ON THE DIFERERENTIABILITY OF LOOPMULTIPLICATION IN CANONICAL COORDINATE-SYSTEM

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Dedicated to Prof. Béla Gyires on his 80th birthday

1. Introduction

It is well known in the theory of Lie-groups that if a coordinate-system of class C^k is given for a Lie-group, then there exists a canonical coordinate-system of the first kind in which the functions, corresponding to the group-multiplication are analytic.

The classical proof of this theorem consists of two parts. In the first one it is proved, with the help of the discussion of differential equations of one-parameter subgroups that the group-multiplication is of class $\mathcal{C}^{k-1}$ (cf. L. Pontrjagin [7], Satz 59). In the second part the $\mathcal{C}^{\omega}$-property is shown by using the correspondence between Lie-groups and Lie-algebras and tracing these differential equations back to equations of constant coefficients.

The first part of the proof can be immediately extended to certain classes of local differentiable loops (see e.g. E. N. Kuz'min [6]).

In this apaper we are discussing the question how the notion of canonical coordinate-system can be extended to a more general class of loops without decreasing the order of differentiability.

The existence of a special canonical coordinate-system, with respect to a local analytical loop, was stated by M. A. Akivis [2], and proved by M. A. Akivis and A. M. Shelekhov [3], later on. An analogous statement for local analytic n-ary loops was proved by V. V. Goldberg [5]. A proof of the existence of a canonical coordinate-system of the first kind for a special loop-class (strong power-associative case, with $\mathcal{C}^2$-property) was given by E. N. Kuz'min [6].

For the case of a strong power-associative loop-class in one dimension, the solution of the above problem can be derived simply from a result of J. Aczél [1], which states that in this case the loop is isomorphic to the additive group of real numbers.

In the course of classification of $(p+1)$-webs, J. P. Dufour and P. Jean [4], give a method which is applicable for our purposes, as well.

Our main purpose is to prove the following statement: if a loop is continuously differentiable k-times in a coordinate-system, then there exist a so-called canonical canonical coordinate-system (in the sense of the definition in [3]), in which the loop-operation is continuously differentiable k-times. Furthermore, as a simple corollary, we shall find that the canonical coordinate-system of the first kind have the same property.

Date: June 26, 1987.

2. Basic concepts

In this section the most important concepts which are in the centre of our considerations will be introduced.

Definition 2.1. Let $\mathcal{F}$ be a differentiable manifold of dimension n. A map of class C^k

$$f\colon \mathcal{F}\times\mathcal{F}\to\mathcal{F};\quad (x,y)\mapsto z$$

where $x,y,z\in\mathcal{F}$, is called a *local differentiable loop of class C^k, if*

a) there exists an element e of $\mathcal{F}$ and a neighbourhood U of e such that

$$f(e,x)=f(x,e)=x \tag{2.1}$$

for every $x\in U$, and

b) for this neighbourhood U, there exists a neighbourhood V of the element e $(V\subset U)$ such that for these U and V, and for all $x,y\in V$

$$f\colon V\times V\to U'\subset U\colon\quad f(x,y)=z, \tag{2.2}$$

where $y\in U$, furthermore for arbitrary two elements $x\in U$, $z\in U$ $(y\in V$, $z\in U)$, (2.2) has a unique solution in U, for x (for y),

c) for each neighbourhood U, satisfying conditions a) and b), there exists a chart (U,φ) of dimension n, where

$$\varphi\colon U\to W\subset\mathbf{R}^n$$

and

$$\varphi\colon e\mapsto 0,$$

where 0 is the origin of $\mathbf{R}^n$, W is a neighbourhood of 0, furthermore

$$\varphi\colon x\mapsto X, y\mapsto Y, z\mapsto Z\quad (x,y,z\in U;\ X,Y,Z\in W).$$

The loop-operation f can be written in its usual coordinate form:

$$f^i(x^j,y^k)=z^i\quad (i,j,k=1,\dots,n),$$

where f^i is the i-th component of the function

$$F=\varphi\circ f\circ(\varphi^{-1}\times\varphi^{-1}), \tag{2.3}$$

for which obviously

$$F\colon\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n;\quad W\times W\to W,$$
$$F(X,Y)=Z, \tag{2.2'}$$

and x^j,y^k,z^i are the coordinates of x,y,z, respectively.

