A Note on Rational $L^p$ Approximation on Jordan Curves

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A Note on Rational $L^p$ Approximation on Jordan Curves

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Abstract The precise asymptotics for the error of best rational approximation of meromorphic functions in integral norm is shown to be a consequence of a result of Gonchar and Rakhmanov. This reproves and extends a recent result of Baratchart, Stahl and Yattselev.

Keywords Rational approximation · Jordan curves · Meromorphic functions · Condenser capacity

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Let $T$ be a rectifiable Jordan curve, $G$ and $O$ the interior and exterior domains of $T$, respectively, with respect to $\mathbb{C}$. Let $A(G)$ denote the set of functions $f$ such that

- $f$ vanishes at infinity and admits holomorphic and single-valued continuation from infinity to an open neighborhood of $\overline{O}$,
- $f$ admits meromorphic, possibly multi-valued, continuation along any arc in $G \setminus E_f$ starting from $T$, where $E_f$ is a finite set of points in $G$.

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E_f is non-empty, the meromorphic continuation of f from infinity has a branch point at each element of E_f.

Examples of such functions are algebraic functions with branch points. See the paper [1] for other examples, motivation and history.

In the recent landmark paper Baratchart et al. [1] have developed the theory of rational approximation of functions f ∈ A(G) in the L^2(s_T) norm on T, where s_T is the arc measure on T, and where the approximation is done from the set R_n(G) of rational functions p_{n-1}/q_n of degree (n−1, n) which have all their poles in G. Let the error of best approximation in L^p(s_T) be denoted by ρ_{n,p}(f, O). The theory in [1] gave, besides a lot of information on the best approximants, the p = 2 case of the asymptotic formula

\[ \lim_{n \to \infty} \rho^{1/2n}_{n,p}(f, O) = \exp\left(-\frac{1}{\text{cap}(K_T, T)}\right) \]  

[see below for the definition of the minimal condenser capacity cap(K_T, T)]. For p = ∞ the same formula follows from a result of Gonchar and Rakhmanov [2, Theorem 1’]. As a consequence, (1) has been established for all 2 ≤ p ≤ ∞.

In this note we derive (1) for all 1 ≤ p < ∞ directly from the p = ∞ case proven in [2, Theorem 1’].

To have a basis of discussion, let g_G(z, ζ) denote the Green’s function of G with pole at ζ ∈ G, and if K ⊂ G is a compact set, then consider the minimal energy

\[ I_G(K) := \inf_{\omega} I_G(\omega) := \inf_{\omega} \int \int g_G(z, t)d\omega(z)d\omega(t), \]

where the infimum is taken for all unit Borel-measures on K. In the case when K is not polar (has positive logarithmic capacity) there is a unique minimizing measure ω_K,T, called the Green equilibrium measure of K (with respect to Ω). cap(K, T) := 1/I_G(K) is called the condenser capacity of the condenser (K, T).

Next, we need the notion of a set of minimal condenser capacity. We say that a compact K ⊂ G is admissible for f ∈ A(G) if \( \overline{G} \setminus K \) is connected, and f has a meromorphic and single-valued extension there. The collection of all admissible sets for f is denoted by \( K_f(G) \). A compact \( K_T \in K_f(G) \) is said to be a set of minimal condenser capacity for f if

- \( \text{cap}(K_T, T) ≤ \text{cap}(K, T) \) for any \( K \in K_f(G) \),
- \( K_T \subseteq K \) for any \( K \in K_f(G) \) for which \( \text{cap}(K, T) = \text{cap}(K_T, T) \).

See [1] for the existence and unicity of such a K_T. The set K_T of minimal condenser capacity is the complement of the “largest” (regarding capacity) domain containing O on which \( f \) is single-valued and meromorphic. It turns out (see [1, Theorem 5]) that \( K_T = E_0 \cup E_1 \cup (\bigcup_{j} \gamma_j) \), where \( \bigcup \gamma_j \) is a finite union of open analytic arcs, \( E_0 \subset E_f \), each point in \( E_0 \) is the endpoint of exactly one \( \gamma_j \), while \( E_1 \) consists of those finitely many points where at least three arcs \( \gamma_j \) meet.

