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Ground state sign-changing solutions for critical Choquard equations with steep well potential

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Abstract. In this paper, we study sign-changing solution of the Choquard type equation

$$-\Delta u + (\lambda V(x) + 1)\, u = \left(I_\alpha * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \mu |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in ((N-4)^+, N)$, I_α is a Riesz potential, $p \in \left[2^*_\alpha, \frac{2N}{N-2}\right)$, $2^*_\alpha := \frac{N+\alpha}{N-2}$ is the upper critical exponent in terms of the Hardy–Littlewood–Sobolev inequality, $\mu > 0$, $\lambda > 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative and has a potential well. By combining the variational methods and sign-changing Nehari manifold, we prove the existence and some properties of ground state sign-changing solution for λ, μ large enough. Further, we verify the asymptotic behaviour of ground state sign-changing solutions as $\lambda \to +\infty$ and $\mu \to +\infty$, respectively.

Keywords: Choquard equation, upper critical exponent, steep well potential, ground state sign-changing solution, asymptotic behaviour.

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1 Introduction and main results

The Choquard equation has a physical prototype, namely the Hartree type evolution equation

$$-i\partial_t \psi = \Delta \psi + \left(I_2 * |\psi|^2\right)\psi, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.1}$$

where $\mathbb{R}_+ = [0, +\infty)$, $I_2(x) = \frac{1}{4\pi|x|}$, $\forall\, x \in \mathbb{R}^3 \backslash \{0\}$, and $*$ is convolution in $\mathbb{R}^3$. Eq. (1.1) was firstly proposed by Pekar to describe a resting polaron in [24]. Two decades later, Choquard [16] introduced Eq. (1.1) as a certain approximation to Hartree–Fock theory of one component plasma, and used it to characterize an electron trapped in its own hole. Afterwards, viewing the quantum state reduction as a gravitational phenomenon in quantum gravity, Penrose et al. [20] proposed Eq. (1.1) in the form of Schrödinger–Newton system to model a single particle moving in its own gravitational field.

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As we know, standing wave solution of Eq. (1.1) corresponds to solution of the Choquard equation

$$-\Delta u + u = \left(I_2 * |u|^2\right) u \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

In detail, with a suitable scaling, the wave function $\psi(x,t) = e^{-it}u(x)$ is a solution of Eq. (1.1) once u is a solution of Eq. (1.2). Lieb demonstrated the seminal work on Eq. (1.2) in [16], in which he certified the existence and uniqueness (up to translations) of positive radial ground state solution by applying symmetrically decreasing rearrangement inequalities. After this, Lions [18] studied the same problem and further proved the existence of infinitely many radial solutions via the variational methods.

From mathematical perspective, scholars prefer to study the general Choquard equation

$$-\Delta u + W(x)u = \gamma\left(I_\alpha * G(u)\right) g(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N \geq 3$, $\gamma \in \mathbb{R}^+$, I_α is the Riesz potential of order $\alpha \in (0, N)$ defined for $x \in \mathbb{R}^N \backslash \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{\frac{N}{2}}},$$

Γ is the Gamma function, $*$ is convolution, $W \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$ and $G(u) = \int_0^u g(s)ds$.

To establish the variational framework for Choquard equations, we need the following celebrated Hardy–Littlewood–Sobolev inequality.

Proposition 1.1 ([17, Theorem 4.3]). *Let $r, s > 1$, $0 < \alpha < N$ satisfy $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$. Then there exists a sharp constant $C(N, \alpha, r, s) > 0$ such that, for all $f \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$, there holds*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}} dxdy \right| \leq C(N, \alpha, r, s)|f|_r|h|_s. \tag{1.4}$$

In particular, if $r = s = \frac{2N}{N+\alpha}$, then the constant $C(N, \alpha, r, s)$ admits a precise expression, namely,

$$C(N, \alpha) := C(N, \alpha, r, s) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right]^{-\frac{\alpha}{N}}.$$

Thanks to (1.4), the integral $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$ is well defined in $H^1(\mathbb{R}^N)$ once $p \in [2_*^\alpha, 2_\alpha^*]$, where $2_\alpha^* := \frac{N+\alpha}{N-2}$ and $2_*^\alpha := \frac{N+\alpha}{N}$ are usually called upper and lower critical exponents with respect to the Hardy–Littlewood–Sobolev inequality, respectively. It is easy to clarify that the critical terms $\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*} dx$ and $\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_*^\alpha})|u|^{2_*^\alpha} dx$ are invariant under the scaling actions $\sigma^{\frac{N-2}{2}} u(\sigma \cdot)$ and $\sigma^{\frac{N}{2}} u(\sigma \cdot)$ $(\sigma > 0)$, respectively, and these two scaling actions served as group actions are noncompact on $H^1(\mathbb{R}^N)$. Consequently, from the perspective of variational methods, the critical exponents 2_*^α and 2_α^* may provoke two kinds of lack of compactness. However, fortunately, similar to the Sobolev critical case studied in [3], these two kinds of loss of compactness can be recovered to some extent by using the extremal functions of the Hardy–Littlewood–Sobolev inequality.

In [21], Moroz and Van Schaftingen studied the case of Eq. (1.3) that $W(x) \equiv 1$, $\gamma = \frac{1}{p}$ and $G(u) = |u|^p$ $(p > 1)$, they proved the existence, regularity, radially symmetry and decaying property at infinity of ground state solution when $p \in (2_*^\alpha, 2_\alpha^*)$. Meanwhile, based on the regularity of solutions, they established a Nehari–Pohožaev type identity and then showed the nonexistence of nontrivial solutions for Eq. (1.3) when $p \notin (2_*^\alpha, 2_\alpha^*)$. Afterwards, in [22], they extended the existence results in [21] to the case of Eq. (1.3) that g satisfies the so-called almost necessary conditions of Berestycki–Lions type. For the critical cases of Eq. (1.3), with the nonexistence result of [21] in hand, an increasing number of scholars devote to studying Eq. (1.3) with critical term and a noncritical perturbed term. We refer the interested readers to [4,9,14,30] for upper critical case, [23,26] for lower critical case and [15,25,31] for doubly critical case.

When it comes to the case $W(x) \not\equiv$ const., we focus our attention on steep well potential of the form $\lambda V(x) + b$, where $\lambda > 0$, $b \in \mathbb{R}$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the following hypotheses:

(V_1) V is bounded from below, $\Omega := \operatorname{int} V^{-1}(0)$ is nonempty and $\overline{\Omega} = V^{-1}(0)$,

(V_2) there exists some constant $M > 0$ such that $\left|\{x \in \mathbb{R}^N : V(x) \leq M\}\right| < +\infty$.

This type of potential was firstly introduced by Bartch and Wang in [2] to study the existence and multiplicity of nontrivial solutions for subcritical Schrödinger equations in the case of $b > 0$. Later, Ding and Szulkin further considered the case $b = 0$ in [8]. Since $|\Omega| < +\infty$, then $-\Delta$ possesses a sequence of positive Dirichlet eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_n \to +\infty$. Assuming $b < 0$ and $b \neq -\mu_i$ for any $i \in \mathbb{N}_+$, Clapp and Ding [6], together with Tang [27], studied the existence and concentration of ground state solution for critical Schrödinger equation. Recently, the pre-existing results on Schrödinger equations have been extended to the Choquard equations, see e.g. [1,14,15,19] and the references therein.

As we concerned here, sign-changing solution of elliptic equation is a focusing topic due to its wide application in biology and physics etc. In [7], Clapp and Salazar investigated the Choquard equation

$$-\Delta u + W(x)u = (I_\alpha * |u|^p)\, |u|^{p-2}u \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an exterior domain, $p \in [2, 2_\alpha^*)$, $\alpha \in ((N-4)^+, N)$ and $W \in C(\mathbb{R}^N, \mathbb{R})$. Under symmetrical assumptions on Ω and decaying properties on W, they derived multiple sign-changing solutions. After this, many scholars considered the same topic in the whole Euclidean space, namely,

$$-\Delta u + W(x)u = (I_\alpha * |u|^p)\, |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