Introducing the chart (U,φ), we get the coordinate-system $\mathcal{D}$,

d) the functions f^i $(i=1,dots,n)$ are of class C^k. Then $F\in C^k$ holds, as well $(k\geq 2)$

Introducing a new coordinate-system $\tilde{\mathcal{D}}$ by the chart $(\tilde U,\tilde\varphi)$, the corresponding form of the loop-operation f in these new coordinates is different from (2.2') in general. That is we can write (2.2) in $\tilde{\mathcal{D}}$ as

$$\tilde F(\tilde X,\tilde Y)=\tilde Z,$$

where

$$\tilde{\varphi}\colon \tilde{U} \to \tilde{W} \subset \mathbf{R}^n,$$
$$\tilde{\varphi}\colon e \mapsto 0; x \mapsto \tilde{X}, y \mapsto \tilde{Y}, z \mapsto \tilde{Z},$$

$\tilde{W}$ is a neighbourhood of 0, and

$$\tilde{F} = \tilde{\varphi} \circ f \circ (\tilde{\varphi}^{-1} \times \tilde{\varphi}^{-1}).$$

Now let us introduce a coordinate-system with the property that the loop-operation f has an especially simple form in these coordinates. The loop f and the coordinate-system $\tilde{\mathcal{D}}$ are such as in Definition 2.1.

Definition 2.2. Let f be a local differentiable loop of class C^k. The coordinate-system $\tilde{\mathcal{D}}$ is called *Canonical coordinate-system* (or C-coordinate-system) with respect to f, if in $\tilde{\mathcal{D}}$ for the function $\tilde{F}$ we have

$$\tilde{F}(\tilde{X}, \tilde{X}) = 2\tilde{X},$$

for all $\tilde{X} \in \tilde{W}$.

Notation 2.3. The canonical coordinate-systems of the first kind used in Lie group theory (cf. Pontrjagin [7]) we shall denote by 'C-1-K-coordinate-system'.

It is easy to see that every C-1-K-coordinate-system is a C-coordinate-system.

3. Existence of a C-coordinate-system

In this section will be investigated the question of the existence of a C-coordinate-system with respect to f.

Theorem 3.1. *Let f be a local differentiable loop of class C^k $(k \geq 2)$. Then there exists a canonical coordinate-system with respect to f of class C^k.*

Proof. Let $\mathcal{D}$ be a coordinate-system and F the loop-operation in $\mathcal{D}$, $F \in C^k$. Let $J(F)$ denote the Jacobian of F at the point $(0,0)$. What can we say about the form of $J(F)$?

In consequence of (2.1) we have

$$F(X, 0) = F(0, X) = X, \quad (X \in W),$$

and so

$$\begin{aligned} D_j f^i(0,0) &= \delta^i_j \\ D_{n+j} f^i(0,0) &= \delta^i_j \end{aligned} \qquad \begin{pmatrix} i = 1, \dots, n \\ j = 1, \dots, n \end{pmatrix}$$

where D_j are the j-th partial derivatives. Thus we obtain the matrix form for $J(F)$:

$$J(F) = \left(\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Let us introduce the following functions in some suitable small neighbourhood of 0. First the diagonal function Q:

$$Q\colon U \to U \times U$$
$$x \mapsto (X, X),$$

then the function T:

$$T = F \circ Q$$

$$T\colon U \to U; \qquad X \mapsto F \circ Q(X). \tag{3.1}$$

It is clear that

$$T(0) = 0.$$

Computing the matrix of $DQ(0)$, it can be easily seen that it has the following from

$$\big(DQ(F)\big) = \begin{pmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Since

$$\begin{aligned} D_j Q^i(0) &= \delta^i_j \\ D_{n+j} Q^i(0) &= \delta^i_j \end{aligned} \qquad \begin{pmatrix} i = 1, \dots, n \\ j = 1, \dots, n \end{pmatrix}$$

Differentiating (3.1) at 0, we obtain

$$DT(0) = DF\big(Q(0)\big) \circ DQ(0) = DF(0,0) \circ DQ(0).$$

In this way the matrix of $DT(0)$ has the form

$$\big(DT(0)\big) = \left(\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right) \begin{pmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \hline 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix},$$

that is

$$\big(DT(0)\big) = 2(E),$$

where is the identity map in $\mathbf{R}^n$. It means that the linear map $DT(0)$ has only one (but multiple: n-times) eigenvalue: 2.

