



Infinitely many homoclinic solutions for a class of damped vibration problems

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Abstract. In this paper, we consider the multiplicity of homoclinic solutions for the following damped vibration problems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + H_x(t, x(t)) = 0,$$

where $A(t) \in (\mathbb{R}, \mathbb{R}^N)$ is a symmetric matrix for all $t \in \mathbb{R}$, $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, and $H(t, x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ is only locally defined near the origin in x for some $\delta > 0$. With the nonlinearity $H(t, x)$ being partially sub-quadratic at zero, we obtain infinitely many homoclinic solutions near the origin by using a Clark's theorem.

Keywords: homoclinic solutions, Clark's theorem, critical points, Palais–Smale condition.

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1 Introduction

The homoclinic orbit is an important kind of trajectory in dynamical systems recognized by Poincaré at the end of the 19th century. Their presence often means the occurrence of chaos or the bifurcation behavior of periodic orbits, see [4, 7, 10, 12, 14] and references therein. In recent decades, the existence and multiplicity of homoclinic orbits has been studied in depth via variational methods. In this paper, we consider the existence of infinitely many homoclinic solutions for the following damped vibration problems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + H_x(t, x(t)) = 0, \tag{1.1}$$

where $x(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$, $A(t) = [a_{ij}(t)]$ is a symmetric and positive $N \times N$ matrix-valued function with $a_{ij} \in L^\infty(\mathbb{R}, \mathbb{R}) (\forall i, j = 1, 2, \dots, N)$, $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, $H(t, x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ with $B_\delta = \{x \in \mathbb{R}^N \mid |x| \leq \delta\}$ for some $\delta > 0$, $H_x(t, x)$ denote its derivative with respect to the x variable.

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When $B = 0$, the system (1.1) is the classical second-order Hamiltonian systems which has been extensively studied in the past, see [1, 5, 6, 8, 11, 13, 15, 16] and references therein. When $B \neq 0$, many authors have studied the existence and multiplicity of homoclinic solutions for (1.1) under various growth conditions, see [2, 3, 17–19] and references therein. In [17], Wu and Zhang obtained the existence and multiplicity of homoclinic solutions by using a symmetric mountain pass theorem and a generalized mountain pass theorem under the local (AR) superquadratic growth condition. In [2], by using a variant fountain theorem, Chen obtained infinitely many nontrivial homoclinic orbits for non-periodic damped vibration systems when $H(t, x)$ satisfies the subquadratic condition at infinity. In [19], Zhang and Yuan studied the existence of the homoclinic solutions via the genus properties in critical point theory when $H(t, x)$ is of subquadratic growth as $|x| \rightarrow +\infty$. In [3], Chen and Tang obtained infinitely many homoclinic solutions for (1.1) by using a fountain theorem when $H(t, x)$ satisfies a new subquadratic condition. In [18], Zhu obtained the existence of nontrivial homoclinic solutions using the mountain pass theorem when $H(t, x)$ satisfies asymptotically quadratic condition.

In this paper, we study the existence of homoclinic solutions for (1.1) when the nonlinearity $H(t, x)$ is only defined near the origin with respect to x and $H(t, x)$ is partially subquadratic at zero. To the best of our knowledge, the existence of homoclinic solutions for damped vibration systems in this case has not been considered before. Our work is motivated by [9], where the authors improved and extended Clark's theorem and applied it to the problems on solutions of elliptic equations and periodic solutions of Hamiltonian systems. Here by using the Clark's theorem in [9], we prove that (1.1) has infinitely many homoclinic solutions near the origin. Furthermore, we make the following assumptions:

(H₁) $H(t, x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ is even in x , $H(t, x) = H(t, -x)$ for all $t \in \mathbb{R}$ and $x \in B_\delta$, and $H(t, 0) = 0$ for all $t \in \mathbb{R}$;

(H₂) There exists constants $\alpha > 0$, such that $(A(t)x, x) \geq \alpha|x|^2$ and $\|B\| < 2\sqrt{\alpha}$ for all $(t, x) \in (\mathbb{R}, \mathbb{R}^N)$;

