BEHAVIOUR OF LOOPS IN A CANONICAL COORDINATE SYSTEM

JÓZSEF KOZMA

1. Introduction

This paper deals with the basic theory of local loops of class C^k.

First we set up the framework of our further considerations. The following definition is standard.

Definition 1.1. Let $\mathcal{F}$ be an n-dimensional differentiable manifold. Apartial mapping f of class C^k

$$f\colon \mathcal{F}\times\mathcal{F}\to\mathcal{F}\colon (x,y)\mapsto z \quad (x,y,z\in\mathcal{F})$$

is called a *local loop-multiplication of class* C^k and $(\mathcal{F};f)$ is called a *local loop of class* C^k if the following conditions are satisfied.

a) The multiplication is local quasigroup, that is, there exist open neighbourhoods $\mathcal{V}$,$\mathcal{U}$ $(\mathcal{V}\subseteq\mathcal{U}\mathcal{F})$such that $f\colon\mathcal{V}\times\mathcal{V}\to\mathcal{U}$ and $f(x,y)=z\in\mathcal{U}$ for all $x,y\in\mathcal{V}$. Furthermore, for arbitrary elements $x\in\mathcal{V}, z\in mcv$ (respectively, $y\in\mathcal{V}, z\in\mathcal{V}$) there exists one and only one $y\in\mathcal{U}$ (respectively, $x\in\mathcal{V}$) for which $f(x,y)=z$.
b) The loop has a unit element, that is, there is an element $e\in\mathcal{V}$ such that $f(x,e)=f(e,x)=x$ for all $x\in\mathcal{V}$.
c) The loop-multiplication is of class C^k.

We shall consider charts $(\mathcal{U}_i, qvarphi_i)$ for which $\varphi_i\colon\mathcal{U}_i\to\mathcal{W}_i\subseteq\mathbf{R}^n; e\mapsto 0$, where 0 is the origin of $\mathbf{R}^n$.

A loop on an m-dimensional manifold $\mathcal{F}$ is called an *m-parameter loop.* Instead of $(\mathcal{F};f)$ we shall frequently write f.

Since the canonical coordinate-systemdefined in [1] takes a prominent part in our investigations now, we recall its definition.

Definition 1.2. Let us consider a loop of class C^k $(k\geq 2)$. We shall say that a coordinate-systemφ given by the chart $(\mathcal{U},\varphi)$ $f\colon\mathcal{U}\to\mathbf{R}^n$, $\varphi(e)=0$ is a *canonical coordinate-system(CCS)* with respect to f if in these coordinates we have

$$f(x,x)=2x$$

for all $x\in\varphi(\mathcal{V})$, where

$$F=\varphi\circ f(\varphi^{-1}\times\varphi^{-1}).$$

This definition is feasible on account of the following result.

Theorem 1.3 ([1], Theorem 1)**.** *If f is a loop of class C^k $(k\geq 2)$, then there exists a* CCS *wirh respect to f, of class C^k.*

Date: April 19, 1989.

2. Morphisms and differentiable subloops

Further on, by a loop we mean a local loop of class C^k $(k \geq 2)$. Finally, we give the definition of morphisms and differentiable subloops.

Definition 2.1. Let $(\mathcal{F}; f)$ and $(\mathcal{G}; g)$ be two loops, and let ξ be a local map $\xi\colon \mathcal{F} \to \mathcal{G}$ of class C^k. If

$$f\big(\xi(x), \xi(y) = \xi\big(f(x, y)\big)$$

for all $x, y \in \mathcal{F}$ for which the product $f(x, y)$ is defined, then ξ is called a *local morphism from* $(\mathcal{F}; f)$ *to* $(\mathcal{G}; g)$. If, in particular, ξ is a local embedding of class C^k, then $(\mathcal{F}; f)$ is called a local *m-parameter subloop of* $(\mathcal{G}; g)$.

Our main goal now to show that any morphism of local loops, when trnsposed to a canonical coordinate-system; is the restriction of a linear map.

We first explain what we mean by transposing morphism to canonical coordinate system. Let $(\mathcal{F}; f)$ be a loop with the CCS given by $(\mathcal{U}; \varphi)$, and $(\mathcal{G}; g)$ a loop with CCSgiven by $(\mathcal{L}; \psi)$, and let $\xi : (\mathcal{F}; f) \to (\mathcal{G}; g)$ be a C^k-morphism, $\xi \in C^k$. We can dfine loops $(\mathbf{R}^n; \sigma)$ is $\mathbf{R}^n$ and $(\mathbf{R}^n; \tau)$ in $\mathbf{R}^n$ by diffeomorphisms φ and ψ. The loop-multiplication σ is defined by

$$\sigma\colon \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \to \tilde{\mathcal{U}}, \quad \sigma = \varphi \circ f \circ (\varphi^{-1} \times \varphi^{-1}),$$

where $0 \in \tilde{\mathcal{V}} \subseteq \mathbf{R}^m$.