In [11], Ghimenti and Van Schaftingen studied the case that $N \geq 1$, $\alpha \in ((N-4)^+, N)$, $W(x) \equiv 1$ and $p \in (2, 2_\alpha^*)$ of Eq. (1.5). There, by introducing a new minimax principle and concentration-compactness lemmas for sign-changing Palais–Smale sequences, they obtained a ground state sign-changing solution. Also, they proved that the least energy in the sign-changing Nehari manifold has no minimizers when $p \in (2_*^\alpha, \max\{2, 2_\alpha^*\})$. Further, Ghimenti, Moroz and Van Schaftingen [10] constructed a ground state sign-changing solution of Eq. (1.5) when $p = 2$ by approaching the case $p = 2$ with the cases $p \in (2, 2_\alpha^*)$. Van Schaftingen and Xia [28] assumed that $N \geq 1$, $\alpha \in ((N-4)^+, N)$, $p \in [2, 2_\alpha^*)$ and $W \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the coercive condition $\lim_{|x|\to\infty} W(x) = +\infty$. By using a constrained minimization argument in sign-changing Nehari manifold, they derived a ground state sign-changing solution of Eq. (1.5) (see the similar result in [32]). Moreover, Zhong and Tang [33] studied the following Choquard equation

$$-\Delta u + (\lambda V(x) + 1)\, u = \left(I_\alpha * (K|u|^p)\right) K(x)|u|^{p-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, $\alpha \in ((N-4)^+, N)$, $p \in (2, 2_\alpha^*)$, $\lambda < 0$ and the functions V, K satisfy

(V_3) $V \in L^{\frac{N}{2}}(\mathbb{R}^N)\backslash\{0\}$ is nonnegative,

(V_4) there exist constants $\rho, \beta, C > 0$ such that $V(x) \geq C|x|^{-\beta}$ for all $|x| < \rho$,

(K_1) $K \in L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\backslash\{0\}$ for some $r \in \big[\frac{2N}{N+\alpha-p(N-2)}, +\infty\big)$ and K is nonnegative.

It follows from (V_3) that the first eigenvalue λ_1 of $-\Delta u + u = \lambda V(x)u$ in $H^1(\mathbb{R}^N)$ is positive. When $\lambda \in (-\lambda_1, 0)$ and $\beta \in \big(2 - \min\{\frac{N+\alpha}{2p} - \frac{N-2}{2}, \frac{N-2}{2}\}, 2\big)$, following the ideas in [5], they derived a ground state sign-changing solution by using minimization arguments in sign-changing Nehari manifold.

Motivated by the above works, in the present paper, we study the Choquard equation

$$-\Delta u + (\lambda V(x) + 1)\, u = \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*-2}u + \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where $\lambda > 0$, $\mu > 0$, $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2_\alpha^*, 2^*)$, and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the hypotheses

(V_5) $V(x) \geq 0$ in $\mathbb{R}^N$ and there exists some $M > 0$ such that $\big|\{x \in \mathbb{R}^N : V(x) \leq M\}\big| < +\infty$,

(V_6) $\Omega := \operatorname{int} V^{-1}(0)$ is a nonempty set with smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

Let $E_\lambda := \big\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) u^2 dx < +\infty\big\}$ be equipped with the inner product

$$(u, v)_\lambda := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + (\lambda V(x) + 1)\, uv dx, \qquad \forall\, u, v \in E_\lambda,$$

and the norm $\|\cdot\|_\lambda = (\cdot,\cdot)_\lambda^{\frac{1}{2}}$ for any $\lambda > 0$. Since $V \geq 0$ in $\mathbb{R}^N$, it is easy to see that $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ and, for any $s \in [2, 2^*]$, there is some constant $\nu_s > 0$ such that, for all $\lambda > 0$,

$$|u|_s \leq \nu_s \|u\| \leq \nu_s \|u\|_\lambda, \qquad \forall\, u \in E_\lambda. \tag{1.7}$$

By (1.4) and (1.7), we deduce the energy functional $\mathcal{J}_{\lambda,\mu}$ of Eq. (1.6) belongs to $C^1(E_\lambda, \mathbb{R})$, where

$$\mathcal{J}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2 \cdot 2_\alpha^*} \int_{\mathbb{R}^N} \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*} dx - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

Now we are prepared to state our main results.

Theorem 1.2. *Assume that $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2_\alpha^*, 2^*)$ and (V_5), (V_6) hold. Then there exist $\Lambda > 0$ and $\mu_* > 0$ such that Eq. (1.6) admits a ground state sign-changing solution $u_{\lambda,\mu}$ for any $\lambda \geq \Lambda$ and $\mu \geq \mu_*$. Further, for any $\mu \geq \mu_*$ and sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ satisfying $\lambda_n \to +\infty$, the sequence $\{u_{\lambda_n,\mu}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to some u_μ in $H^1(\mathbb{R}^N)$ in the sense of subsequence, where u_μ is a ground state sign-changing solution of*

$$\begin{cases} -\Delta u + u = A_\alpha \displaystyle\int_\Omega \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dy |u|^{2_\alpha^*-2}u + \mu|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Moreover, for any $\lambda \geq \Lambda$ and sequence $\{\mu_n\} \subset [\mu_*, +\infty)$ with $\mu_n \to +\infty$, the sequence $\{u_{\lambda,\mu_n}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to 0 in $H^1(\mathbb{R}^N)$ up to a subsequence.

Remark 1.3. Similar to the proof of Theorem 1.1 in [14], by minimizing $\mathcal{J}_{\lambda,\mu}$ on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \left\{ u \in E_\lambda \backslash \{0\}, \left\langle \mathcal{J}'_{\lambda,\mu}(u), u \right\rangle = 0 \right\},$$

we can demonstrate that Eq. (1.6) has a positive ground state solution $v_{\lambda,\mu}$ for any $\lambda, \mu > 0$ large enough. It is easy to show $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) > \mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$. Indeed, if $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) = \mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$, then $|u_{\lambda,\mu}| \in \mathcal{N}_{\lambda,\mu}$ satisfies $\mathcal{J}_{\lambda,\mu}(|u_{\lambda,\mu}|) = \inf_{\mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}$. Thereby, in a standard way, we may deduce $\mathcal{J}'_{\lambda,\mu}(|u_{\lambda,\mu}|) = 0$. Whereas, the strong maximum principle implies $|u_{\lambda,\mu}| > 0$ in $\mathbb{R}^N$, and the regular estimates for Choquard equations (see e.g. [21,22]) implies $u_{\lambda,\mu} \in C(\mathbb{R}^N, \mathbb{R})$, thus $u_{\lambda,\mu}$ has constant sign in $\mathbb{R}^N$, which contradicts with $u^{\pm}_{\lambda,\mu} \neq 0$. Furthermore, due to the presence of the perturbed term $\mu|u|^{p-2}u$, the methods introduced in [11,32] to verify that the least energy of sign-changing solutions is less than twice the least energy of nontrivial solutions seem invalid here, we propose an open question whether $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) < 2\mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$.

Remark 1.4. To our knowledge, there seem to be no results on (ground state) sign-changing solutions for Choquard equations with upper critical exponent, even on the bounded domain. Our present work extends and improves the existence results of sign-changing solutions verified in [7,10,11,28,33]. In [5], the authors studied the ground state sign-changing solutions for a class of critical Schrödinger equations

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 6)$ is a bounded domain and $\lambda \in (0, \lambda_1)$, with λ_1 denoting the first eigenvalue of $-\Delta$ on $\mathcal{D}$. They proved that any sign-changing $(PS)_c$ sequence is relatively compact once $c < c_0 + \frac{1}{N}S^{\frac{N}{2}}$, where c_0 is the least energy of nontrivial solutions. As a counterpart for the work in [5], Zhong and Tang studied a class of Choquard equations with critical Sobolev exponent in [33], where they showed the relative compactness of sign-changing $(PS)_c$ sequence with c less than the similar threshold. However, in this paper, due to the presence of the upper critical nonlocal term $(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2}u$ in Eq. (1.6), the relative compactness of sign-changing $(PS)_c$ sequence with

$$c \in \left[\frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \inf_{\mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu} + \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}} \right)$$

cannot be deduced as expected, where S_α is defined by (2.12) hereinafter. Also, it seems intractable to search for sign-changing $(PS)_c$ sequence such that $c < \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}$ for small $\mu > 0$. Naturally, we attempt to construct a sign-changing $(PS)_c$ sequence with $c < \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}$ by assuming that $\mu > 0$ is sufficiently large. Therefrom, by applying the properties of steep well potential λV, we can standardly prove the relative compactness of this type of sign-changing $(PS)_c$ sequence and then obtain ground state sign-changing solution.

We will give the proof of Theorem 1.2 in the forthcoming section. Throughout this paper, we use the following notations:

- ♠ $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$.
- ♠ $L^\infty(\mathbb{R}^N)$ is the space of measurable functions with the norm $|u|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)|$.