[Publ. Math. Debrecen, **37** (1990), 313-325]

Since $F \in C^k$ and $Q \in C^k$ obvioulsly, the map T is of class C^k in a neighbourhood of 0. Then there exists the inverse of the map T in a neighbourhood of 0, and

$$D(T^{-1})(0) = \big(DT(0)\big)^{-1},$$

furthermore $T^{-1} \in C^k$ in this neighbourhood.

As the matrix of $\big(DT(0)\big)^{-1}$ is

$$\begin{pmatrix} 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/2 & 0 \\ 0 & 0 & \cdots & 0 & 1/2 \end{pmatrix}$$

the Jacobian $J(T^{-1})$ of the map T^{-1} at the origin has the only eigenvalue $1/2$.

When looking for a transformation to obtain a suitable new coordinate-system $\tilde{\mathcal{D}}$, we shall apply a result of S. Sternberg ([8], Theorem 2), with some modification.

Let T^k denote the set of all C^k homeomorphisms defined in some neighbourhood of the origin in n-space, keeping the origin fixed and having non-vanishing Jacobian there. Let $JT(0)$ denote the Jacobian of T at the origin.

Theorem 3.2. *Let T be a transformation in T^k such that $JT(0) = \frac{1}{2}(E)$, and $k \geq 2$. Then there exists a transformation R in T^k such that*

$$(*) \qquad JT(0) = R \circ T \circ R^{-1}.$$

The proof of this modified version of S. Sternberg's theorem is in the Appendix of the present paper.

If we apply this theorem for the inverse of the transformation T, introduced by (3.1), we get a transformation $R \in T^k$ such that

$$R(0) = R^{-1}(0) = 0, \quad \text{and} \quad D(T^{-1})(0) = R \circ T \circ R^{-1}.$$

If we compute the inverse of both sides, we find

$$(3.2) \qquad DT(0) = R \circ T \circ R^{-1},$$

or what is the same

$$R^{-1} \circ DT(0) \circ R = T.$$

That is, the map T is linearisable by an inner automorphism R of class C^k.

Let us now introduce new coordinates by R, as follows:

$$R\colon X \mapsto \tilde{X}, \quad Y \mapsto \tilde{Y}, \quad Z \mapsto \tilde{Z}.$$

So we have been given a new coordinate-system $\tilde{\mathcal{D}}$ by the chart $(\tilde{U}, \tilde{\varphi})$, where

$$\tilde{\varphi} = R \circ \varphi$$

and the loop-operation f has in $\tilde{\mathcal{D}}$ the form (see (2.3)):

$$(3.3) \qquad \begin{aligned} \tilde{F} &= \tilde{\varphi} \circ f \circ (\tilde{\varphi}^{-1} \times \tilde{\varphi}^{-1}) \\ &= R \circ \varphi \circ f \circ (\varphi^{-1} \circ R^{-1} \times \varphi^{-1} \circ R^{-1}) \\ &= R \circ \varphi \circ f \circ (\varphi^{-1} \times \varphi^{-1}) \circ (R^{-1} \times R^{-1}) \\ &= R \circ F \circ (R^{-1} \times R^{-1}). \end{aligned}$$

That is, if

$$F(X, Y) = Z,$$

then

$$\tilde{F}(\tilde{X}, \tilde{Y}) = R \circ F \circ (R^{-1} \times R^{-1})(\tilde{X}, \tilde{Y}) = R \circ F(X, Y) = R(Z) = \tilde{Z}.$$

There remains only to compute $\tilde{F}(\tilde{X}, \tilde{Y})$. Using (3.3), we get

$$\tilde{F}(\tilde{X}, \tilde{X}) = R \circ F \circ (R^{-1} \times R^{-1})(\tilde{X}, \tilde{X}).$$