These definitions explain the notation in (1), and with these we claim

**Theorem 1** \( (1) \) holds for all \( 1 \leq p \leq \infty \).
Proof The \( p = \infty \) case is covered by the Gonchar–Rakhmanov theorem from [2], so it is left to show

\[
\liminf_{n \to \infty} \rho_{n,1}^{1/2n}(f, O) \geq \exp \left( -\frac{1}{\text{cap}(K_T, T)} \right).
\]

Let \( G_1 \supset G_2 \supset \cdots \) be a nested sequence of Jordan domains with boundaries \( T_1, T_2, \ldots \) such that \( T_{j+1} \subset G_j \), each \( T_j \) lies outside \( \overline{G} \), the maximal distance from a point of \( T_j \) to \( T \) is less than \( 1/j \) and \( \text{length}(T_j) \to \text{length}(T) \) (say some level line of the conformal mapping of \( O \) onto the exterior of the unit disk suffices as \( T_j \)). Then there is a compact set \( K \subset G \) and a \( j_0 \) such that \( K_{T_j} \subset K \) for \( j \geq j_0 \) (see Lemma 2 below), and for \( z, t \in K \) we have \( g_{G_j}(z, t) \leq g_G(z, t) + \eta_j \) where \( \eta_j \to 0 \) (see Lemma 3 below). If \( r \in \mathcal{R}_n(G) \) is any rational function from \( \mathcal{R}_n(G) \) and if we apply Cauchy’s formula for \( (f - r_n)(z) \), \( z \in T_j \), in \( O \) using integration on \( T \), we obtain

\[
\sup_{z \in T_j} |f(z) - r_n(z)| \leq \| f - r_n \|_{L^1(sT)} \frac{1}{\text{dist}(T_j, T)},
\]

so

\[
\liminf_{n \to \infty} \rho_{n,1}^{1/2n}(f, O) \geq \liminf_{n \to \infty} \rho_{n,\infty}^{1/2n}(f, O_j) = \exp \left( -I_{G_j} \left( \omega_{K_{T_j},T_j} \right) \right),
\]

where the equality follows by the aforementioned Gonchar–Rakhmanov theorem. Here for \( j \geq j_0 \) we have

\[
I_{G_j} \left( \omega_{K_{T_j},T_j} \right) \leq I_{G_j} \left( \omega_{K_{T_j},T} \right)
\]

by the definition of the Green equilibrium measure \( \omega_{K_{T_j},T_j} \), and clearly \( g_{G_j}(z, t) \leq g_G(z, t) + \eta_j \), \( t \in K \) and \( K_{T_j} \subset K \) imply

\[
I_{G_j} \left( \omega_{K_{T_j},T} \right) \leq I_G \left( \omega_{K_{T_j},T} \right) + \eta_j.
\]

Finally, since \( K_T \) is the set of minimal condenser capacity for \( G \), it maximizes the energies \( I_G(\omega_{K_{S,T}}) \) for all \( S \subset G \). Hence it follows that

\[
I_G \left( \omega_{K_{T_j},T} \right) \leq I_G \left( \omega_{K_T,T} \right).
\]

Putting all these together we get

\[
\liminf_{n \to \infty} \rho_{n,1}^{1/2n}(f, O) \geq \exp \left( -I_G \left( \omega_{K_T,T} \right) \right) e^{-\eta_j} = \exp \left( -\frac{1}{\text{cap}(K_T, T)} \right) e^{-\eta_j},
\]

which proves (2) if we let \( j \to \infty \). \( \square \)

The proof above used the following two facts.
Lemma 2 There is a compact set \( K \subset G \) and a \( j_0 \) such that \( K_{T_j} \subset K \) for \( j \geq j_0 \).

Lemma 3 For \( z, t \in K \) we have \( g_{G_j}(z, t) \leq g_G(z, t) + \eta_j \) where \( \eta_j \to 0 \).

Proof of Lemma 2 Let \( H_a = \{ z | \Re z > a \} \), and fix a neighborhood \( S \) around \( T \) to which \( f \) has a single-valued analytic continuation.

Assume to the contrary that there is a sequence of points \( P_j \in K_{T_j}, j = 1, 2, \ldots \), such that

\[
\liminf_{j \to \infty} \text{dist}(P_j, \overline{C \setminus G}) = 0.
\]