In the CCS $(\mathcal{L}; \psi)$ of the loop $(\mathcal{G}; g)$ the loop-multiplication τ is defined as follows

$$\tau\colon \tilde{\mathcal{W}} \times \tilde{\mathcal{W}} \to \tilde{\mathcal{L}}, \quad \tau = \psi \circ g \circ (\psi^{-1} \times \psi^{-1}).$$

By the defintion of the morphism ξ, for the loop $(\mathcal{G}; g)$ we have

$$g\colon \tilde{\mathcal{W}} \times \tilde{\mathcal{W}} \to \tilde{\mathcal{L}}, \quad \xi \circ f = g(\xi \times \xi),$$

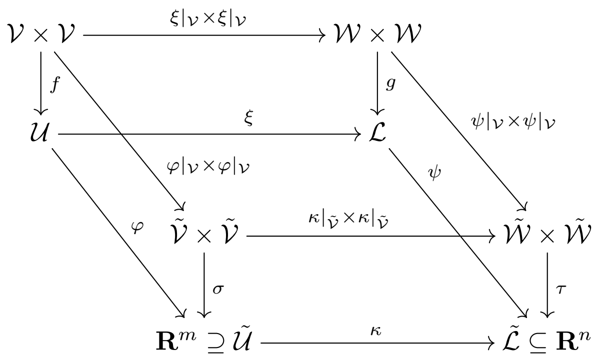
and the unit element of g is e_g: $e_g = \xi(e_f)$, where e_f is the unit element of

$$g\big(\xi(e_f), \xi(x)\big) = \xi\big(g(e_f), x\big)(\xi(x).)$$

It is clear that the mapping $\kappa = \varphi \circ \xi \circ \psi^{-1}$ is a ,orphism from the loop $(\mathbf{R}^n; \tau)$ to the loop $(\mathbf{R}^n; \sigma)$, that is,

$$\kappa \circ \sigma(x, y) = \tau \circ (\kappa \times \kappa)(x, y) \tag{2.1}$$

holds for any $x, y \in \mathcal{V}$. Furthermore if, in particular, ξ is an embedding, then κ is a local embedding, as well.



The mapping κ is said to be the *transposed mapping of the morphism* ξ *by the mappings* φ *and* ψ.

[Arch. Math., **55** (1990), 498-502]

3. Linearity in a CCS

Now we ae going to show that κ is linear. With the notations introduced above the theorem in question can be stated as follows.

Theorem 3.1. *The transposed mapping κ of the morphism ξ by the homeomorphisms φ and ψ is linear, that is, there exists a linear mapping $T\colon \mathbf{R}^m \to \mathbf{R}^m$ such that κ is the restriction of T to $\tilde{\mathcal{U}}$.*

In order to prove this theorem we require the following lemma.

Main lemma. *Let κ be a differentiable map of a star shaped neighbourhood $\mathcal{W}^*$ in $\mathbf{R}^m$ onto a neighbourhood of $\mathbf{R}^n$ keeping the origin fixed. If there exists a sequence of real numbers $\{r_m\}$, $r_m \neq 0$, converging to 0 an such that for each m*

$$\kappa(r_m x) = r_m \cdot \kappa(x) \tag{3.1}$$

holds for all x in $\mathcal{V}^*$, then κ is a linear mapping (that is, the restriction of a linear map).

Proof of the Main lemma. As κ is differentiable at 0, we have

$$\kappa(x) = D\kappa(0)(x) + \|x\| \cdot \varepsilon(x)$$

for all $x \in \mathcal{V}^*$, where $\|\dot{}\|$ denotes the usual Euclidean norm, and $\lim_{x\to 0} \varepsilon(x) = 0$. We can substitute $r_m x$ for x:

$$\kappa(r_m x) = D\kappa(0)(r_m x) + \|r_m x\| \cdot \varepsilon(r_m x).$$

Taking into account of linearity of of $D\kappa(0)$, and relation (3.1), we obtain the following

$$r_m \cdot \kappa(x) = r_m \cdot D\kappa(0)(x) + r_m \cdot \|x\| \cdot \varepsilon(r_m x)$$

for all $x \in \mathcal{V}^*$. Multiplying both sides with r_m^{-1}, and comparing the two equalities for $\kappa(x)$ we get

$$\|x\| \cdot \varepsilon(x) = \|x\| \cdot \varepsilon(r_m x)$$

for all $x \in \mathcal{V}^*$ and for all m. It follows that

$$\varepsilon(x) = \lim_{m\to\infty} \varepsilon(r_m x) = \lim_{x\to 0} \varepsilon(x) = 0$$

which shows that $\kappa = DK(0)|_{\mathcal{V}^*}$, that is, κ is a restriction of a linear map, which was to be shown. □