Since from (3.1) it follows

$$F(T \circ Q^{-1}),$$

we find

$$\begin{aligned}\tilde{F}(\tilde{X}, \tilde{X}) &= R \circ T \circ Q^{-1} \circ (R^{-1} \times R^{-1})(\tilde{X}, \tilde{X}) \\ &= R \circ T \circ Q^{-1}\big(R^{-1}(\tilde{X}) \times R^{-1}(\tilde{X})\big) = R \circ T \circ R^{-1}(\tilde{X}).\end{aligned}$$

Finally by (3.2) we obtain

$$\tilde{F}(\tilde{X}, \tilde{X}) = DT(0)(\tilde{X}) = 2\tilde{X},$$

which completes the proof. □

4. Relation between C-1-K-coordinate-systems

As it was mentioned in the first section, the existence, for some special classes of loops, of C-1-K-coordinate-systems which are canonical coordinate-systems as well, is already known, In connection with this there arises the question: what kind of connection can we obtain among the C-coordinate-systems with respect to local differentiable loops of class C^k? More precisely, there is any classification of these coordinate-systems?

Let us verify a simple statement first:

Lemma 4.1. *Let Φ be a linear transformation, which is a homeomorphism in $\mathbf{R}^n$. If $\mathcal{D}$ denotes the C-coordinate-system with respect to the local differentiable loop f of class C^k and $\tilde{\mathcal{D}}$ denotes the new coordinate-system which we get from $\mathcal{D}$ by the transformation Φ, then $\tilde{\mathcal{D}}$ is a C-coordinate-system with respect to f, as well.*

Proof. For the coordinate-system $\tilde{\mathcal{D}}$ we have

$$\tilde{F} = \Phi \circ F \circ (\Phi^{-1} \times \Phi^{-1}).$$

If X and $\tilde{X}$ are such that

$$\Phi \colon X \mapsto \tilde{X},$$

we can calculate $\tilde{F}(\tilde{X}, \tilde{X})$:

$$\begin{aligned}\tilde{F}(\tilde{X}, \tilde{X}) &= \Phi \circ F \circ (\Phi^{-1} \times \Phi^{-1})(\tilde{X}, \tilde{X}) \\ &= \Phi \circ F(\Phi^{-1}(\tilde{X}), \Phi^{-1}(\tilde{X}) \\ &= \Phi \circ F(X, X) = \Phi(2X) = 2\Phi(X) = 2\tilde{X}.\end{aligned}$$

This equation shows that $\tilde{\mathcal{D}}$ is a canonical coordinate-system with respect to f. □

The following theorem is a modified version for loops of a statement to be found in [5].

Theorem 4.2. *The coordinate-transformation Φ between two different C-coordinate-systems is a linear transformation.*

Proof. Let F and $\tilde{F}$ denote the forms of loop-operation f in the two canonical coordinate-systems, respectively. Then we have

$$\Phi \circ F = \tilde{F} \circ (\Phi \times \Phi). \tag{4.1}$$

Let W be a neighbourhood of 0, according to Definition 2.1. If $X \in W$ $(X \neq 0)$ and n is an arbitrary positive integer, then from $X \in W$ there follows $\frac{X}{2^n} \in W$, too. Since $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are C-coordinate-systems, furthermore applying (4.1), we get

$$\begin{aligned} 2\Phi\left(\frac{X}{2}\right) &= \tilde{F}\left(\Phi\left(\frac{X}{2}\right), \Phi\left(\frac{X}{2}\right)\right) = \tilde{F} \circ (\Phi \times \Phi)\left(\frac{X}{2}, \frac{X}{2}\right) \\ &= \Phi \circ F\left(\frac{X}{2}, \frac{X}{2}\right) = \Phi(X), \end{aligned}$$

that is

$$\Phi\left(\frac{X}{2}\right) = \frac{1}{2}\Phi(X).$$

Replacing X by $\frac{X}{2}$, from the above equation we get

$$\Phi\left(\frac{X}{2^2}\right) = \frac{1}{2}\Phi\left(\frac{X}{2}\right) = \frac{1}{2^2}\Phi(X).$$

