

Comparing Figure 5.2 to Figure 5.1, we let $R_1(\alpha) := V_1$, $R_1(\alpha+\pi) := U_1$, $R_2(\beta) := V_2$, $R_2(\beta+\pi) := U_2$, and $H(\varphi) := H$, furthermore introduce the point $C(\varphi)$ as the intersection of $\overline{O}H$ with $\mathcal{C}$. Finally, for $j \in \{1,2\}$, we let C_j and R_j be the points where $\overline{F_j}H$ intersects $\mathcal{C}$ and $\partial\mathcal{I}$, respectively, and introduce distances $c_1(\alpha) = d_e(F_1, C_1(\alpha))$, and $c_2(\beta) = d_e(F_2, C_2(\beta))$

Proposition 5.3. *If $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ is the intersection of a hyperbolic quadric with $\mathcal{I}$, and $\partial\mathcal{I}$ is analytic around the points I and J, then*

$$1 = \frac{1+\tanh a}{1-\tanh a}\left|\frac{\left(\frac{\tanh f+\tanh a}{\tanh f-\tanh a}\right)^{k-1} + \left(\frac{\tanh f-1}{\tanh f+1}\right)^{-(k-1)}}{\left(\frac{\tanh f-1}{\tanh f+1}\right)^{-(k-1)}\left(\frac{\tanh f+\tanh a}{\tanh f-\tanh a}\right)^{k-1} + 1}\right| \tag{5.3}$$

or $\partial\mathcal{I}$ coincides with $\mathcal{C}$ in a neighborhood of points I and J.

Proof. Let $\delta_1(\alpha) := r_1(\alpha) - c_1(\alpha)$ and $\delta_2(\beta) := r_2(\beta) - c_2(\beta)$, and, in case of non-vanishing denominators, let

$$\sigma_1(\alpha) := \frac{\delta_1(\alpha+\pi)}{\delta_1(\alpha)} \quad \text{and} \quad \sigma_2(\beta) := \frac{\delta_2(\beta)}{\delta_2(\beta+\pi)}, \tag{5.4}$$

$$\tau(\alpha) := \frac{\delta_1(\alpha)}{\delta_2(\beta+\pi)} \quad \text{and} \quad \varrho(\alpha) := \frac{\sigma_1(\alpha)}{\sigma_2(\beta)}. \tag{5.5}$$

Notice that $\varphi \to 0$ implies

$$c_1(\alpha), r_1(\alpha) \to 1 + \tanh f, \quad c_1(\alpha+\pi), r_1(\alpha+\pi) \to 1 - \tanh f,$$
$$c_2(\beta), r_2(\beta) \to 1 - \tanh f, \quad c_2(\beta+\pi), r_2(\beta+\pi) \to 1 + \tanh f,$$

and for non-vanishing denominators, by Lemma 3.3, these give

$$\tau(\alpha) \to (F_1, F_2; J, B)^k \ \text{ and } \ \varrho(\alpha) \to \frac{(F_1, F_2; I, B)^k}{(F_1, F_2; J, B)^k}, \qquad (k \geq 2). \tag{5.6}$$

Following (5.2) for for $\mathcal{C}$, we have

$$\Big(1 + \frac{h_1(\alpha)}{c_1(\alpha+\pi)}\Big)\Big(1 - \frac{h_2(\beta)}{c_2(\beta)}\Big) = e^{4a}\Big(1 - \frac{h_1(\alpha)}{c_1(\alpha)}\Big)\Big(1 + \frac{h_2(\beta)}{c_2(\beta+\pi)}\Big). \tag{5.7}$$

Using δ_1 and δ_2 in (5.2) leads to

$$\begin{aligned}\Big(1+\frac{h_1(\alpha)}{c_1(\alpha+\pi)+\delta_1(\alpha+\pi)}\Big)\Big(1-\frac{h_2(\beta)}{c_2(\beta)+\delta_2(\beta)}\Big)&\\ =e^{4a}\Big(1-\frac{h_1(\alpha)}{c_1(\alpha)+\delta_1(\alpha)}\Big)\Big(1+\frac{h_2(\beta)}{c_2(\beta+\pi)+\delta_2(\beta+\pi)}\Big).&\end{aligned}\tag{5.8}$$