- ♠ $C_0^\infty(\mathbb{R}^N)$ consists of infinitely times differentiable functions with compact support in $\mathbb{R}^N$.
- ♠ $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$ endowed with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + uv dx \quad \text{and} \quad \|u\| = (u, u)^{\frac{1}{2}}.$$

- ♠ $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ with the norm $\|u\|_\Omega = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$.
- ♠ $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_D = |\nabla u|_2$.
- ♠ The best Sobolev constant $S = \inf\left\{\|u\|_D^2 : u \in D^{1,2}(\mathbb{R}^N) \text{ and } |u|_{2^*} = 1\right\}$.
- ♠ $u^\pm(x) := \pm\max\{\pm u(x), 0\}$ and $(E^*, \|\cdot\|_*)$ is the dual space of Banach space $(E, \|\cdot\|)$.
- ♠ $o(1)$ is a quantity tending to 0 as $n \to \infty$ and $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^N$.
- ♠ $\mathbb{B}_r(y) = \{x \in \mathbb{R}^N : |x - y| < r\}$, $\mathbb{B}_r^c(y) = \mathbb{R}^N \backslash \mathbb{B}_r(y)$ and $\mathbb{B}_r(0) = \mathbb{B}_r$ for $r > 0$, $y \in \mathbb{R}^N$.

2 Proof of Theorem 1.2

For the limiting problem of Eq. (1.6) as $\lambda \to +\infty$, namely Eq. (1.8), its energy functional is

$$\mathcal{J}_{\infty,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 dx - \frac{A_\alpha}{2 \cdot 2_\alpha^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy - \frac{\mu}{p}\int_\Omega |u|^p dx.$$

Due to (1.4) and $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, $\mathcal{J}_{\infty,\mu} \in C^1(H_0^1(\Omega), \mathbb{R})$. Define the sign-changing Nehari manifolds

$$\mathcal{M}_{\lambda,\mu} = \left\{u \in E_\lambda : u^\pm \neq 0, \left\langle \mathcal{J}'_{\lambda,\mu}(u), u^\pm \right\rangle = 0\right\},$$
$$\mathcal{M}_{\infty,\mu} = \left\{u \in H_0^1(\Omega) : u^\pm \neq 0, \left\langle \mathcal{J}'_{\infty,\mu}(u), u^\pm \right\rangle = 0\right\}.$$

Clearly, $\mathcal{M}_{\lambda,\mu}$ and $\mathcal{M}_{\infty,\mu}$ contain all of the sign-changing solutions of Eqs. (1.6) and (1.8), respectively. To search for ground state sign-changing solutions, we consider the following minimization problems:

$$m_{\lambda,\mu} = \inf\left\{\mathcal{J}_{\lambda,\mu}(u) : u \in \mathcal{M}_{\lambda,\mu}\right\},$$
$$m_{\infty,\mu} = \inf\left\{\mathcal{J}_{\infty,\mu}(u) : u \in \mathcal{M}_{\infty,\mu}\right\}.$$

Before completing the proof of Theorem 1.2, we establish several preliminary lemmas.

Lemma 2.1. *For any $\lambda > 0$, $\mu > 0$ and $u \in E_\lambda$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u})$ of positive numbers such that $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}} u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}} u^- \in \mathcal{M}_{\lambda,\mu}$, also,*

$$\mathcal{J}_{\lambda,\mu}\left(s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}} u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}} u^-\right) = \max_{s,t\geq 0} \mathcal{J}_{\lambda,\mu}\left(s^{\frac{1}{2_\alpha^*}} u^+ + t^{\frac{1}{2_\alpha^*}} u^-\right).$$

Proof. Firstly, we certify the existence of such pair of numbers. For any $\lambda > 0$, $\mu > 0$ and

$u \in E_\lambda$ with $u^\pm \neq 0$, define the function $\mathcal{F}_{\lambda,\mu,u}(s,t)$ for any $(s,t) \in [0,+\infty)^2$ by

$$\begin{aligned}\mathcal{F}_{\lambda,\mu,u}(s,t) &= \mathcal{J}_{\lambda,\mu}(s^{\frac{1}{2_\alpha^*}}u^+ + t^{\frac{1}{2_\alpha^*}}u^-)\\ &= \frac{s^{\frac{2}{2_\alpha^*}}}{2}\|u^+\|_\lambda^2 - \frac{s^2}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^+|^{2_\alpha^*}\right)|u^+|^{2_\alpha^*}dx - \frac{\mu s^{\frac{p}{2_\alpha^*}}}{p}\int_{\mathbb{R}^N}|u^+|^p dx\\ &\quad + \frac{t^{\frac{2}{2_\alpha^*}}}{2}\|u^-\|_\lambda^2 - \frac{t^2}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^-|^{2_\alpha^*}\right)|u^-|^{2_\alpha^*}dx - \frac{\mu t^{\frac{p}{2_\alpha^*}}}{p}\int_{\mathbb{R}^N}|u^-|^p dx\\ &\quad - \frac{st}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^+|^{2_\alpha^*}\right)|u^-|^{2_\alpha^*}dx.\end{aligned}$$

It is easy to derive $\lim_{|(s,t)|\to 0}\mathcal{F}_{\lambda,\mu,u}(s,t) = 0$ and $\lim_{|(s,t)|\to+\infty}\mathcal{F}_{\lambda,\mu,u}(s,t) = -\infty$. Then there exists some point $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) \in [0,+\infty)^2$ such that

$$\mathcal{F}_{\lambda,\mu,u}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = \max_{(s,t)\in[0,+\infty)^2}\mathcal{F}_{\lambda,\mu,u}(s,t).$$

Since $\mathcal{F}_{\lambda,\mu,u}(s, t_{\lambda,\mu,u})$ is increasing in s for $s > 0$ small enough, there results $s_{\lambda,\mu,u} \neq 0$. Similarly, we deduce $t_{\lambda,\mu,u} \neq 0$. Thereby, $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) \in (0,+\infty)^2$. Then

$$\frac{\partial\mathcal{F}_{\lambda,\mu,u}}{\partial s}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = \frac{\partial\mathcal{F}_{\lambda,\mu,u}}{\partial t}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = 0.$$

Naturally, $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\lambda,\mu}$.

Further, we claim such pair of numbers is unique. For brevity, we introduce the notation

$$B(u,v) := \frac{1}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u|^{2_\alpha^*}\right)|v|^{2_\alpha^*}dx, \qquad \forall\, u,v \in E_\lambda.$$

Through direct calculation, we deduce that the Hessian matrix of $\mathcal{F}_{\lambda,\mu,u}$ at $(s,t) \in (0,+\infty)^2$ is

$$\begin{aligned}H_{\lambda,\mu,u}(s,t) &= \frac{2-2_\alpha^*}{(2_\alpha^*)^2}\begin{pmatrix} s^{\frac{2}{2_\alpha^*}-2}\|u^+\|_\lambda^2 & 0\\ 0 & t^{\frac{2}{2_\alpha^*}-2}\|u^-\|_\lambda^2\end{pmatrix}\\ &\quad - \begin{pmatrix} B(u^+,u^+) & B(u^+,u^-)\\ B(u^+,u^-) & B(u^-,u^-)\end{pmatrix} - \frac{\mu(p-2_\alpha^*)}{(2_\alpha^*)^2}\begin{pmatrix} s^{\frac{p}{2_\alpha^*}-2}|u^+|_p^p & 0\\ 0 & t^{\frac{p}{2_\alpha^*}-2}|u^-|_p^p\end{pmatrix}.\end{aligned}$$

It follows from [17, Theorem 9.8] that $B(u^+,u^-)^2 < B(u^+,u^+)B(u^-,u^-)$. Then, noting $p \geq 2_\alpha^*$, we conclude that $H_{\lambda,\mu,u}(s,t)$ is negative defined for any $(s,t) \in (0,+\infty)^2$. Thereby, it is easy to know that $\mathcal{F}_{\lambda,\mu,u}$ has at most one critical point on $(0,+\infty)^2$. Thus, $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u})$ is the unique pair of positive numbers such that $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\lambda,\mu}$, and this lemma is proved. □

As a by-product, we may derive $\mathcal{M}_{\infty,\mu} \neq \emptyset$. Indeed, since $\mathcal{J}_{\lambda,\mu} = \mathcal{J}_{\infty,\mu}$ in $H_0^1(\Omega)$, we have

Remark 2.2. For any $\mu > 0$ and $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\mu,u}, t_{\mu,u})$ of positive numbers such that $s_{\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\infty,\mu}$ and

$$\mathcal{J}_{\infty,\mu}(s_{\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\mu,u}^{\frac{1}{2_\alpha^*}}u^-) = \max_{s,t\geq 0}\mathcal{J}_{\infty,\mu}(s^{\frac{1}{2_\alpha^*}}u^+ + t^{\frac{1}{2_\alpha^*}}u^-).$$

To facilitate the subsequent discussion, we show some properties of $\mathcal{M}_{\lambda,\mu}$ in the following