Reiterating this still $(n-2)$-times we obtain

$$\Phi\left(\frac{X}{2^n}\right) = \frac{1}{2^n}\Phi(X), \tag{4.2}$$

for any positive integer n. (More precisely we can get it by a simple induction.) As Φ is differentiable,

$$\Phi(X) = D\Phi(0)(X) + \|X\| \cdot \varepsilon(X), \tag{4.3}$$

where $\|\cdot\|$ denotes the usual Euclidean norm, and

$$\lim_{X \to 0} \varepsilon(X) = 0.$$

Applying (4.3) for $X/2^n$, we can write

$$\Phi\left(\frac{X}{2^n}\right) = D\Phi(0)\left(\frac{X}{2^n}\right) + \left\|\frac{X}{2^n}\right\| \varepsilon\left(\frac{X}{2^n}\right).$$

Now using linearity of $D\Phi(0)$ and the norm $\|\cdot\|$, then multiplying both sides with 2^n, furthermore taking (4.1) into account, we get

$$\Phi(X) = D\Phi(0)(X) + \|X\| \varepsilon\left(\frac{X}{2^n}\right).$$

Comparing it with (4.3), we obtain

$$\|X\|\varepsilon(X) = \|X\|\varepsilon\left(\frac{X}{2^n}\right),$$

for all n. From which

$$\varepsilon(X) = \lim_{n \to \infty} \varepsilon\left(\frac{X}{2^n}\right) = \lim_{X \to 0} \varepsilon(X) = 0$$

follows, and this, by (4.3), gives linearity of Φ. $\square$

[Publ. Math. Debrecen, **37** (1990), 313-325]

In this section we have shown that the group of linear homeomorphisms classifies the C-coordinate-systems: two C-coordinate-systems are in the same class (are equivalent) if and only if there exists a lienar homeomorphism, mapping the first canonical coordinate-system into the other.

5. Subgroups in canonical coordinate-systems

In this last section we prove a theorem, applying a C-coordinate-system.

Theorem 5.1. *Let $\mathcal{D}$ be a C-coordinate-system with respect to the local differentiable loop f. Then every one-parameter subgroup $X(t)$ os f, defined in some neighbourhood of e, can be expressed in the coordinates of $\mathcal{D}$ as*

$$X(t) = A \cdot t,$$

where $t \in \mathcal{I}$ (a suitable small interval around 0), and $A \in \mathbf{R}^n$.

Proof. If F is the form of the loop-operation f in $\mathcal{D}$, then

$$F\big(X(t), X(t)\big) = 2X(t).$$

On the other hand

$$F\big(X(t), X(t)\big) = X(t+t) = X(2t),$$

or in another form (inserting $t/2$ instead of t):

$$X\left(\frac{t}{2}\right) = \frac{1}{2} X(t).$$

If $X(t) \in W$, a neighbourhood of 0 (according to Definition 2.1) then $X(t/2^n) \in W$ for any positive integer n. From the above equation

$$X\left(\frac{t}{2^n}\right) = \frac{1}{2^n} X(t) \tag{5.1}$$

follows for any positive integer n. Indeed, we simply have to write $t/2$ instead of t, $(n-1)$-times successively, afterwards to make an induction on n.

Since $X(t)$ is differentiable, it can be written:

$$X(t) = DX(0)t + |t| \cdot \varepsilon(t), \tag{5.2}$$

where $\lim_{t \to 0} \varepsilon(t) = 0$.

If we apply (5.2) for $X(t/2^n)$, we get

$$X\left(\frac{t}{2^n}\right) = DX(0)\frac{t}{2^n} + \left|\frac{t}{2^n}\right| \cdot \varepsilon\left(\frac{t}{2^n}\right),$$

where $\lim_{t/2^n \to 0} \varepsilon(t/2^n) = 0$. Multiplying this equation by 2^n and taking into account (5.1), we get

$$X(t) = DX(0)t + |t| \cdot \varepsilon\left(\frac{t}{2^n}\right).$$

Comparing the last equation with (5.2)

$$\varepsilon(t) = \varepsilon\left(\frac{t}{2^n}\right)$$

follows for any positive integer n.

Since $\lim_{t/2^n \to 0} \varepsilon(t/2^n) = \lim_{n\to\infty} \varepsilon(t/2^n) = 0$, then

$$\varepsilon(t) = 0,$$

that is, denoting $DX(0)$ by A, from (5.2) we obtain

$$X(t) = A \circ t.$$

□

Finally let us mention an important corollary of Theorem 5.1.