Subtracting (5.7) from this, then dividing by $\delta_2(\beta+\pi)\neq 0$, and using (5.4) and (5.5), we arrive at

$$\begin{aligned}&\Big(1+\frac{h_1(\alpha)}{c_1(\alpha+\pi)}\Big)\frac{h_2(\beta)\sigma_2(\beta)}{c_2(\beta)\big(c_2(\beta)+\delta_2(\beta)\big)}-\\ &\quad-\frac{h_1(\alpha)\varrho(\alpha)\sigma_2(\beta)\tau(\alpha)}{c_1(\alpha+\pi)\big(c_1(\alpha+\pi)+\delta_1(\alpha+\pi)\big)}\Big(1-\frac{h_2(\beta)}{c_2(\beta)}\Big)-\\ &\quad-\frac{h_1(\alpha)\delta_1(\alpha+\pi)}{c_1(\alpha+\pi)\big(c_1(\alpha+\pi)+\delta_1(\alpha+\pi)\big)}\frac{h_2(\beta)\sigma_2(\beta)}{c_2(\beta)\big(c_2(\beta)+\delta_2(\beta)\big)}\\ &=-e^{4a}\Big(1-\frac{h_1(\alpha)}{c_1(\alpha)}\Big)\frac{h_2(\beta)}{c_2(\beta+\pi)\big(c_2(\beta+\pi)+\delta_2(\beta+\pi)\big)}+\\ &\quad+e^{4a}\frac{h_1(\alpha)\tau(\alpha)}{c_1(\alpha)\big(c_1(\alpha)+\delta_1(\alpha)\big)}\Big(1+\frac{h_2(\beta)}{c_2(\beta+\pi)}\Big)-\\ &\quad-e^{4a}\frac{h_1(\alpha)\delta_1(\alpha)}{c_1(\alpha)\big(c_1(\alpha)+\delta_1(\alpha)\big)}\frac{h_2(\beta)}{c_2(\beta+\pi)\big(c_2(\beta+\pi)+\delta_2(\beta+\pi)\big)}.\end{aligned}\tag{5.9}$$

Let us now take the limit of this for $\varphi\to 0$, which involves $\alpha\to 0$ and $\beta\to\pi$. Using (5.6), we obtain for any point of accumulation $\hat\sigma_2$ of σ_2 in $\mathbb{R}\cup\{-\infty,\infty\}$ that

$$\begin{aligned}&\frac{1+\tanh a}{1-\tanh f}\frac{\tanh f-\tanh a}{(1+\tanh f)^2}\hat\sigma_2-\frac{(\tanh f+\tanh a)(F_1,F_2;I,B)^k}{(1-\tanh f)^2}\frac{1+\tanh a}{1+\tanh f}\hat\sigma_2\\ &=e^{4a}\frac{(\tanh f+\tanh a)(F_1,F_2;J,B)^k}{(1+\tanh f)^2}\frac{1-\tanh a}{1-\tanh f}-e^{4a}\frac{1-\tanh a}{1+\tanh f}\frac{\tanh f-\tanh a}{(1-\tanh f)^2},\end{aligned}\tag{5.10}$$

hence, by taking into account that $\frac{1+\tanh a}{1-\tanh a}=e^{2a}$, we obtain that

$$\hat\sigma_2=-e^{2a}\frac{\frac{\tanh f-\tanh a}{1-\tanh f}-\frac{(\tanh f+\tanh a)}{1+\tanh f}(F_1,F_2;J,B)^k}{\frac{\tanh f-\tanh a}{1+\tanh f}-\frac{(\tanh f+\tanh a)}{1-\tanh f}(F_1,F_2;I,B)^k}.$$