Proof of the Theorem. Since φ and ψ determine a CCS, we have

$$\sigma(x,x) = 2x$$

and

$$\sigma\big(\kappa(x), \kappa(x)\big) = 2\kappa(x)$$

for every $x \in \tilde{\mathcal{V}}$. In virtue of Equation (2.1), from the previous relations obtain that

$$\kappa(2x) = \kappa \circ \sigma(x,x) = \sigma\big(\kappa(x), \kappa(x)\big) = 2\kappa(x)$$

for every $x \in \tilde{\mathcal{V}}$. Let us consider a star shaped neighbourhood $\mathcal{W}^* \subseteq \tilde{\mathcal{V}}$ of the origin. Then for every point $x \in \mathcal{V}^*$ we have $x_1 = \frac{1}{2}x \in \mathcal{V}^*$, and $\tau(\frac{1}{2}x, \frac{1}{2}x) = x$. By induction we get $x_m = \frac{1}{2^m}x \in \mathcal{V}^*$ for all natural numbers m. Thus we can write

$$\kappa\left(\frac{1}{2^m}x\right) = \frac{1}{2^m}\kappa(x)$$

[Arch. Math., **55** (1990), 498-502]

for all $x \in \mathcal{V}^*$.

In such a way we have found a sequence of real numbers $\left\{\frac{1}{2^m}\right\}_{m=1}^{\infty}$ in a star shaped neighbourhood $\mathcal{V}^*$ such that the conditions of the Main lemma are satisfied. Thus the morphism κ is a restriction of a linear map which was to be proved. □

Let us now observe some consequences of the theorem. First we formulate a statement regarding to the subloops of a local differentiable loop.

Corollary 3.2. *In a* CCS *any m-parameter subloop is an m-plane, and the restriction of a* CCS *of the loop is a* CCS *of the subloop.*

Proof. Firstly, by the Main lemma κ is a linear mapping in an appropriate neighbourhood $\mathcal{V}^*$. Now taking into account the linearity of κ in $\mathcal{V}^*$ obtain that $\kappa(\mathcal{V}^*)$ is locally a linear m-dimensional subspace of $\mathbf{R}^n$ near 0. That is, $\kappa = \psi \circ \xi \circ \varphi^{-1}$ is a mapping of $\mathcal{V}^*$ onto the linear subspace $\kappa(\mathcal{V}^*) \subseteq \tilde{\mathcal{L}}$, in other words, a local m-parameter subloop is locally an m-dimensional plane in $\mathbf{R}^n$.

Secondly, let us consider the local loop $(\mathbf{R}^n; \tau)$ in the coordinate-system $\big(\psi \circ \xi(\mathcal{U}), \mathrm{id}\big)$ which is, by definition, a CCS. We have seen that the subloop $(\mathbf{R}^m; \sigma)$ is locally an m-dimensional plane in $\mathbf{R}^n$, through the origin. Then a usual linear parametrization of this plane is exactly a CCS for this subloop. □

Let the morphism ξ be the identity map. Then the following statement is a straightforward consequence of the theorem(see also [1]).

Corollary 3.3. *A* CCS *is* (*locally*) *unique up to a linear isomorphism.*

Remarks. 1. There is a former theorem ([1], Theorem 3) analogous to our Corollary 3.2. It states that one-parameter subgroups are linear in a CCS.

2. The formulation of Corollary 3.2 might have been complemented with the following: *'if the loop has any subloop at all'.* In fact, there are loops without one-parameter subloops (see [2]).

The author expresses his sincere thanks to K. H. Hofmann and P. T. Nagy for their interest and their suggestions.

References

[1] J. Kozma, On the differentiability of loop-multiplication in canonical coordinate syste. Publ Math. Debrecen, **37**, (1990), 313-325.

[2] J. Kozma, Loops with an without subloops, Acta Sci. Math., **55**, (1991), 21-31.

Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6725 Szeged (Hungary)

E-mail address: kozma@math.u-szeged.hu

[Arch. Math., **55** (1990), 498-502]