We may assume that here the \( \liminf \) is actually a limit and \( P_j \to P \in T \) (select a subsequence). Select a \( \tilde{P}_j \in T_j \) with \( \text{dist}(P_j, \tilde{P}_j) \to 0 \). Fix a \( z_0 \in G \) and let \( \varphi^*, \varphi^*_j \) be the conformal maps that map the unit disk onto \( G, G_j \) such that \( \varphi^*(0) = \varphi^*_j(0) = z_0 \) and \( (\varphi^*)'(0) > 0 \), \( (\varphi^*_j)'(0) > 0 \). It is known (see e.g. [3, Theorem 6.12 and Exercise 6.3/4]) that \( \varphi^*_j \to \varphi^* \) uniformly on the closed unit disk, therefore \( (\varphi^*_j)^{-1}(P_j) \to (\varphi^*)^{-1}(P), (\varphi^*_j)^{-1}(\tilde{P}_j) \to (\varphi^*)^{-1}(P). \) Combine these with some fixed mapping of the unit disk onto the right-half plane \( H_0 \) to deduce the following: if \( \varphi_j, \varphi \) are conformal maps of \( G_j, G \) onto \( H_0 \) such that \( \varphi_j(z_0) = \varphi(z_0) = 1, \varphi_j(\tilde{P}_j) = 0, \varphi(P) = 0 \), then \( \varphi_j \to \varphi \) uniformly on compact subsets of \( G \) and \( \varphi_j(P_j) \to \varphi(P) = 0 \). Therefore, there is an \( a > 0 \) such that \( \varphi_j(E_f) \subset \overline{H_a} \) for all large \( j \) and at the same time \( \varphi_j(P_j) \notin \overline{H_a} \). Hence, if \( B_j := \varphi_j(K_{T_j}) \), then

\[
B_j = \varphi_j(K_{T_j}) \not\subset \overline{H_a} \quad \text{for} \quad j \geq j_0
\]

with some \( j_0 \). We may also assume \( a > 0 \) to be so small and \( j_0 \) so large that \( \varphi_j(G \setminus S) \subset H_a \) for \( j \geq j_0 \) (note that \( \varphi(G \setminus S) \) is a compact subset of \( H_0 \)). Fix a \( j \geq j_0 \), and with this \( j \) we get a contradiction as follows.

Consider the mapping

\[
z = x + iy \to z' = \max(x, a) + iy
\]

(the projection onto \( \overline{H_a} \)) and set \( B'_j = \{ z' | z \in B_j \} \). Then

\[
g_{H_0}(z, w) = \log \left| \frac{z + \overline{w}}{z - w} \right| \leq \log \left| \frac{z' + \overline{w'}}{z' - w'} \right| = g_{H_0}(z', w')
\]

(just note that the imaginary parts are the same, while the real parts increase resp. decrease when we go from \( z + \overline{w} \) resp. \( z - w \) to \( z' + \overline{w}' \) resp. \( z' - w' \)).

We need

Lemma 4 There is a Borel-mapping \( \Phi : B'_j \to B_j \) such that \( \Phi(x)' = x \) for all \( x \in B'_j \). For every Borel-measure \( \mu \) on \( B'_j \) this generates a Borel-measure \( \nu \) on \( B_j \) via

\[
\nu(E) = \mu(\Phi^{-1}[E]) \quad \text{for all Borel-sets} \quad E \subset B_j \quad \text{(here \( \Phi^{-1}[E] \) is the complete inverse image of \( E \)) such that}
\]

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\[
\int \log \left| \frac{z + w}{z - w} \right| \, d\nu(z) d\nu(w) = \int \log \left| \frac{\Phi(u) + \Phi(v)}{\Phi(u) - \Phi(v)} \right| \, d\mu(u) d\mu(v).
\]

**Proof** With this lemma at hand we continue the proof of Lemma 2. We have

\[
I_{H_0}(v) = \int \log \left| \frac{z + w}{z - w} \right| \, d\nu(z) d\nu(w) = \int \log \left| \frac{\Phi(u) + \Phi(v)}{\Phi(u) - \Phi(v)} \right| \, d\mu(u) d\mu(v)
\]

\[
\leq \int \log \left| \frac{u + \overline{v}}{u - v} \right| \, d\mu(u) d\mu(v) = I_{H_0}(\mu),
\]

where, at the second inequality, we used (4).

Let \( \Omega_j \) be the unbounded component of \( \overline{C} \setminus B_j' \) and \( \text{Pc}(B_j') : \overline{C} \setminus \Omega_j \) be the so called polynomial convex hull of \( B_j' \). Next we show that \( \text{Pc}(B_j') \) is an admissible set for the function \( F := f(\varphi^{-1}_j) \) in \( H_0 \). To see this let \( \Gamma \) be a polygonal curve in \( \Omega_j \cap H_0 \) starting and ending at the origin, i.e. \( \Gamma \) is a closed curve that lies in the right-half plane \( H_0 \) except for the point \( 0 \in \Gamma \), and \( \Gamma \) does not intersect \( \text{Pc}(B_j') \). Let \( F^* \) be the continuation of \( F \) along (a neighborhood of) \( \Gamma \) as we traverse \( \Gamma \) once from 0 to 0. We need to show that after traversing \( \Gamma \) we get back to the same function element, i.e. \( F^* = F \) in a neighborhood of the origin.