Substitution of

$(F_1,F_2;I,B)=-\frac{1-\tanh f}{1+\tanh f}\frac{\tanh f-\tanh a}{\tanh f+\tanh a}$ and $(F_1,F_2;J,B)=-\frac{1+\tanh f}{1-\tanh f}\frac{\tanh f-\tanh a}{\tanh f+\tanh a}$

results in

$$\hat\sigma_2=-e^{2a}\frac{1+\tanh f}{1-\tanh f}\frac{1+(F_1,F_2;J,B)^{k-1}}{1+(F_1,F_2;I,B)^{k-1}}=-e^{2a}\frac{1+(F_1,F_2;J,B)^{k-1}}{1+(F_1,F_2;I,B)^{k-1}}(I,J;F_2).$$

By (5.5) and (5.6) we have

$$\frac{\hat{\sigma}_1}{\hat{\sigma}_2}=\lim_{\varphi\to 0}\frac{\sigma_1(\alpha)}{\sigma_2(\beta)}=\lim_{\varphi\to 0}\varrho(\alpha)=(F_1,F_2;I,J)^k=\frac{(I,J;F_1)^k}{(I,J;F_2)^k},$$

hence

$$\frac{\hat{\sigma}_1}{(I,J;F_1)^k}=\frac{\hat{\sigma}_2}{(I,J;F_2)^k}=\frac{-e^{2a}}{(I,J;F_2)^{k-1}}\frac{1+(F_1,F_2;J,B)^{k-1}}{1+(F_1,F_2;I,B)^{k-1}}.$$

As $\hat{\sigma}_1\hat{\sigma}_2=1$, $(I,J;F_1)(I,J;F_2)=1$, and $(I,J;F_2)=-(F_1,F_2;I)^{-1}=-(F_1,F_2;J)$ by the central symmetry of the configuration in point O, we deduce

$$\begin{aligned}1=\hat{\sigma}_1\hat{\sigma}_2&=\Big(\frac{e^{2a}}{(I,J;F_2)^{k-1}}\frac{1+(F_1,F_2;J,B)^{k-1}}{1+(F_1,F_2;I,B)^{k-1}}\Big)^2\\&=\Big(\frac{e^{2a}}{(F_1,F_2;I)^{1-k}}\frac{(F_1,F_2;B)^{k-1}+(F_1,F_2;J)^{k-1}}{(F_1,F_2;B)^{k-1}+(F_1,F_2;I)^{k-1}}\Big)^2\\&=e^{2a}\left|\frac{(F_1,F_2;B)^{k-1}+(F_1,F_2;I)^{-(k-1)}}{(F_1,F_2;I)^{-(k-1)}(F_1,F_2;B)^{k-1}+1}\right|.\end{aligned}$$

Substituting the expressions of e^{2a} and the affine ratios in terms of $\tanh a$ and $\tanh f$ yields (5.3) for the case of the existence of a non-vanishing sequence $\alpha_i\to 0$ for which $\delta_1(\alpha_i)\neq 0$, $\delta_2(\beta_i)\neq 0$, and $\delta_2(\beta_i+\pi)\neq 0$.

To finish the proof of the proposition it remains to consider the case when $\delta_1(\alpha_i)\delta_2(\beta_i+\pi)\delta_2(\beta_i)=0$ for any non-vanishing sequence $\alpha_i\to 0$. In this case one of the factors $\delta_1(\alpha_i)$, $\delta_2(\beta_i+\pi)$, and $\delta_2(\beta_i)$ vanishes for any non-vanishing sequence $\alpha_i\to 0$ at infinitely many i, which, by the analyticity of $\partial\mathcal{I}$ and $\mathcal{C}$, implies that there is an $\varepsilon>0$ such that for $|\alpha|<\varepsilon$ either $\delta_1(\alpha)=0=\delta_2(\beta+\pi)$ or $\delta_1(\alpha+\pi)=0=\delta_2(\beta)$.