Lemma 2.3. *For any $\lambda > 0$ and $\mu > 0$, if $\{u_n\} \subset \mathcal{M}_{\lambda,\mu}$ and $\lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(u_n) = m_{\lambda,\mu}$, then $m_{\lambda,\mu} > 0$ and there exist some constants $C_{\lambda,\mu,1}, C_{\lambda,\mu,2} > 0$ such that $C_{\lambda,\mu,2} \le \|u_n^\pm\|_\lambda, \|u_n\|_\lambda \le C_{\lambda,\mu,1}$ for all n.*

Proof. From $\mathcal{M}_{\lambda,\mu} \neq \emptyset$, we know $m_{\lambda,\mu} < +\infty$ for any $\lambda, \mu > 0$. Since $\{u_n\} \subset \mathcal{M}_{\lambda,\mu}$, there holds

$$m_{\lambda,\mu} + o(1) = \mathcal{J}_{\lambda,\mu}(u_n) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n \right\rangle \ge \frac{p-2}{2p}\|u_n\|_\lambda^2. \tag{2.1}$$

Then there is constant $C_{\lambda,\mu,1} > 0$ such that $\sup_n \|u_n\|_\lambda \le C_{\lambda,\mu,1}$. Thereby, (1.4) and (1.7) imply

$$\begin{aligned}\|u_n^\pm\|_\lambda^2 &= \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n^\pm|^{2_\alpha^*} dx + \mu \int_{\mathbb{R}^N} |u_n^\pm|^p dx \\ &\le A_\alpha C(N,\alpha) \nu_{2^*}^{2\cdot 2_\alpha^*} \|u_n\|_\lambda^{2_\alpha^*} \|u_n^\pm\|_\lambda^{2_\alpha^*} + \mu \nu_p^p \|u^\pm\|_\lambda^p \\ &\le A_\alpha C(N,\alpha) \nu_{2^*}^{2\cdot 2_\alpha^*} C_{\lambda,\mu,1}^{2_\alpha^*} \|u_n^\pm\|_\lambda^{2_\alpha^*} + \mu \nu_p^p \|u^\pm\|_\lambda^p.\end{aligned}$$

As a consequence, there exists some constant $C_{\lambda,\mu,2} > 0$ such that $\inf_n \|u_n^\pm\|_\lambda \ge C_{\lambda,\mu,2}$. Further, we deduce from (2.1) that $m_{\lambda,\mu} > 0$. Thus we complete the proof of this lemma. □

Next, following [5], we construct a sign-changing $(PS)_c$ sequence $\{u_n\}$ for $\mathcal{J}_{\lambda,\mu}$, (i.e. $u_n^\pm \neq 0$ for any n, $\mathcal{J}_{\lambda,\mu}(u_n) \to c$ and $\mathcal{J}'_{\lambda,\mu}(u_n) \to 0$ in E_λ^* as $n \to \infty$). Let P_λ be the cone of nonnegative functions in E_λ, $Q = [0,1]^2$ and $\Gamma_{\lambda,\mu}$ be the set of continuous maps $\gamma : Q \to E_\lambda$ such that, for any $(s,t) \in Q$,

(a) $\gamma(s,0) = 0$, $\gamma(0,t) \in P_\lambda$ and $\gamma(1,t) \in -P_\lambda$,

(b) $(\mathcal{J}_{\lambda,\mu} \circ \gamma)(s,1) \le 0$ and

$$\frac{\int_{\mathbb{R}^N} \left[\left(I_\alpha * |\gamma(s,1)|^{2_\alpha^*}\right)|\gamma(s,1)|^{2_\alpha^*} + \mu|\gamma(s,1)|^p\right] dx}{\|\gamma(s,1)\|_\lambda^2} \ge 2.$$

For any $u \in E_\lambda$ with $u^\pm \neq 0$, define $\gamma_{\sigma,u}(s,t) = \sigma t(1-s)u^+ + \sigma tsu^-$ for $\sigma > 0$ and $(s,t) \in Q$. It is easy to show $\gamma_{\sigma,u} \in \Gamma_{\lambda,\mu}$ for $\sigma > 0$ large enough. Therefore, $\Gamma_{\lambda,\mu} \neq \emptyset$. Define the functional

$$\mathcal{L}_{\lambda,\mu}(u,v) = \begin{cases} \dfrac{\int_{\mathbb{R}^N} \left[(I_\alpha * |u|^{2_\alpha^*})(|u|^{2_\alpha^*} + |v|^{2_\alpha^*}) + \mu|u|^p\right] dx}{\|u\|_\lambda^2}, & u \neq 0, \\ 0, & u = 0. \end{cases}$$

Clearly, $\mathcal{L}_{\lambda,\mu} > 0$ if $u \neq 0$. Moreover, $u \in \mathcal{M}_{\lambda,\mu}$ if and only if $\mathcal{L}_{\lambda,\mu}(u^+, u^-) = \mathcal{L}_{\lambda,\mu}(u^-, u^+) = 1$.

As a start point, we display a minimax characterization on $m_{\lambda,\mu}$ for any $\lambda > 0$ and $\mu > 0$.

Lemma 2.4. *For any $\lambda > 0$ and $\mu > 0$, there holds*

$$m_{\lambda,\mu} = \inf_{\gamma\in\Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)). \tag{2.2}$$

Proof. On the one hand, for every $u \in \mathcal{M}_{\lambda,\mu}$, $\gamma_u(s,t) = \sigma t(1-s)u^+ + \sigma tsu^- \in \Gamma_{\lambda,\mu}$ for some $\sigma > 0$ large enough. Then it follows from Lemma 2.1 that

$$\mathcal{J}_{\lambda,\mu}(u) = \max_{s,t\ge 0} \mathcal{J}_{\lambda,\mu}(su^+ + tu^-) \ge \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma_u(s,t)) \ge \inf_{\gamma\in\Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)).$$

Thereby, due to the arbitrariness of $u \in \mathcal{M}_{\lambda,\mu}$, there results

$$m_{\lambda,\mu} \geq \inf_{\gamma \in \Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)).$$

On the other hand, for each $\gamma \in \Gamma_{\lambda,\mu}$ and $t \in [0,1]$, since $\gamma(0,t) \in P_\lambda$ and $\gamma(1,t) \in -P_\lambda$, we conclude

$$\mathcal{L}_{\lambda,\mu}(\gamma(0,t)^+, \gamma(0,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(0,t)^-, \gamma(0,t)^+) = \mathcal{L}_{\lambda,\mu}(\gamma(0,t)^+, \gamma(0,t)^-) \geq 0, \tag{2.3}$$

$$\mathcal{L}_{\lambda,\mu}(\gamma(1,t)^+, \gamma(1,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(1,t)^-, \gamma(1,t)^+) = -\mathcal{L}_{\lambda,\mu}(\gamma(1,t)^-, \gamma(1,t)^+) \leq 0. \tag{2.4}$$

Meanwhile, due to $\gamma(s,0) = 0$ for all $s \in [0,1]$, there holds

$$\mathcal{L}_{\lambda,\mu}(\gamma(s,0)^+, \gamma(s,0)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,0)^-, \gamma(s,0)^+) - 2 = -2, \qquad \forall\, s \in [0,1]. \tag{2.5}$$

And, for each $\gamma \in \Gamma_{\lambda,\mu}$, by the definition of $\mathcal{L}_{\lambda,\mu}$ and the property (b) we have, for all $s \in [0,1]$,

$$\begin{aligned} &\mathcal{L}_{\lambda,\mu}(\gamma(s,1)^+, \gamma(s,1)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,1)^-, \gamma(s,1)^+) - 2 \\ &\quad \geq \frac{\int_{\mathbb{R}^N} \left[(I_\alpha * |\gamma(s,1)|^{2_\alpha^*})|\gamma(s,1)|^{2_\alpha^*} + \mu|\gamma(s,1)|^p\right] dx}{\|\gamma(s,1)\|_\lambda^2} - 2 \geq 0. \end{aligned} \tag{2.6}$$

Moreover, it is easy to verify that, for any $(s,t) \in \partial Q$,

$$\begin{pmatrix} \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^+, \gamma(s,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^-, \gamma(s,t)^+) \\ \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^+, \gamma(s,t)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^-, \gamma(s,t)^+) - 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.7}$$

Then, by combining (2.3)−(2.7) with the Miranda theorem (see e.g. Lemma 2.4 in [13]), we derive that there exists some $(s_\gamma, t_\gamma) \in (0,1)^2$ satisfying

$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) = 0,$$

$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) = 2.$$

In view of this fact, we easily obtain

$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^+, \gamma(s_\gamma, t_\gamma)^-) = \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma, t_\gamma)^-, \gamma(s_\gamma, t_\gamma)^+) = 1,$$

which implies $\gamma(s_\gamma, t_\gamma) \in \mathcal{M}_{\lambda,\mu}$. Consequently, from the arbitrariness of $\gamma \in \Gamma_{\lambda,\mu}$, we deduce

$$\inf_{\gamma \in \Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)) \geq m_{\lambda,\mu}.$$