By assumption, \( F \) has a continuation to the strip \( H_0 \setminus \overline{H_a} \) which we denote by \( F_0 \). Also, by the assumption on \( K_j \), \( F \) has a single-valued continuation \( F_1 \) to the set \( \overline{C} \setminus B_j \). Note that necessarily \( F_1 = F_0 \) on the set \( (H_0 \setminus \overline{H_a}) \setminus B_j \). We may assume that \( \Gamma \) does not contain a vertical segment, and for some small \( \varepsilon > 0 \) let \( Q_1, \ldots, Q_m \) be the points of \( \Gamma \) (in the order of the traverse) that lie on the line \( \set{z \mid \Re z = a - \varepsilon} \). Let here \( \varepsilon > 0 \) be so small that \( \overline{H_{a - \varepsilon}} \cap \Gamma \cap B_j = \emptyset \) (there is such an \( \varepsilon > 0 \) since the preceding relation is true with \( \varepsilon = 0 \)). Then the points \( Q_1, \ldots, Q_m \) lie outside \( B_j \), and let \( D_k \subset H_0 \setminus \overline{H_a} \) be a small disk around \( Q_k \) not intersecting \( B_j \). Note that, as we have just remarked, \( F_1 \equiv F_0 \) on all these disks. Now we can easily prove by induction that \( F^* \equiv F_0 \equiv F_1 \) on each \( D_k \). Indeed, for \( k = 1 \) the equality \( F^* \equiv F_0 \) is true by the monodromy theorem in \( H_0 \setminus \overline{H_a} \). Now assume that we already know the claim for \( D_k \). The portion \( \Gamma_k \) of \( \Gamma \) in between the points \( Q_k \) and \( Q_{k+1} \) either lies in \( H_{a - \varepsilon} \) or in \( H_0 \setminus \overline{H_{a - \varepsilon}} \). In the former case the continuation of \( F^* \equiv F_1 \) along \( \Gamma_k \) is the same as \( F_1 \) (note that \( \Gamma_k \) does not intersect \( B_j \)), hence on \( D_{k+1} \) we have \( F^* \equiv F_1 \equiv F_0 \). On the other hand, if \( \Gamma_k \) lies in \( H_0 \setminus \overline{H_{a - \varepsilon}} \), then the continuation \( F^* \equiv F_0 \) along \( \Gamma_k \) is the same as \( F_0 \) by the monodromy theorem in \( H_0 \setminus \overline{H_a} \), hence in this case we have again \( F^* \equiv F_0 \equiv F_1 \) on \( D_{k+1} \), which completes the induction. Another application of the monodromy theorem along the portion of \( \Gamma \) from \( Q_m \) to 0 shows that, indeed, as we get back at the origin, with \( F^* \) we arrive back to the same function element \( F_0 \) that we started with.

We have thus shown that \( \text{Pc}(B_j') \) is an admissible set for \( f(\varphi^{-1}_j) \) in \( H_0 \), hence \( K^+_j := \varphi^{-1}_j(\text{Pc}(B_j')) \) is an admissible set for \( f \) in \( G_j \), and \( K^+_j \) lies in \( \varphi^{-1}_j(\overline{H_a}) \). If we define the measure \( \mu \) on \( B_j' \) by stipulating \( \mu(E) = \omega_{K^+_j}(\varphi^{-1}_j(E)) \) for all Borel-sets \( E \subset B_j' \), \( \nu \) is the associated measure via Lemma 4, and finally \( \omega \) is the measure defined.

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by $\omega(E) = \nu(\varphi_j(E))$, then $\omega$ is supported on $K_{T_j}$, and has total mass 1 because $\omega_{K_j^*, T_j}$ is supported on the outer boundary of $K_j^*$ (see [1, Sec. 7.1.3]), and hence the interior of $Pc(B_j')$ has zero $\mu$-measure. Now we obtain from Lemma 4 and from the conformal invariance of the Green’s function

$$I_{G_j}(\omega) = I_{H_0}(\nu) \leq I_{H_0}(\mu) = I_{G_j}(\omega_{K_j^*, T_j}) ,$$

which implies

$$I_{G_j}(K_{T_j}) \leq I_{G_j}(\omega) \leq I_{G_j}(\omega_{K_j^*, T_j}) = I_{G_j}(K_j^*).$$

Therefore, by the extremality of $K_{T_j}$ for $G_j$, we must have equality here, and then, by the definition of the set $K_{T_j}$ of minimal condenser capacity, we must have $K_{T_j} \subset K_j^* \subset \varphi^{-1}_j(\overline{H_a})$, which contradicts (3).