Assume first that $\delta_1(\alpha)=0=\delta_2(\beta+\pi)$ for $|\alpha|<\varepsilon$, that is, $\mathcal{C}$ and $\mathcal{I}$ coincide in a neighborhood of J. Subtracting (5.7) from (5.8) gives

$$\begin{aligned}&\Big(1+\frac{h_1(\alpha)}{c_1(\alpha+\pi)}\Big)\frac{h_2(\beta)\delta_2(\beta)}{c_2(\beta)\big(c_2(\beta)+\delta_2(\beta)\big)}-\\&\qquad-\frac{h_1(\alpha)\delta_1(\alpha+\pi)}{c_1(\alpha+\pi)\big(c_1(\alpha+\pi)+\delta_1(\alpha+\pi)\big)}\Big(1-\frac{h_2(\beta)}{c_2(\beta)}\Big)-\\&\qquad-\frac{h_1(\alpha)\delta_1(\alpha+\pi)}{c_1(\alpha+\pi)\big(c_1(\alpha+\pi)+\delta_1(\alpha+\pi)\big)}\frac{h_2(\beta)\delta_2(\beta)}{c_2(\beta)\big(c_2(\beta)+\delta_2(\beta)\big)}=0.\end{aligned}$$

If there is a non-vanishing sequence $\alpha_i\to 0$ such that $\delta_2(\beta_i)\neq 0$ for all indexes i, then division by $\delta_2(\beta_i)$ and application of (3.2) gives

$$\Big(1+\frac{h_1(0)}{c_1(\pi)}\Big)\frac{h_2(\pi)}{c_2^2(\pi)}-\frac{h_1(0)(F_1,F_2;I,B)^k}{c_1^2(\pi)}\Big(1-\frac{h_2(\pi)}{c_2(\pi)}\Big)=0,$$

where the integer k is at least 2. Substitution of the values $h_1(0) = \tanh f + \tanh a$, $h_2(\pi) = \tanh f - \tanh a$, $c_1(\pi) = 1 - \tanh f$, $c_2(\pi) = 1 + \tanh f$, and $(F_1, F_2; I, B) = -\frac{1-\tanh f}{1+\tanh f}\frac{\tanh f - \tanh a}{\tanh f + \tanh a}$, we arrive at

$$1 = (-1)^k \Big(\frac{1 - \tanh f}{1 + \tanh f}\frac{\tanh f - \tanh a}{\tanh f + \tanh a}\Big)^{k-1},$$

which is a contradiction as the absolute value of the right-hand side is less than 1. So $\mathcal{C}$ and $\mathcal{I}$ coincides in a neighborhood of I, as well.

Let us rephrase this result as follows:

$$\text{\textit{if $\mathcal{C}$ and $\mathcal{I}$ coincide in a neighborhood of J, then they coincide in a neighborhood of I, as well.}} \tag{5.11}$$

Secondly, assume now that $\delta_2(\beta) = 0$ (consequently, $\delta_1(\alpha + \pi) = 0$) for $|\alpha| < \varepsilon$, that is, $\mathcal{C}$ and $\mathcal{I}$ coincides in a neighborhood of I. By the symmetry of our given configuration, we immediately deduce by (5.11) that then $\mathcal{C}$ and $\mathcal{I}$ coincide in a neighborhood of J, as well.

The proof is complete. □

Proposition 5.4. *If $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ is the intersection of a hyperbolic quadric $\mathcal{H}$ with $\mathcal{I}$, then coincidence of $\partial\mathcal{I}$ and $\mathcal{C}$ in a neighborhood of I and J implies their identity.*

Proof. For a point $H \in \mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ we call the lines HF_1 and HF_2 *focal*. A focal line F_jH is of type $\ddot{C}\ddot{C}$ if both ot its intersections with $\partial\mathcal{I}$ is also on $\mathcal{C}$. A focal line F_jH is of type $\ddot{C}\ddot{D}$ or $\ddot{D}\ddot{C}$ if the intersections of $\overline{F_j}H$ or $F_j\overline{H}$, respectively, with $\partial\mathcal{I}$ is on $\mathcal{C}$, and the other intersection of F_jH with $\partial\mathcal{I}$ is not on $\mathcal{C}$.

The proof will be proceeded in several steps.