Corollary 5.2. *If for the loop f there exist one-parameter subgroups in every direction, then there exists a C-1-K-coordinate-system. Moreover, every C-coordinate-system is a C-1-K-coordinate-system. It means that in this case C-1-K-coordinate-systems and C-coordinate-systems are the same.*

6. Appendix

For completeness sake we give here a proof of a version of Sternberg's theorem on normal forms of contractions used in the proof of our Theorem 3.1. We follow the original proof, but the different assumptions involve some modifications.

Proof. a) The statement of the theorem will be proved first for special transformations. Let F^k be the space of n-tuples of real polynomials without constant terms of n real variables which have terms of degree at most k, and whose matrix of linear terms is non-singular. The multiplication in F^k is defined by substitution followed by truncation of order k, that is terms of degree at least k are omitted.

Lemma 6.1. *Let T be an element of F^k, whose matrix of linear terms is $\frac{1}{2}(E)$. Then T is equivalent to its matrix of linear terms by an inner automorphism R_0 of F^k.*

Proof of Lemma 6.1. Since the matrix of linear terms of T is actually the Jacobian at 0, we can write

$$JT(0) = R_0 \circ T \circ R_0^{-1}, \tag{6.1}$$

or

$$R_0 \circ T = JT(0) \circ R_0. \tag{6.1'}$$

The transformation T has the following form

$$\begin{aligned} T &= (t^1, \dots, t^n) \\ &= \left(\frac{1}{2}x_1 + \sum_{2 \le \sum_j i_j \le k} t^1_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \dots, \frac{1}{2}x_n + \sum_{2 \le \sum_j i_j \le k} t^n_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}\right). \end{aligned}$$

We wish to find an R_0, the matrix of linear terms of which is the unit matrix.

$$R_0 = (r^1, \dots, r^n) = \left(\dots, x_i + \sum_{2 \le \sum_j i_j \le k} r^i_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \dots\right).$$

If we substitute these forms into (6.1') we obtain for i-th component:

$$\frac{1}{2}\left(x_i + \sum_{2\leq\sum_j i_j\leq k} r^i_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}\right) = \frac{1}{2}x_i + \sum_{2\leq\sum_j i_j\leq k} t^i_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n} + \\ + \sum_{2\leq\sum_j i_j\leq k} r^i_{i_1,\dots,i_n}\left(\frac{1}{2}x_1 + \sum_{2\leq\sum_j i_j\leq k} t^1_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}\right)\cdots \\ \cdots\left(\frac{1}{2}x_n + \sum_{2\leq\sum_j i_j\leq k} t^n_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}\right).$$

Comparing the coefficients of $x_1^{i_1}\cdots x_n^{i_n}$ we can compute the constants $r^i_{i_1,\dots,i_n}$ step by step , starting with the terms of lowest degree, and therefore obtain R_0 itself.

b) To find an R belonging to T^k, but not necessarily to F^k, we introduce a special norm. Let V^k_N be the space of all C^k mappings which are defined on a neighbourhood N of the origin and which vanish at 0 with order k. Let us define a norm on V^k_N as follows. For a mapping $f \in V^k_N$:

$$\|f\|^k_N = \sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k f(X)(V_1,\dots,V_k)\|}{\|V_1\|\dots\|V_k\|},$$

where the supremums are taken over all X and $V_1,\dots,V_k$ such that $X, V_1,\dots,V_k \in N$ and $\|X\|$ is the ordinary Euclidean norm. It is clear that $\|\cdot\|^k_N$ is a norm, furthermore from $f \in V^{k'}_{N'}$ follows $f \in V^k_N$ for any $k' \leq k$ and $N' \subseteq N$. □

Lemma 6.2. *Let f be a transformation in V^k_N. Then for arbitrary $\varepsilon > 0$ there exists a sufficiently small neighbourhood N^ε such that*

$$\|f\|^l_{N^\varepsilon} < \varepsilon \|f\|^k_{N^\varepsilon} \tag{6.2}$$

for $1 \leq l \leq k-1$.