This contradiction proves the claim in Lemma 3.

Proof of Lemma 4  In this proof we use the special structure of the sets $K_{T_j}$ described just before the statement of Theorem 1.

For $z \in H_a \cap B_j' = H_a \cap B_j$ set $\Phi(z) = z$, and for $z = a + iy \in B_j' \cap \{x = a\}$ let $\Phi(z) = x(z) + iy \in B_j$ be the point in $B_j$ with the smallest possible $x$-coordinate $x(z)$. In the latter case $\Phi(z) \in H_0 \setminus H_a$, and clearly $\Phi(z)' = z$ for all $z \in B_j'$, so it is left to verify that $\Phi$ is a Borel-map. To obtain this it is sufficient to show that for a dense set of $B < C$ and for a dense set of $A \in [0, a)$ the inverse image $\Phi^{-1}[R]$ is a Borel-set, where $R = [0, A] \times [B, C]$. In order to show this, note that if the boundary of $R$ does not contain either endpoints of an open analytic arc $\gamma \subset B_j$ which is not a vertical or horizontal segment, then $\partial R \cap \gamma$ is a finite set. Therefore, in this case $R \cap \gamma$ consists of a finite number of analytic arcs, and hence $(R \cap \gamma)'$ is the union of finitely many closed segments on $\partial H_a$. Since $B_j$ is the union of finitely many points and finitely many open analytic arcs, it follows that $(R \cap B_j)'$ consists of a finite number of closed segments on $\partial H_a$, provided $\partial R$ does not contain any of the endpoints of these arcs. Since $\Phi^{-1}[R] = (R \cap B_j)'$, we are done.

Proof of Lemma 3  Let $\epsilon > 0$ and select a Jordan curve $\sigma$ separating $K$ and $T$ so that $g_G(z, \tau) \leq \epsilon$ for all $z \in \sigma$, $\tau \in K$ (there is such a $\sigma$: if $\sigma_1$ separates $T$ and $K$ then $g_G(z, t) \leq M$ for all $z \in \sigma_1$, $t \in K$ with some constant $M$). Map now the strip in between $T$ and $\sigma_1$ into a ring $R = \{1 \leq |z| \leq r\}$ by a conformal map $\varphi$. Then the three-circle-theorem gives

$$g_G(z, t) \leq M \frac{\log |\varphi(z)|}{\log r} ,$$

so

$$\sigma = \left\{z \left| |\varphi(z)| = \exp \left(\epsilon \frac{\log r}{M}\right) \right\}.$$
suffices for small $\varepsilon$. Now $g_{G,j}(z, \tau) \searrow g_G(z, \tau)$ for all $z \in \sigma$ and $\tau \in K$, so, by Dini’s theorem, this convergence is uniform in $z \in \sigma$ for all fixed $\tau \in K$, i.e. $g_{G,j}(\zeta, \tau) < 2\varepsilon$ for $j \geq j_\tau$ and all $\zeta \in \sigma, \tau \in K$. Then $g_{G,j}(z, t) < 2\varepsilon$ is true for all $z \in \sigma$ and $t \in K$ lying sufficiently close to some $\zeta \in \sigma$ and $\tau \in K$, and by compactness of $\sigma$ we get $g_{G,j}(z, t) < 2\varepsilon$ for all $z \in \sigma$ and $t$ lying sufficiently close to $\tau$. Then for the same values $g_{G,j}(z, t) < 2\varepsilon$ automatically holds for $j \geq j_\tau$ because the Green’s function $g_{G,j}$ decrease. Finally, by the compactness of $K$ there is a $j_0$ such that this inequality holds for all $z \in \sigma, t \in K$ and $j \geq j_0$.

As a consequence, $g_{G,j}(z, t) - g_G(z, t) \leq 2\varepsilon$ for $z \in \sigma, t \in K$ and $j \geq j_0$, and then, by the maximum theorem, this inequality persists for all $t \in K$ and $z$ lying inside $\sigma$.

\[\square\]

References