Equation (5.9) clearly shows that if any three of $\delta_2(\beta)$, $\delta_2(\beta + \pi)$, $\delta_1(\alpha)$, and $\delta_1(\alpha+\pi)$ vanish, then the fourth one vanishes, too. Therefore we have the following:

$$\text{\textit{There does not exist a point $H \in \mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ such that one of its focal lines is $\ddot{C}\ddot{C}$, while the other is $\ddot{C}\ddot{D}$.}} \tag{5.12}$$

As there are common arcs of $\partial\mathcal{I}$ and $\mathcal{C}$ through I and J, either $R(\varphi) = C(\varphi)$ for every $\varphi \in [0, \pi]$, or there are angles $\varphi_+ < \varphi^+$ in $(0, \pi)$ such that $R(\varphi) = C(\varphi)$ for all $\varphi \in [0, \varphi_+] \cup [\varphi^+, \pi]$, and $R(\varphi) \neq C(\varphi)$ for some $\varphi \in (\varphi_+, \varphi^+)$ in all neighborhoods of φ_+ and φ^+.

Further, either $R(\varphi) = C(\varphi)$ for every $\varphi \in [\pi, 2\pi]$, or there are angles $\varphi^- < \varphi_-$ in $(\pi, 2\pi)$ such that $R(\varphi) = C(\varphi)$ for all $\varphi \in [\pi, \varphi^-] \cup [\varphi_-, 2\pi]$, and $R(\varphi) \neq C(\varphi)$ for some $\varphi \in (\varphi^-, \varphi_-)$ in all neighborhoods of φ^- and φ_-.

Both intervals $[\varphi^+, \varphi^-]$ and $[\varphi_-, \varphi_+]$ will be called interval of *initial coincidence* (IC for short) and every corresponding common arc of $\partial\mathcal{I}$ and $\mathcal{C}$ will be called arc of *initial coincidence* (IC for short). We shall use the shorter notations $R_+ = R(\varphi_+)$, $R_- = R(\varphi_-)$, $R^+ = R(\varphi^+)$, and $R^- = R(\varphi^-)$, and regard the arcs on $\partial\mathcal{I}$ or $\mathcal{C}$ positively oriented, so $[R^+R^-]$ and $[R_-R_+]$ are the arcs of IC.

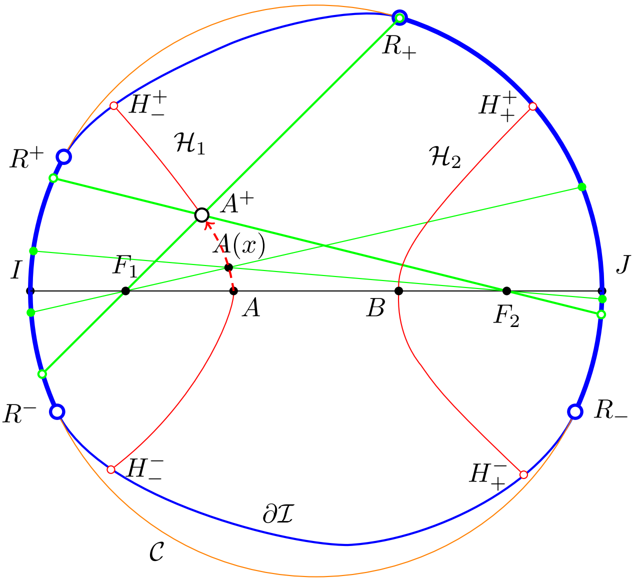


FIGURE 5.3. $A^+ = A(x^+)$: critical point of the "upper" segment of $\mathcal{H}_-$