Proof of Lemma 6.2. It is sufficient to give a proof only for the following case:

$$\|f\|^l_{N^\varepsilon} < \varepsilon\|f\|^{l+1}_{N^\varepsilon}$$

because from this the general case follows immediately. For the estimation we use the mean-value theorem. Then for some $0 \leq t \leq 1$, in consequence of $D^l f(0) = 0$, we obtain

$$\|f\|^l_{N^\varepsilon} \sup_X \sup_{V_1,\dots,V_l} \frac{\|D^l f(X)(V_1,\dots,V_l)\|}{\|V_1\|\dots\|V_l\|} \\ = \sup_X \sup_{V_1,\dots,V_l} \frac{\|D^{l+1} f(tX)(X,V_1,\dots,V_l)\|}{\|V_1\|\dots\|V_l\|} \\ \leq \sup_X \sup_{V_1,\dots,V_l} \frac{\|D^{l+1} f(X)(X,V_1,\dots,V_l)\|}{\|X\|\|V_1\|\dots\|V_l\|}\cdot\sup_X\|X\| < \varepsilon\|f\|^{l+1}_{N^\varepsilon}$$

since $\sup_X \|X\| < \varepsilon$, if N^ε is sufficiently small. □

c) For the desired $R \in T^k$ the relation $(*)$ can be rewritten in the form

$$R = \mathcal{D}_T(R),$$

where the operation $\mathcal{D}_T$ is defined as follows. If R is a transformation in T^k, then

$$\mathcal{D}_T \colon R \to [JT(0)]^{-1} \circ R \circ T.$$

If we define this operator in the space V_N^k, then there holds the following

Lemma 6.3. *Let f be in V_N^k, and T in T^k for which $JT(0) = \frac{1}{2}E$. Then there exists a neighbourhood N' of the 0 such that*

$$\|\mathcal{D}_T f\|_{N'}^k < K\|f\|_{N'}^k,$$

for some $K < 1$.

Proof of Lemma 6.3.

(6.3)

$$\begin{aligned}
\|\mathcal{D}_T f\|_{N'}^k &= \sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k(2f\circ T)(X)(V_1,\dots,V_k)\|}{\|V_1\|\dots\|V_k\|} \\
&= 2\sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k(f\circ T)(X)(V_1,\dots,V_k)\|}{\|V_1\|\dots\|V_k\|} \\
&\le 2\left\{\sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k f\big(T(X)\big)\big(DT(X)(V_1),\dots,DT(X)(V_k)\big)\|}{\|V_1\|\dots\|V_k\|} + \right. \\
&\qquad\qquad \left. + \sum_\lambda \sup_X \sup_{V_1,\dots,V_k} \frac{\|P_\lambda(X;V_1,\dots,V_k)\|}{\|V_1\|\dots\|V_k\|}\right\}.
\end{aligned}$$

$P_\lambda(X;V_1,\dots,V_k)$ denotes the following term:

$$\begin{aligned}
&D^\lambda f\big(T(X)\big)\big(D^{m_1}T(X)(V_1,\dots,V_{m_1}),(D^{m_2}T(X)(V_{m_1+1},\dots,V_{m_1+m_2}),\dots \\
&\qquad\qquad \dots,(D^{m_\lambda}T(X)(V_{m_1+\dots+m_{\lambda-1}+1},\dots,V_{m_1+\dots+m_\lambda})\big),
\end{aligned}$$

where $1 \le \lambda \le k$ and $\sum_{i=1}^{\lambda} m_i = k$. Thus for the second term on the right hand side of inequality (6.1) we have