(H₃) There exist $t_0 \in \mathbb{R}$ and $r > 0$ such that uniformly in $t \in [t_0 - r, t_0 + r]$,

$$\lim_{|x| \rightarrow 0} \frac{H(t, x)}{|x|^2} = +\infty;$$

(H₄) For all $(t, x) \in \mathbb{R} \times B_\delta$,

$$|H_x(t, x)| \leq b(t),$$

where $b(t) : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $b \in L^\zeta(\mathbb{R})$ for some $1 \leq \zeta \leq 2$.

Now, we state the main result as follows.

Theorem 1.1. *Assume that (H₁)–(H₄) hold, then (1.1) has infinitely many homoclinic solution x_k with $\|x_k\|_{L^\infty} \rightarrow 0$ as $k \rightarrow \infty$.*

Remark 1.2. Now we give some comparisons between our result and other results on the system (1.1). Firstly, in the previous works [2, 3, 17–19], the authors needed to make assumptions about the behavior of the nonlinearity $H(t, x)$ as $|x| \rightarrow +\infty$. They assumed that $H(t, x)$ satisfies the subquadratic condition, superquadratic condition or asymptotically quadratic condition at infinity. Compared with these works, we do not need the behavior of the nonlinearity $H(t, x)$ for $|x|$ large. Secondly, our subquadratic conditions near zero are also weaker than the related papers [2, 3]. In [2, 3], the authors assumed that $H(t, x)$ satisfies $\lim_{|x| \rightarrow 0} \frac{H(t, x)}{|x|^2} = +\infty$ for

all $t \in \mathbb{R}$. By contrast, we only assume that $\lim_{|x| \rightarrow 0} \frac{H(t,x)}{|x|^2} = +\infty$ in a interval $t \in [t_0 - r, t_0 + r]$. Thirdly, in the literature [2, 3, 17–19], the authors did not give the information for the obtained homoclinic solutions. However, we can prove that the homoclinic solutions found here converge to the null solution in L^∞ norm.

Example 1.3. Let $H(t, x) = \eta(t)|x|^\mu$, where $1 < \mu < 2$, $\eta(t) \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies that $\eta(t) = 1$, $\forall |t| \leq 1$, and $\eta(t) = 0$, $\forall |t| \geq 2$. It is not difficult to see that $H(t, x)$ satisfies all conditions of Theorem 1.1. It is worth noting that $H(t, x)$ does not satisfies $\lim_{|x| \rightarrow 0} \frac{H(t,x)}{|x|^2} = +\infty$ for all $t \in \mathbb{R}$.

The remainder of this paper is organized as follows. In Section 2, we give the variational framework for (1.1). In Section 3, we prove our main result in detail.

2 Preliminaries

In this section, we establish the variational framework for (1.1) and give a preliminary result.

Let $E = H^1(\mathbb{R}, \mathbb{R}^N)$ be a Hilbert space where the function is from \mathbb{R} to \mathbb{R}^N with the inner product

$$\langle x, y \rangle_0 = \int_{\mathbb{R}} \left((x(t), y(t)) + (\dot{x}(t), \dot{y}(t)) \right) dt, \quad \forall x, y \in E_0, \quad (2.1)$$

where (\cdot, \cdot) means the standard inner product in \mathbb{R}^N . The corresponding norm is

$$\|x\|_0 = \left(\int_{\mathbb{R}} (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}, \quad \forall x \in E_0. \quad (2.2)$$

For simplicity, we define a new norm on E . Let

$$\|x\| = \left(\int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))] dt \right)^{\frac{1}{2}}, \quad \forall x \in E. \quad (2.3)$$