For the sake of brevity, a point P of $\mathcal{R} = \{R_+, R^+, R^-, R_-\}$ is called an $\mathring{\mathrm{R}}$-point (or we say that P is $\mathring{\mathrm{R}}$), and a point Q of arcs of IC not in $\mathcal{R}$ is called a $\mathring{\mathrm{C}}$-point (or we say that Q is $\mathring{\mathrm{C}}$). A focal line m will be called an $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$-line (respectively $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$-line) if it intersects $\partial\mathcal{I}$ in two $\mathring{\mathrm{R}}$-points (respectively $\mathring{\mathrm{C}}$-points). A focal line F_jH will be called an $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$-line (respectively $\mathring{\mathrm{C}}\mathring{\mathrm{R}}$-line) if $\overline{F_jH}$ intersects $\partial\mathcal{I}$ in $\mathring{\mathrm{R}}$-point (respectively in $\mathring{\mathrm{C}}$-point), but $\overline{HF_j}$ intersects $\partial\mathcal{I}$ in $\mathring{\mathrm{C}}$-point (respectively in $\mathring{\mathrm{R}}$-point).

Hyperbola $\mathcal{H}$ is composed from two convex curves $\mathcal{H}_- \ni A$ and $\mathcal{H}_+ \ni B$, the branches, so each of them intersects $\partial\mathcal{I}$ in two points $H_\pm^+$ and $H_\pm^-$. Let x and y be (Euclidean) arc length parameter on the branch $\mathcal{H}_-$ and $\mathcal{H}_+$, respectively, and define $A\colon x \mapsto A(x) \in \mathcal{H}_-$ and $B\colon y \mapsto B(y) \in \mathcal{H}_+$ be such that $A(0) = A$, $B(0) = B$, and $\overline{F_1}A(x)$ rotates anti clockwise if x increases, while $\overline{F_2}B(y)$ rotates clockwise if y increases. (See Figure 5.3.)

For sufficiently small x, both focal lines $A(x)F_1$ and $A(x)F_2$ are clearly $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$. Let x^+ be the supremum of $x > 0$ such that both of the focal lines of $A(x)$ is $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$. Let x^- be the infimum of $x > 0$ such that both of its focal line of $A(x)$ is $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$. Define similarly the values y^+ and y^-. Points $A^+ = A(x^+)$, $A^- = A(x^-)$, $B^+ = B(y^+)$, and $B^- = B(y^-)$ are the so called *critical* points of $\mathcal{H}$. Clearly, at least one of the focal lines of a critical point is not $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, but either $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$ or $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$.

If $A^+ = H_-^+$, then $R_+ = R^+$ and $\varphi_+ = \varphi^+$, and more importantly arc $[R_-R^-]$ is common in $\partial\mathcal{I} \cap \mathcal{C}$. Further, $A(x) = F_1R_- \cap \mathcal{H}_-$ is a critical point with focal line $A(x)F_1$ of type $\mathring{\mathrm{C}}\mathring{\mathrm{R}}$, and, therefore, with focal line $A(x)F_2$ that is $\ddot{\mathrm{C}}\ddot{\mathrm{C}}$ as can not be $\ddot{\mathrm{D}}\ddot{\mathrm{C}}$, by (5.12). If $A(x)F_2$ is $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, then a small, appropriate decrease of x keeps $A(x)F_2$ being $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, but changes $A(x)F_1$ to $\ddot{\mathrm{D}}\ddot{\mathrm{C}}$ which contradicts (5.12), hence we deduce $A(x)F_2$ is $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$, i.e. $A(x) = F_2R^- \cap \mathcal{H}_-$. Letting $B(y) = F_2R^- \cap \mathcal{H}_+$ we see that $B(y)F_1$ is $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, and by the very same reasoning again, a small, appropriate decrease of y produces a contradiction with (5.12).

Thus, if even just one of the critical points is not in $\mathcal{I}$, then $\partial\mathcal{I} \equiv \mathcal{C}$.

Therefore, from now on we assume that there are exactly four critical points, and so $0 < \varphi_+ < \varphi^+ < \pi$ and $\pi < \varphi^- < \varphi_- < 2\pi$.

Our aim is to rule out the impossible configurations of the critical points.

No critical point can have focal lines of type $\mathring{C}\mathring{C}$ and $\mathring{R}\mathring{C}$ at once. (5.13)

If $A(x)F_2$ is $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$ and $A(x)F_1$ is $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$, then an small, appropriate increase of x keeps $A(x)F_2$ being $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, but changes $A(x)F_1$ to $\ddot{\mathrm{D}}\ddot{\mathrm{C}}$ which contradicts (5.12).