Now, by combining the above two sides, we know (2.2) holds. Thus this lemma is showed. □

Lemma 2.5. *For any $\lambda > 0$ and $\mu > 0$, $\mathcal{J}_{\lambda,\mu}$ possesses a sign-changing $(PS)_{m_{\lambda,\mu}}$ sequence $\{u_n\} \subset E_\lambda$.*

Proof. We will end the proof in two steps. Firstly, we construct a $(PS)_{m_{\lambda,\mu}}$ sequence for $\mathcal{J}_{\lambda,\mu}$. Take a minimizing sequence $\{w_n\} \subset \mathcal{M}_{\lambda,\mu}$ for $m_{\lambda,\mu}$ and set $\gamma_{\sigma,n}(s,t) = \sigma t(1-s)w_n^+ + \sigma t s w_n^-$. By Lemma 2.3, it is easy to choose a sufficiently large constant $\bar{\sigma} > 0$ such that $\{\gamma_{\bar{\sigma},n}\} \subset \Gamma_{\lambda,\mu}$. Due to Lemmas 2.1 and 2.4, there holds

$$\lim_{n\to\infty} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma_{\bar{\sigma},n}(s,t)) = \lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(w_n) = m_{\lambda,\mu}. \tag{2.8}$$

We assert that there exists some sequence $\{u_n\} \subset E_\lambda$ such that, as $n \to \infty$,

$$\mathcal{J}_{\lambda,\mu}(u_n) \to m_{\lambda,\mu}, \qquad \mathcal{J}'_{\lambda,\mu}(u_n) \to 0, \qquad \min_{(s,t)\in Q} \|u_n - \gamma_{\bar{\sigma},n}(s,t)\|_\lambda \to 0. \tag{2.9}$$

If not, there exists some constant $\delta_{\lambda,\mu} > 0$ such that, for n suitably large, $\gamma_{\bar{\sigma},n}(Q) \cap U_{\delta_{\lambda,\mu}} = \emptyset$, in which

$$U_{\delta_{\lambda,\mu}} \triangleq \{u \in E_\lambda : \exists\, v \in E_\lambda \text{ s.t. } \|v-u\|_\lambda \le \delta_{\lambda,\mu}, \|\nabla \mathcal{J}_{\lambda,\mu}(v)\| \le \delta_{\lambda,\mu}, |\mathcal{J}_{\lambda,\mu}(v) - m_{\lambda,\mu}| \le \delta_{\lambda,\mu}\}.$$

Then, by a variant of the classical deformation lemma due to Hofer (see [12, Lemma 1]), there exists a continuous map $\eta_{\lambda,\mu} : [0,1] \times E_\lambda \to E_\lambda$, which satisfies that, for some $\varepsilon_{\lambda,\mu} \in (0, \frac{m_{\lambda,\mu}}{2})$,

(i) $\eta_{\lambda,\mu}(0,u) = u,\ \eta_{\lambda,\mu}(\tau,-u) = -\eta_{\lambda,\mu}(\tau,u),\ \forall\, \tau \in [0,1],\ u \in E_\lambda,$

(ii) $\eta_{\lambda,\mu}(\tau,u) = u,\ \forall\, u \in \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\varepsilon_{\lambda,\mu}} \cup \left(E_\lambda \backslash \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\varepsilon_{\lambda,\mu}}\right),\ \forall\, \tau \in [0,1],$

(iii) $\eta_{\lambda,\mu}\left(1, \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \backslash U_{\delta_{\lambda,\mu}}\right) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}},$

(iv) $\eta_{\lambda,\mu}\left(1, (\mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \cap P_\lambda) \backslash U_{\delta_{\lambda,\mu}}\right) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}} \cap P_\lambda,$

where the sublevel set $\mathcal{J}_{\lambda,\mu}^d := \{u \in E_\lambda : \mathcal{J}_{\lambda,\mu}(u) \le d\}$ for $d \in \mathbb{R}$. By (2.8), we choose large n such that

$$\gamma_{\bar{\sigma},n}(Q) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \quad \text{and} \quad \gamma_{\bar{\sigma},n}(Q) \cap U_{\delta_{\lambda,\mu}} = \emptyset. \tag{2.10}$$

Set the continuous map $\widetilde{\gamma}_{\lambda,\mu,n}(s,t) = \eta_{\lambda,\mu}(1, \gamma_{\bar{\sigma},n}(s,t))$ for any $(s,t) \in Q$. We claim $\widetilde{\gamma}_{\lambda,\mu,n} \in \Gamma_{\lambda,\mu}$.

Indeed, from $\gamma_{\bar{\sigma},n}(s,0) = 0$ and (ii), it follows that $\widetilde{\gamma}_{\lambda,\mu,n}(s,0) = \eta_{\lambda,\mu}(1,0) = 0$ for any $s \in [0,1]$. Since $\gamma_{\bar{\sigma},n}(0,t), -\gamma_{\bar{\sigma},n}(1,t) \in P_\lambda$ and (2.10) implies $\gamma_{\bar{\sigma},n}(0,t), -\gamma_{\bar{\sigma},n}(1,t) \in \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \backslash U_{\delta_{\lambda,\mu}}$, we deduce from (i), (iv) that $\widetilde{\gamma}_{\lambda,\mu,n}(0,t) \in P_\lambda$ and $\widetilde{\gamma}_{\lambda,\mu,n}(1,t) \in -P_\lambda$ for all $t \in [0,1]$. Also, $\mathcal{J}_{\lambda,\mu}(\gamma_{\bar{\sigma},n}(s,1)) \le 0$ and (ii) imply $\widetilde{\gamma}_{\lambda,\mu,n}(s,1) = \eta_{\lambda,\mu}(1, \gamma_{\bar{\sigma},n}(s,1)) = \gamma_{\bar{\sigma},n}(s,1)$ for any $s \in [0,1]$. Then, by $\gamma_{\bar{\sigma},n} \in \Gamma_{\lambda,\mu}$, we know $\widetilde{\gamma}_{\lambda,\mu,n}$ satisfies the property (b). From the above arguments, we derive our claim $\widetilde{\gamma}_{\lambda,\mu,n} \in \Gamma_{\lambda,\mu}$.

Thereby, since (2.10) and (iii) imply $\widetilde{\gamma}_{\lambda,\mu,n}(Q) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}}$, we conclude

$$m_{\lambda,\mu} \le \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\widetilde{\gamma}_{\lambda,\mu,n}(s,t)) \le m_{\lambda,\mu} - \frac{\varepsilon_{\lambda,\mu}}{2},$$

which is a contradiction. Thus there is a sequence $\{u_n\} \subset E_\lambda$ possessing the properties in (2.9).

Secondly, we prove $u_n^\pm \ne 0$ for all large n. By (2.9), there exists a sequence $\{v_n\}$ such that

$$v_n = \alpha_n w_n^+ + \beta_n w_n^- \in \gamma_{\bar{\sigma},n}(Q) \quad \text{and} \quad \|v_n - u_n\|_\lambda \xrightarrow{n} 0. \tag{2.11}$$

Due to $\{w_n\} \subset \mathcal{M}_{\lambda,\mu}$ and $p \in (2, 2^*)$, from (1.4), Lemma 2.3 and the Young inequality we have

$$\|w_n^\pm\|_\lambda^2 \le A_\alpha C(N,\alpha)(\nu_{2^*} C_{\lambda,\mu,1})^{2^*_\alpha} |w_n^\pm|_{2^*}^{2^*_\alpha} + \frac{2^*-p}{2^*-2} |w_n^\pm|_2^2 + \frac{\mu^{\frac{2^*-2}{p-2}}(p-2)}{2^*-2} |w_n^\pm|_{2^*}^{2^*}.$$