And the corresponding inner product is denoted by $\langle \cdot, \cdot \rangle$. Now we show that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Since $\|B\| < 2\sqrt{\alpha}$ from (H_2) , then $\frac{\|B\|^2}{2\alpha} < 2$. Hence we can choose a constant ε_0 such that

$$\frac{\|B\|^2}{2\alpha} < \varepsilon_0 < 2. \quad (2.4)$$

Set

$$C_0 = \min \left\{ 1 - \frac{\varepsilon_0}{2}, \alpha - \frac{\|B\|^2}{2\varepsilon_0} \right\}. \quad (2.5)$$

By (2.4), we see that $C_0 > 0$. Then by (H_2) and mean inequality, we have

$$\begin{aligned} \|x\|^2 &= \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))] dt \\ &\geq \int_{\mathbb{R}} \left[\left(1 - \frac{\varepsilon_0}{2}\right) |\dot{x}(t)|^2 + \left(\alpha - \frac{\|B\|^2}{2\varepsilon_0}\right) |x(t)|^2 \right] dt \\ &\geq C_0 \|x\|_0^2. \end{aligned} \quad (2.6)$$

On the other hand,

$$\begin{aligned} \|x\|^2 &= \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))] dt \\ &\leq \int_{\mathbb{R}} [|\dot{x}(t)|^2 + \|A(t)\|_{L^\infty(\mathbb{R})}|x|^2 + \|B\|(|\dot{x}(t)|^2 + |x|^2)] dt \\ &\leq C_1 \|x\|_0^2, \end{aligned} \quad (2.7)$$

where $C_1 = (1 + \|A(t)\|_{L^\infty(\mathbb{R})} + \|B\|)$ is a constant. Therefore, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

To obtain the homoclinic solution of (1.1), we consider the following systems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + \hat{H}_x(t, x(t)) = 0, \quad (2.8)$$

where $\hat{H} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies that \hat{H} is even in u , $\hat{H}(t, x) = H(t, x)$ for $t \in \mathbb{R}$ and $|x| < \frac{\delta}{2}$, and $\hat{H}(t, x) = 0$ for $t \in \mathbb{R}$ and $|x| > \delta$.

Define the functional Φ on E by

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))] dt - \int_{\mathbb{R}} \hat{H}(t, x(t)) dt \\ &= \frac{1}{2} \|x\|^2 - \int_{\mathbb{R}} \hat{H}(t, x(t)) dt. \end{aligned} \quad (2.9)$$

By (H_1) , $\Phi \in C^1(E, \mathbb{R})$ and the critical points of Φ correspond to the homoclinic solutions of (2.8) (see [17]). We can get that

$$\begin{aligned} \langle \Phi'(x), y \rangle &= \int_{\mathbb{R}} [(\dot{x}(t), \dot{y}(t)) + (A(t)x(t), y(t)) - (B\dot{x}(t), y(t))] dt \\ &\quad - \int_{\mathbb{R}} (\hat{H}_x(t, x(t)), y(t)) dt. \end{aligned} \quad (2.10)$$

Now we introduce a Clark's theorem established by Liu and Wang [9]. Clark's theorem is a classical theorem in the critical point theory and has a large number of applications in differential equations. In [9], Liu and Wang improved and extended Clark's theorem, and applied it to elliptic equations and Hamiltonian systems.

Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. We say that Φ satisfies (PS) condition if any sequence $\{x_j\}$ such that $\Phi(x_j)$ is bounded and $\Phi'(x_j) \rightarrow 0$ as $j \rightarrow \infty$ contains a convergent subsequence.

Theorem 2.1 ([9]). *Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k -dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{x \in X \mid \|x\| = \rho\}$, then at least one of the following conclusions holds.*

- (1) *There exists a sequence of critical points $\{x_k\}$ satisfying $\Phi(x_k) < 0$ for all n and $\|x_k\| \rightarrow 0$ as $k \rightarrow \infty$.*
- (2) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point x such that $\|x\| = a$ and $\Phi(x) = 0$.*

Remark 2.2. Clearly, under the assumptions of Theorem 2.1 there exist infinitely many critical points x_k of Φ that satisfies $\Phi(x_k) \leq 0$, $\Phi(x_k) \rightarrow 0$ and $\|x_k\| \rightarrow 0$ as $k \rightarrow \infty$.