No critical point can have focal lines of type $\mathring{R}\mathring{R}$ and $\mathring{C}\mathring{R}$ at once. (5.14)

If $A(x)F_2$ is $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$ and $A(x)F_1$ is $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$, then letting $B(y) = A(x)F_2 \cap \mathcal{H}_+$ we see that $B(y)$ is a critical point with focal lines being $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$ and $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$ that contradicts (5.13).

No critical point can have two focal lines of type $\mathring{R}\mathring{C}$ at once. (5.15)

If $B(y)F_1$ and $B(y)F_2$ are $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$, the point $A(x) = B(y)F_1 \cap \mathcal{H}_-$ has the focal line $A(x)F_1$ of type $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$, but its other focal line $A(x)F_2$ is of type $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$ that contradicts (5.13).

No two critical points can be on a focal line of type $\mathring{R}\mathring{R}$. (5.16)

A focal line of type $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$ can pass only one of the focuses, say it is F_1. Assume that this focal line is $R^-R_+ = A(x)F_1 = B(y)F_1$. Consider the critical point $B(z)$ for $z < 0$. Both of its focal lines have intersection with $\partial\mathcal{I}$ above the line IJ of type $\mathring{\mathrm{C}}$. The other two intersections can only be of type $\mathring{\mathrm{R}}$ or $\mathring{\mathrm{C}}$, but then (5.15) or (5.13) leads to contradiction.

There are at most two critical points with focal line of type $\mathring{R}\mathring{R}$. (5.17)

If there were three such critical points, then two of them would have a common focal line of type $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$. This contradicts (5.16).

By (5.17), there are at least two critical points such that none of their four focal lines is $\mathring{\mathrm{R}}\mathring{\mathrm{R}}$. One of the focal lines of such a critical point must have an $\mathring{\mathrm{R}}$-endpoint, so it is an $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$ focal line. The other focal line can only be $\mathring{\mathrm{R}}\mathring{\mathrm{C}}$ or $\mathring{\mathrm{C}}\mathring{\mathrm{C}}$, but these contradict (5.15) and (5.13), respectively.

Thus, we conclude that $\partial\mathcal{I} = \mathcal{C}$. □

Proposition 5.5. *Equation* (5.3) *is never satisfied.*

Proof. Substitution of $x = \tanh f$ and $y = \tanh a$ into (5.3), and making some rearrangement we get

$$\pm 1 = \frac{1+y}{1-y}\frac{1+(\frac{x+y}{x-y})^{k-1}(\frac{x-1}{x+1})^{k-1}}{(\frac{x+y}{x-y})^{k-1}+(\frac{x-1}{x+1})^{k-1}},$$

where $0 < y < x < 1$. Introducing $A = \frac{x+y}{x-y}$ and $B = \frac{1-x}{1+x}$, we first obtain that $\frac{1+y}{1-y} = \frac{A+B}{1+AB}$, and then

$$\pm 1 = \frac{A+B}{1+AB}\frac{1+(-1)^{k-1}A^{k-1}B^{k-1}}{A^{k-1}+(-1)^{k-1}B^{k-1}},$$

where $0 < B < 1 < A$. Easy rearrangements of this equation lead to

$$A(A^{k-2}\mp 1)(-1\pm(-1)^{k-1}B^k) = B(1\mp(-1)^{k-1}B^{k-2})(A^k\mp 1).$$

As $0 < B < 1 < A$, the left-hand side is negative and the right-hand side is positive, hence this equation can not be valid. □

Propositions 5.3, 5.4 and 5.5 imply our main result.

Theorem 5.6. *If $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is the intersection of a hyperbolic quadric with $\mathcal{I}$, and $\partial\mathcal{I}$ is analytic around the points I and J, then $\partial\mathcal{I}$ is an ellipse and consequently Hilbert plane $(\mathcal{I}, d_\mathcal{I})$ is hyperbolic.*

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