Then, by (1.7), there holds

$$\frac{p-2}{(2^*-2)\nu_{2^*}^2}|w_n^\pm|_{2^*}^2 \le A_\alpha C(N,\alpha)(\nu_{2^*}C_{\lambda,\mu,1})^{2_\alpha^*}|w_n^\pm|_{2^*}^{2_\alpha^*} + \frac{\mu^{\frac{2^*-2}{p-2}}(p-2)}{2^*-2}|w_n^\pm|_{2^*}^{2^*},$$

which implies $\inf_n |w_n^\pm|_{2^*} > 0$. In view of this fact, the second limiting formula in (2.11) and (1.7), to show $u_n^\pm \neq 0$ for n large enough, it suffices to verify that $\alpha_n \nrightarrow 0$ and $\beta_n \nrightarrow 0$ up to subsequences. Suppose inversely $\alpha_n \to 0$ up to a subsequence. Then it follows from $\mathcal{J}_{\lambda,\mu} \in C(E_\lambda, \mathbb{R})$ and Lemma 2.3 that

$$m_{\lambda,\mu} = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(v_n) = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\alpha_n w_n^+ + \beta_n w_n^-) = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\beta_n w_n^-),$$

which together with $m_{\lambda,\mu} > 0$ implies $\bar{\beta} := \sup_n \beta_n < +\infty$. Further, by Lemma 2.1, the Fubini theorem, Lemma 2.3, (1.4) and (1.7), we deduce

$$\begin{aligned}
m_{\lambda,\mu} &= \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(w_n)\\
&= \lim_{n\to\infty}\max_{s,t\ge 0}\mathcal{J}_{\lambda,\mu}(sw_n^+ + tw_n^-)\\
&\ge \lim_{n\to\infty}\max_{s\ge 0}\mathcal{J}_{\lambda,\mu}(sw_n^+ + \beta_n w_n^-)\\
&= \lim_{n\to\infty}\max_{s\ge 0}\left[\frac{s^2}{2}\|w_n^+\|_\lambda^2 - \frac{s^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |w_n^+|^{2_\alpha^*}\right)|w_n^+|^{2_\alpha^*}dx - \frac{\mu s^p}{p}\int_{\mathbb{R}^N}|w_n^+|^p dx\right.\\
&\quad + \frac{\beta_n^2}{2}\|w_n^-\|_\lambda^2 - \frac{\beta_n^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |w_n^-|^{2_\alpha^*}\right)|w_n^-|^{2_\alpha^*}dx - \frac{\mu\beta_n^p}{p}\int_{\mathbb{R}^N}|w_n^-|^p dx\\
&\quad \left. - \frac{s^{2_\alpha^*}\beta_n^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |w_n^+|^{2_\alpha^*}\right)|w_n^-|^{2_\alpha^*}dx\right]\\
&= \lim_{n\to\infty}\max_{s\ge 0}\left[\frac{s^2}{2}\|w_n^+\|_\lambda^2 - \frac{s^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |w_n^+|^{2_\alpha^*}\right)|w_n^+|^{2_\alpha^*}dx\right.\\
&\quad \left. - \frac{s^{2_\alpha^*}\beta_n^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |w_n^+|^{2_\alpha^*}\right)|w_n^-|^{2_\alpha^*}dx - \frac{\mu s^p}{p}\int_{\mathbb{R}^N}|w_n^+|^p dx + \mathcal{J}_{\lambda,\mu}(\beta_n w_n^-)\right]\\
&\ge \max_{s\ge 0}\left[\frac{1}{2}C_{\lambda,\mu,2}^2 s^2 - \frac{1}{2_\alpha^*}A_\alpha C(N,\alpha)\left(\nu_{2^*}C_{\lambda,\mu,1}\right)^{2\cdot 2_\alpha^*}\bar{\beta}^{2_\alpha^*}s^{2_\alpha^*} - \frac{\mu}{p}\left(\nu_p C_{\lambda,\mu,1}\right)^p s^p\right.\\
&\quad \left. - \frac{1}{2\cdot 2_\alpha^*}A_\alpha C(N,\alpha)\left(\nu_{2^*}C_{\lambda,\mu,1}\right)^{2\cdot 2_\alpha}s^{2\cdot 2_\alpha}\right] + \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\beta_n w_n^-)\\
&> m_{\lambda,\mu},
\end{aligned}$$

a contradiction. Naturally, $\{\alpha_n\}$ has no subsequence tending to 0. Similarly, we can show $\{\beta_n\}$ has no subsequence tending to 0. Thus $u_n^\pm \neq 0$ for n large enough. This lemma is proved. □

Now, we estimate the least energy $m_{\lambda,\mu}$ from above. By [9, Lemma 1.2], the best constant

$$S_\alpha := \inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N}\left(I_\alpha * |u|^{2_\alpha^*}\right)|u|^{2_\alpha^*}dx = 1\right\} \tag{2.12}$$

is attained by the functions

$$U_\varepsilon(\cdot) = \frac{\left[N(N-2)\varepsilon^2\right]^{\frac{N-2}{4}}}{\left[C(N,\alpha)A_\alpha S^{\frac{\alpha}{2}}\right]^{\frac{N-2}{4+2\alpha}}(\varepsilon^2 + |\cdot|^2)^{\frac{N-2}{2}}}, \qquad \varepsilon > 0.$$

Take $\delta > 0$ such that $\mathbb{B}_{5\delta} \subset \Omega$, and extract two cut-off functions $\varphi, \psi \in C_0^\infty(\Omega, [0,1])$ satisfying

$$\varphi(x) = \begin{cases} 1, & x \in \mathbb{B}_\delta, \\ 0, & x \in \mathbb{B}_{2\delta}^c \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 0, & x \in \mathbb{B}_{2\delta}, \\ 1, & x \in \mathbb{B}_{4\delta} \backslash \mathbb{B}_{3\delta}, \\ 0, & x \in \mathbb{B}_{5\delta}^c. \end{cases}$$

Define $u_\varepsilon = \varphi U_\varepsilon$ and $v_\varepsilon = \psi U_\varepsilon$. As in [3,4], through direct computation, we obtain, as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{N-2}), \tag{2.13}$$

$$\int_\Omega |u_\varepsilon|^2 dx = \begin{cases} O(\varepsilon), & N = 3, \\ O(\varepsilon^2 |\ln \varepsilon|), & N = 4, \\ O(\varepsilon^2), & N \geq 5 \end{cases} \tag{2.14}$$

and

$$\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2_\alpha^*} |u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dxdy = A_\alpha^{-1} S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{\frac{N+\alpha}{2}}). \tag{2.15}$$

Additionally, as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla v_\varepsilon|^2 + v_\varepsilon^2 dx = O(\varepsilon^{N-2}) \quad \text{and} \quad \int_\Omega |v_\varepsilon(x)|^p dx \geq d_p \varepsilon^{\frac{(N-2)p}{2}} \text{ for some } d_p > 0. \tag{2.16}$$

Lemma 2.6. *There exists some $\mu_* > 0$ independent of λ such that, for any $\lambda > 0$ and $\mu \geq \mu_*$,*

$$m_{\lambda,\mu} \leq m_{\infty,\mu} < m_* := \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

Proof. Since $\mathcal{M}_{\infty,\mu} \subset \mathcal{M}_{\lambda,\mu}$ and $\mathcal{J}_{\lambda,\mu} = \mathcal{J}_{\infty,\mu}$ on $\mathcal{M}_{\infty,\mu}$, we easily derive $m_{\lambda,\mu} \leq m_{\infty,\mu}$. For any $\varepsilon > 0$ and $\mu > 0$, by Remark 2.2, there exist some constants $s_{\mu,\varepsilon} > 0, t_{\mu,\varepsilon} > 0$ such that $s_{\mu,\varepsilon} u_\varepsilon - t_{\mu,\varepsilon} v_\varepsilon \in \mathcal{M}_{\infty,\mu}$ and $\mathcal{J}_{\infty,\mu}(s_{\mu,\varepsilon} u_\varepsilon - t_{\mu,\varepsilon} v_\varepsilon) = \max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon)$. It suffices to show $\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) < m_*$ for $\varepsilon > 0$ small enough. Noting $\operatorname{spt} u_\varepsilon \cap \operatorname{spt} v_\varepsilon = \emptyset$, we deduce

$$\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) \leq \max_{s>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon) + \max_{t>0} \mathcal{J}_{\infty,\mu}(tv_\varepsilon). \tag{2.17}$$

It easily follows from (2.13)−(2.15) that, for $\varepsilon > 0$ sufficiently small and all $\mu > 0$, $s > 0$,

$$\mathcal{J}_{\infty,\mu}(su_\varepsilon) \leq S_\alpha^{\frac{N+\alpha}{2+\alpha}} \left(s^2 - \frac{1}{4 \cdot 2_\alpha^*} s^{2 \cdot 2_\alpha^*} \right).$$

In view of this, there exist some sufficiently small $s_1 > 0$ and sufficiently large $s_2 > 0$ independent of ε, μ such that, for $\varepsilon > 0$ small enough and all $\mu > 0$,

$$\max_{s \in (0,s_1)} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < m_* \quad \text{and} \quad \max_{s \in (s_2,+\infty)} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < 0.$$