$$\begin{aligned}
&\sum_\lambda \sup_X \sup_{V_1,\dots,V_k} \frac{\|P_\lambda(X;V_1,\dots,V_k)\|}{\|V_1\|\dots\|V_k\|} \le \\
&\sum_\lambda \left\{\sup_X \sup_{V_1,\dots,V_k} \frac{\|D^\lambda f(X)\big(D^{m_1}T(X)(V_1,\dots,V_{m_1}),\dots,D^{m_\lambda}T(X)(V_{m_1+\dots+m_{\lambda-1}+1},\dots,V_{m_1+\dots+m_\lambda})\big)\|}{\|D^{m_1}T(X)(V_1,\dots,V_{m_1})\|\cdots\|D^{m_\lambda}T(X)(V_{m_1+\dots+m_{\lambda-1}+1},\dots,V_{m_1+\dots+m_\lambda})\|}\cdot\right. \\
&\qquad \cdot \sup_X \sup_{V_1,\dots,V_k} \frac{\|D^{m_1}T(X)(V_1,\dots,V_{m_1})\|}{\|V_1\|\dots\|V_{m_1}\|}\cdots \\
&\qquad \left. \cdot \sup_X \sup_{V_{m_1+\dots+m_{\lambda-1}+1},\dots,V_{m_1+m_\lambda}} \frac{\|D^{m_1}T(X)(V_{m_1+\dots+m_{\lambda-1}+1},\dots,V_{m_1+m_\lambda})\|}{\|V_{m_1+\dots+m_{\lambda-1}+1}\|\cdots\|V_{m_1+m_\lambda}\|}\right\} \\
&\le \sum_\lambda \big\{\|f\|_{N'}^\lambda \|T\|_{N'}^{m_1}\cdots\|T\|_{N'}^{m_\lambda}\big\} \le \sum_\lambda M_\lambda \|f\|_{N'}^\lambda < \sum_\lambda \varepsilon_\lambda \|f\|_{N'}^k < \varepsilon\|f\|_{N'}^k,
\end{aligned}$$

taking Lemma 6.2 into account, and requiring ε_λ be sufficiently small, furthermore choosing N' according to ε_λ.

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The estimate of the first term of (6.3) is similar:

$$\begin{aligned}
&\sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k f\big(T(X)\big)\big(DT(X)(V_1),\dots,DT(X)(V_k)\big)\|}{\|V_1\|\dots\|V_k\|}\\
&\quad\le \sup_X \sup_{V_1,\dots,V_k} \frac{\|D^k f(X)\big(DT(X)(V_1),\dots,DT(X)(V_k)\big)\|}{\|V_1\|\dots\|V_k\|}\times\\
&\quad\times \sup_X \sup_{V_1,\dots,V_k} \frac{\|DT(X)(V_1)\|}{\|V_1\|}\cdots \sup_X \sup_{V_1,\dots,V_k} \frac{\|DT(X)(V_k)\|}{\|V_k\|}\\
&\quad= \|f\|^k_{N'}\left(\sup_X \|DT(X)\|\right)^k \le \|f\|^k_{N'}\left(\frac{1}{2}+\delta\right)^k,
\end{aligned}$$

since T is smooth in a neighbourhood of the 0. Thus we have

$$\|\mathcal{D}_T\|^k_{N'} \le \|f\|^k_{N'}\cdot 2\Big[\left(\frac{1}{2}+\delta\right)^k+\varepsilon\Big] < K\|f\|^k_{N'},$$

for some $K<1$, for a sufficiently small neighbourhood N', consequently for small ε and δ, and for $k\ge 2$. □

d) For a transformation T in T^k which satisfies the conditions of Theorem 5.1, Lemma 6.1 implies the existence of an $R_0\in F^k\subset T^k$ such that the transformation $([JT(0)]^{-1}\circ R_0\circ T-R_0)$ is in V^k_N. Indeed, in T^k the coefficients of the power series of R_0 and the $[JT(0)]^{-1}\circ R_0\circ T$ at the origin equal up to order k. Thus we can use norm $\|\cdot\|^k_N$ for the sequence R_n-R_0, where

$$R_n=[JT(0)]^{-n}\circ R_0\circ T^n=\left\{\sum_{k=0}^{n-1}\mathcal{D}^k_T([JT(0)]^{-1}\circ R_0\circ T-R_0)\right\}+R_0.$$

Owing to Lemma 6.3, the sequence R_N-R_0 is uniformly convergent in some neighbourhood of 0, and tends to a transformation R' in T^k, and R^n tends to a transformation $R=R'+R_0$.

Thus $\lim_{n\to\infty}\big([JT(0)]^{-1}\circ R_n\circ T\big)=[JT(0)]^{-1}\circ R\circ T$. On the other hand $[JT(0)]^{-1}\circ R_n\circ T=R_{n+1}$, that is

$$\lim_{n\to\infty}\big([JT(0)]^{-1}\circ R_n\circ T\big)=\lim_{n\to\infty}R_{n+1}=R,$$

which completes the proof. □

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[Publ. Math. Debrecen, **37** (1990), 313-325]