3 Proof of the main result

In this section, we use Theorem 2.1 to prove the main result of this paper.

Proof. Step 1. We prove that Φ is bounded from below. Let $\|\cdot\|_{L^p(\mathbb{R})}$ denote the norm of $L^p(\mathbb{R}, \mathbb{R}^N)$ ($1 \leq p \leq \infty$). By (H_4) , we have that

$$|\hat{H}(t, x)| \leq b(t)|x|, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (3.1)$$

where $b \in L^\xi(\mathbb{R})$ is from (H_4) . If $\xi = 1$, we have

$$\int_{\mathbb{R}} \hat{H}(t, x(t)) dt \leq \int_{\mathbb{R}} b(t)|x| dt \leq \|x\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} b(t) dt \leq C'_1 \|x\| \|b(t)\|_{L^1(\mathbb{R})}, \quad (3.2)$$

where the Sobolev inequality $\|x\|_{L^\infty(\mathbb{R})} \leq C'_1 \|x\|$ has been used. If $1 < \xi \leq 2$, by the Hölder inequality and the Sobolev inequality, we have

$$\int_{\mathbb{R}} \hat{H}(t, x(t)) dt \leq \left(\int_{\mathbb{R}} (b(t))^\xi dt \right)^{\frac{1}{\xi}} \left(\int_{\mathbb{R}} |x|^{\frac{\xi}{\xi-1}} dt \right)^{\frac{\xi-1}{\xi}} \leq C'_\xi \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \quad (3.3)$$

Then, by (3.2), (3.3) we can see that

$$\int_{\mathbb{R}} \hat{H}(t, x(t)) dt \leq C'_\xi \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \quad (3.4)$$

Therefore by (2.3) and (3.4), we have

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) + (Bx(t), \dot{x}(t))] dt \\ &\quad - \int_{\mathbb{R}} \hat{H}(t, x(t)) dt \\ &\geq \frac{1}{2} \|x\|^2 - C'_\xi \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \end{aligned} \quad (3.5)$$

Consequently, Φ is bounded from below.

Step 2. We prove that $\Phi(x)$ satisfies the (PS) condition. Let $\{x_n\}$ be a (PS) sequence, that is $\Phi(x_n)$ is bounded and $\Phi'(x_n) \rightarrow 0$ as $n \rightarrow \infty$. By (3.5), we see that $\{x_n\}$ is bounded in E . Hence, there exists a subsequence of $\{x_n\}$ (for simplicity still denoted by $\{x_n\}$) and some $x_0 \in E$ such that $x_n \rightharpoonup x_0$ in E , and $x_n \rightarrow x_0$ strongly in $C_{loc}(\mathbb{R}^1)$ as $n \rightarrow \infty$. Then $\Phi'(x_0) = 0$.

Notice that

$$\begin{aligned} \|x_n - x_0\|^2 &= \langle (\Phi'(x_n) - \Phi'(x_0)), (x_n - x_0) \rangle \\ &\quad + \int_{\mathbb{R}} ((\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)), (x_n - x_0)) dt \end{aligned} \quad (3.6)$$

Since $x_n \rightharpoonup x_0$ in E and $\Phi'(x_n) \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\langle (\Phi'(x_n) - \Phi'(x_0)), (x_n - x_0) \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.7)$$

By (3.1), the Hölder inequality and the Sobolev inequality, for every $R > 0$ we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}} ((\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)), (x_n - x_0)) dt \right| \\
& \leq \int_{\mathbb{R}} |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| |x_n - x_0| dt \\
& \leq \int_{\mathbb{R} \setminus [-R, R]} |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| (|x_n| + |x_0|) dt \\
& \quad + \int_{-R}^R |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| |x_n - x_0| dt \\
& \leq 2 \int_{\mathbb{R} \setminus [-R, R]} b(t) (|x_n| + |x_0|) dt + \int_{-R}^R |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| |x_n - x_0| dt \\
& \leq 2 \|b(t)\|_{L^\xi(\mathbb{R} \setminus [-R, R])} (\|x_n\| + \|x_0\|) + \int_{-R}^R |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| |x_n - x_0| dt.
\end{aligned} \tag{3.8}$$