Moreover, from (2.13)−(2.15) again we conclude, for $\varepsilon > 0$ sufficiently small and any $\mu > 0$,

$$\begin{aligned}\max_{s\in[s_1,s_2]} \mathcal{J}_{\infty,\mu}(su_\varepsilon) &\le \max_{s>0}\left(\frac{s^2}{2}\int_\Omega |\nabla u_\varepsilon|^2dx - \frac{s^{2\cdot 2_\alpha^*}A_\alpha}{2\cdot 2_\alpha^*}\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2_\alpha^*}|u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy\right)\\ &\quad + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p}{p}\int_\Omega |u_\varepsilon|^p dx\\ &\le \frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}}\left[1+O(\varepsilon^{N-2})\right]\left[1-O(\varepsilon^{\frac{N+\alpha}{2}})\right]\\ &\quad + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1}|U_1|^p dx\\ &= \frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{N-2}) + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1}|U_1|^p dx.\end{aligned}$$

If $N \ge 4$, or $N = 3$ and $\alpha \in (1,3)$, by (2.14) and $p \ge 2_\alpha^*$ we deduce, for $\varepsilon > 0$ small enough and $\mu > 0$,

$$\eta_N(\varepsilon) := O(\varepsilon^{N-2}) + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1}|U_1|^p dx < 0.$$

If $N = 3$ and $\alpha \in (0,1]$, take $\mu = \varepsilon^{\frac{\alpha-3}{2}}$, by (2.14), there exists small $\varepsilon_1 > 0$ such that $\eta_3(\varepsilon) < 0$ for all $\varepsilon \in (0,\varepsilon_1]$. Based on the above discussion, for $\varepsilon > 0$ small enough and any $\mu \ge \varepsilon_1^{\frac{2}{\alpha-3}}$ if $N = 3$ and $\alpha \in (0,1)$, also, for $\varepsilon > 0$ small enough and any $\mu > 0$ if $N \ge 4$ or $N = 3$ and $\alpha \in (1,3)$, we conclude

$$\max_{s>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < m_*. \tag{2.18}$$

In addition, due to (2.16), there exists some $C_1 > 0$ such that, for $\varepsilon > 0$ small enough and any $\mu > 0$,

$$\max_{t>0} \mathcal{J}_{\infty,\mu}(tv_\varepsilon) \le \max_{t>0}\left[C_1\varepsilon^{N-2}t^2 - \mu d_p\left(\varepsilon^{N-2}t^2\right)^{\frac{p}{2}}\right] \le \frac{(p-2)(2C_1)^{\frac{p}{p-2}}}{2p(\mu p d_p)^{\frac{2}{p-2}}}. \tag{2.19}$$

Now, by combining (2.17), (2.18) and (2.19), there exists some large $\mu_* \in \left[\frac{1}{\varepsilon_1}, +\infty\right)$ such that $\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) < m_*$ for any $\mu \ge \mu_*$ and small $\varepsilon > 0$. Thus this lemma is proved. □

In the forthcoming lemma, we show that $\mathcal{J}_{\lambda,\mu}$ satisfies the local $(PS)_c$ condition for λ large.

Lemma 2.7. *There exists some $\Lambda > 0$ independent of μ such that, for any $\lambda \ge \Lambda$ and $\mu \ge \mu_*$, each $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ for $\mathcal{J}_{\lambda,\mu}$, with level $c \in (0, m_*)$, has a convergent subsequence.*

Proof. From the definition of $\{u_n\}$, there results

$$m_* + o(1) + o(\|u_n\|_\lambda) \ge \mathcal{J}_{\lambda,\mu}(u_n) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n\right\rangle \ge \frac{p-2}{2p}\|u_n\|_\lambda^2.$$

Then there exists some $C_2 > 0$ independent of λ and μ such that $\limsup_n \|u_n\|_\lambda \le C_2$. Naturally, $\{u_n\}$ is bounded in E_λ. Hence, there exists some $u \in E_\lambda$ such that, up to subsequences,

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_\lambda,\\ u_n \to u & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall\ s \in [1, 2^*), \text{ as } n \to \infty.\\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^N,\end{cases} \tag{2.20}$$

Set $v_n = u_n - u$. Clearly, $\limsup_n \|v_n\|_\lambda \leq 2C_2$. We will show $\|v_n\|_\lambda \xrightarrow{n} 0$ up to a subsequence. Define

$$\beta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{B}_1(y)} v_n^2 dx.$$

We assert $\beta = 0$. Otherwise, $\beta > 0$. Due to (V_5), there exists some large $R > 0$ such that

$$\left|\{x \in \mathbb{B}_R^c(0) : V(x) \leq M\}\right| \leq \left(\frac{\beta S}{16C_2^2}\right)^{\frac{N}{2}}.$$

Then it follows from the Hölder and Sobolev inequalities that

$$\limsup_{n\to\infty} \int_{\{x\in\mathbb{B}_R^c(0):V(x)\leq M\}} v_n^2 dx \leq \left|\{x \in \mathbb{B}_R^c(0) : V(x) \leq M\}\right|^{\frac{2}{N}} S^{-1} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \leq \frac{\beta}{4}. \quad (2.21)$$

Moreover, if taking $\Lambda = \frac{1}{M}\left(16C_2^2\beta^{-1} - 1\right)$ and letting $\lambda \geq \Lambda$, we have

$$\limsup_{n\to\infty} \int_{\{x\in\mathbb{B}_R^c(0):V(x)>M\}} v_n^2 dx \leq \frac{1}{\lambda M + 1} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \leq \frac{\beta}{4}. \quad (2.22)$$

Consequently, combining (2.20)−(2.22) leads to

$$\beta \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} v_n^2 dx = \limsup_{n\to\infty} \int_{\mathbb{B}_R^c(0)} v_n^2 dx \leq \frac{\beta}{2},$$

which contradicts $\beta > 0$. That is, our claim $\beta = 0$ is true. Then, thanks to [29, Lemma 1.21],

$$v_n \to 0 \quad \text{in } L^s(\mathbb{R}^N), \qquad \forall\, s \in (2, 2^*). \quad (2.23)$$

By (2.20), it is easy to show $\mathcal{J}'_{\lambda,\mu}(u) = 0$. Further, with $\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n\rangle = o(1)$ in hand, we deduce from (2.20), (2.23) and the nonlocal version of the Brézis–Lieb lemma (see e.g. [4, Lemma 2.2]) that

$$o(1) = \|v_n\|_\lambda^2 - \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{2_\alpha^*}\right)|v_n|^{2_\alpha^*} dx. \quad (2.24)$$

Set $\kappa = \limsup_{n\to\infty} \|v_n\|_\lambda$. Due to (2.24) and the definition of S_α, there results $\kappa = 0$ or $\kappa \geq S_\alpha^{\frac{N+\alpha}{2(2+\alpha)}}$. We claim $\kappa = 0$. If not, because $\mathcal{J}_{\lambda,\mu}(u) \geq 0$, it follows from (2.20), (2.24) and Lemma 2.2 in [4] that

$$c = \lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(u_n) = \mathcal{J}_{\lambda,\mu}(u) + \frac{2+\alpha}{2(N+\alpha)} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \geq \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}},$$

which contradicts $c < m_*$. Thus $u_n \to u$ in E_λ up to a subsequence. This lemma is proved. □

Based on the above preliminary lemmas, we shall complete the proof of main results below.

Proof of Theorem 1.2. Let $\lambda \geq \Lambda$ and $\mu \geq \mu_*$. Thanks to Lemmas 2.5 and 2.6, $\mathcal{J}_{\lambda,\mu}$ has a sign-changing $(PS)_{m_{\lambda,\mu}}$ sequence $\{u_n\} \subset E_\lambda$, with $m_{\lambda,\mu} < m_*$. From Lemma 2.7, we derive that $u_n \to u_{\lambda,\mu}$ in E_λ in the sense of subsequence. Then, there result $\mathcal{J}'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$ in E_λ^* and $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) = m_{\lambda,\mu}$. Further, Lemma 2.3 implies $u_{\lambda,\mu}^\pm \neq 0$. That is, Eq. (1.6) has a ground state sign-changing solution $u_{\lambda,\mu}$.

Next, we show the concentration of ground state sign-changing solutions for Eq. (1.6) as $\lambda \to +\infty$. Given $\mu \geq \mu_*$ arbitrarily. For sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ with $\lambda_n \to +\infty$, let $u_{\lambda_n,\mu} \in E_{\lambda_n}$ be such that

$$u^{\pm}_{\lambda_n,\mu} \neq 0, \qquad \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}) = 0 \quad \text{in } E^*_{\lambda_n}, \qquad \mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) = m_{\lambda_n,\mu}.$$

By Lemma 2.6, it is easy to obtain

$$m_* > \mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle > \frac{p-2}{2p}\|u_{\lambda_n,\mu}\|^2_{\lambda_n}. \tag{2.25}$$

Obviously, $\{u_{\lambda_n,\mu}\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, there exists some $u_\mu \in H^1(\mathbb{R}^N)$ such that, up to subsequences,

$$\begin{cases} u_{\lambda_n,\mu} \overset{n}{\rightharpoonup} u_\mu & \text{in } H^1(\mathbb{R}^N), \\ u_{\lambda_n,\mu} \overset{n}{\to} u_\mu & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall\, s \in [1, 2^*), \\ u_{\lambda_n,\mu}(x) \overset{n}{\to} u_\mu(x) & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{2.26}$$

It follows from the Fatou lemma, (2.25) and (2.26) that

$$0 \leq \int_{\Omega^c} V(x)u_\mu^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x)u^2_{\lambda_n,\mu} dx \leq \liminf_{n\to\infty} \frac{\|u_{\lambda_n,\mu}\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which together with (V_6) implies $u_\mu|_{\Omega^c} = 0$. Then, $u_\mu \in H^1_0(\Omega)$, since $\partial\Omega$ is smooth. Thereby, for any $\omega \in H^1_0(\Omega)$, we derive from $\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), \omega\rangle = 0$ and (2.26) that $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$.