For any $\varepsilon > 0$, since $b(t) \in L^\xi(\mathbb{R})$ and $\{x_n\}$ is bounded in E , there exists $R_0 > 0$ large enough such that

$$(\|x_n\| + \|x_0\|) \|b(t)\|_{L^\xi(\mathbb{R} \setminus [-R_0, R_0])} < \frac{\varepsilon}{4}, \quad \forall n \in \mathbb{Z}^+. \tag{3.9}$$

On the other hand, since $x_n \rightarrow x_0$ strongly in $C([-R_0, R_0])$, there must exist $n_0 \in \mathbb{Z}^+$ such that for $n \geq n_0$

$$\int_{-R_0}^{R_0} |\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0)| |x_n - x_0| dt < \frac{\varepsilon}{2}. \tag{3.10}$$

Then by (3.8), (3.9) and (3.10), for $n \geq n_0$ we have

$$\left| \int_{\mathbb{R}} (\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0), x_n - x_0) dt \right| < \varepsilon,$$

which implies that

$$\left| \int_{\mathbb{R}} (\hat{H}_x(t, x_n) - \hat{H}_x(t, x_0), x_n - x_0) dt \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.11}$$

Hence, by (3.6), (3.7) and (3.11), we have $x_n \rightarrow x_0$ in E as $n \rightarrow \infty$. Therefore, $\Phi(x)$ satisfies the (PS) condition.

Step 3. We show that for every $k \in \mathbb{N}$, there exists a k -dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$. Let X^k be a k -dimensional subspace of $C_0^\infty([t_0 - r, t_0 + r])$. Since X^k is a finite dimensional space and the norms in finite dimensional space are all equivalent, there exists a positive constant $C_k > 0$ such that

$$\|x\|^2 \leq C_k \|x\|_{L^2}^2, \quad \forall x \in X^k. \tag{3.12}$$

By (H_3) and the definition of $\hat{H}(t, x)$, there exists a constant $0 < \delta_k < \frac{\delta}{2}$ such that for $t \in [t_0 - r, t_0 + r]$ and $x \in B_{\delta_k}$, we have

$$\hat{H}(t, x) \geq C_k |x|^2. \tag{3.13}$$

Recall the Sobolev inequality $\|x\|_{L^\infty(\mathbb{R})} \leq C_1 \|x\|$, we take $\rho_k = \frac{\delta_k}{C_1}$. Then for any $x \in S_{\rho_k}$, we have $\|x\|_{L^\infty} < \delta_k$. Thus by (2.9), (3.12) and (3.13), for any $x \in X^k \cap S_{\rho_k}$ we have

$$\begin{aligned}\Phi(x) &\leq \frac{1}{2} \|x\|^2 - C_k \|x\|_{L^2(\mathbb{R})}^2 \\ &< -\frac{1}{2} \|x\|^2 \\ &= -\frac{1}{2} \rho_k^2 < 0,\end{aligned}$$

which implies that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$. Now by Theorem 2.1, we obtain infinitely many solutions $\{x_k\}$ for (2.1) such that $\|x_k\| \rightarrow 0$ as $k \rightarrow \infty$. By Sobolev's inequality, we can get that $\|x_k\|_{L^\infty(\mathbb{R})} \rightarrow 0$ as $k \rightarrow \infty$. Then there exists $k_0 \in \mathbb{N}$ such that $\|x_k\|_{L^\infty(\mathbb{R})} < \frac{\delta}{2}$, $\forall k \geq k_0$. Hence by the definition of $\hat{H}(t, x)$, for $k \geq k_0$, $\{x_k\}$ are also solutions of (1.1). \square

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