Set $v_{\mu,n} = u_{\lambda_n,\mu} - u_\mu$. For any $\varepsilon > 0$, by (V_5), there exists some large $R_\varepsilon > 0$ such that

$$\left|\left\{x \in \mathbb{B}^c_{R_\varepsilon} : V(x) \leq M\right\}\right| < \left[\frac{(p-2)S\varepsilon}{4pm_*}\right]^{\frac{N}{2}}.$$

Then, due to the Hölder and Sobolev inequalities, the weakly lower semicontinuity of norm and (2.25), there holds

$$\int_{\left\{x\in\mathbb{B}^c_{R_\varepsilon}:V(x)\leq M\right\}} v^2_{\mu,n} dx \leq \left|\left\{x \in \mathbb{B}^c_{R_\varepsilon} : V(x) \leq M\right\}\right|^{\frac{2}{N}} S^{-1}\|v_{n,\mu}\|^2_{\lambda_n} < \varepsilon.$$

From the weakly lower semicontinuity of norm and (2.25), it follows that

$$\int_{\left\{x\in\mathbb{B}^c_{R_\varepsilon}:V(x)\geq M\right\}} v^2_{\mu,n} dx \leq \frac{\|v_{n,\mu}\|^2_{\lambda_n}}{\lambda_n M} \leq \frac{4pm_*}{(p-2)M\lambda_n} \to 0 \quad \text{as } n \to \infty.$$

Thereby, we deduce from (2.26) that $|v_{\mu,n}|_2 \overset{n}{\to} 0$. Further, by (2.25), the Hölder and Sobolev inequalities, there holds

$$\begin{aligned} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_{\mu,n}|^p dx &\leq \limsup_{n\to\infty} \left(|v_{\mu,n}|_2^{\frac{2(2^*-p)}{2^*-2}} |v_{\mu,n}|_{2^*}^{\frac{2^*(p-2)}{2^*-2}} \right) \\ &\leq \left[\frac{4pm_*}{(p-2)S}\right]^{\frac{2^*(p-2)}{2(2^*-2)}} \limsup_{n\to\infty} |v_{\mu,n}|_2^{\frac{2(2^*-p)}{2^*-2}} = 0. \end{aligned} \tag{2.27}$$

By (2.26), (2.27), the nonlocal type of the Brézis–Lieb Lemma 2.2 in [4] and $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$, we have

$$0 = \left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle = \|v_{\mu,n}\|^2_{\lambda_n} - \int_{\mathbb{R}^N} \left(I_\alpha * |v_{\mu,n}|^{2^*_\alpha}\right)|v_{\mu,n}|^{2^*_\alpha} dx + o(1). \tag{2.28}$$

Denote $\kappa_\mu = \limsup_{n\to\infty}\|v_{\mu,n}\|_{\lambda_n}$. It follows from (2.28) and the definition of S_α that $\kappa_\mu^2 \le S_\alpha^{-2_\alpha^*}\kappa_\mu^{2\cdot 2_\alpha^*}$. Then, by (2.25), there results $\kappa_\mu = 0$ or $\kappa_\mu \ge S_\alpha^{\frac{N+\alpha}{2(2+\alpha)}}$. We assert $\kappa_\mu = 0$. If not, from Lemma 2.6, (2.25)−(2.28), the nonlocal type of the Brézis–Lieb lemma and $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$, we have

$$\begin{aligned}
m_* &> \lim_{n\to\infty}\mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu})\\
&= \mathcal{J}_{\infty,\mu}(u_\mu) + \frac{2+\alpha}{2(N+\alpha)}\limsup_{n\to\infty}\|v_{\mu,n}\|^2_{\lambda_n}\\
&= \mathcal{J}_{\infty,\mu}(u_\mu) - \frac{1}{p}\left\langle \mathcal{J}'_{\infty,\mu}(u_\mu), u_\mu\right\rangle + \frac{2+\alpha}{2(N+\alpha)}k_\mu^2\\
&\ge m_*,
\end{aligned}$$

a contradiction. Hence, $\|u_{\lambda_n,\mu} - u_\mu\|_{\lambda_n} \xrightarrow{n} 0$. Then, it is easy to show $u_{\lambda_n,\mu} \to u_\mu$ in $H^1(\mathbb{R}^N)$.

From $\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u^\pm_{\lambda_n,\mu}\rangle = 0$, (1.4), the Young and Sobolev inequalities, we deduce that

$$\begin{aligned}
S|u^\pm_{\lambda_n,\mu}|^2_{2^*} \le \|u^\pm_{\lambda_n,\mu}\|^2_{\lambda_n} &= \int_{\mathbb{R}^N}\left(I_\alpha * |u_{\lambda_n,\mu}|^{2_\alpha^*}\right)|u^\pm_{\lambda_n,\mu}|^{2_\alpha^*}dx + \mu|u^\pm_{\lambda_n,\mu}|_p^p\\
&\le A_\alpha C(N,\alpha)|u_{\lambda_n,\mu}|^{2_\alpha^*}_{2^*}|u^\pm_{\lambda_n,\mu}|^{2_\alpha^*}_{2^*} + \frac{2^*-p}{2^*-2}\|u^\pm_{\lambda_n,\mu}\|^2_{\lambda_n} + \frac{p-2}{2^*-2}\mu^{\frac{2^*-2}{p-2}}|u^\pm_{\lambda_n,\mu}|^{2^*}_{2^*},
\end{aligned}$$

which together with (2.25) implies

$$S|u^\pm_{\lambda_n,\mu}|^2_{2^*} \le \frac{A_\alpha C(N,\alpha)(2^*-2)}{p-2}\left[\frac{2pm_*}{S(p-2)}\right]^{\frac{2_\alpha^*}{2}}|u^\pm_{\lambda_n,\mu}|^{2_\alpha^*}_{2^*} + \mu^{\frac{2^*-2}{p-2}}|u^\pm_{\lambda_n,\mu}|^{2^*}_{2^*}.$$

In view of this, there holds $\inf_n |u^\pm_{\lambda_n,\mu}|_{2^*} > 0$. Thereby, $\|u_{\lambda_n,\mu} - u_\mu\| \xrightarrow{n} 0$ implies $|u^\pm_\mu|_{2^*} > 0$. Naturally, $u^\pm_\mu \ne 0$ and then $u_\mu \in \mathcal{M}_{\infty,\mu}$. Thus we derive from (2.26), the Fatou lemma and Lemma 2.6 that

$$\begin{aligned}
m_{\infty,\mu} &\le \mathcal{J}_{\infty,\mu}(u_\mu) - \frac{1}{p}\left\langle \mathcal{J}'_{\infty,\mu}(u_\mu), u_\mu\right\rangle\\
&= \frac{p-2}{2p}\int_\Omega\left(|\nabla u_\mu|^2 + u_\mu^2\right)dx + \frac{(2\cdot 2_\alpha^* - p)A_\alpha}{2p\cdot 2_\alpha^*}\int_\Omega\int_\Omega\frac{|u_\mu(x)|^{2_\alpha^*}|u_\mu(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy\\
&\le \lim_{n\to\infty}\left[\frac{p-2}{2p}\|u_{\lambda_n,\mu}\|^2_{\lambda_n} + \frac{2\cdot 2_\alpha^* - p}{2p\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u_{\lambda_n,\mu}|^{2_\alpha^*}\right)|u_{\lambda_n,\mu}|^{2_\alpha^*}dx\right]\\
&= \lim_{n\to\infty}\left[\mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle\right]\\
&\le m_{\infty,\mu},
\end{aligned}$$

which leads to $\mathcal{J}_{\infty,\mu}(u_\mu) = m_{\infty,\mu}$. Therefore, u_μ is a ground state sign-changing solution for Eq. (1.8).

Further, we certify the asymptotic behavior of ground state sign-changing solutions for Eq. (1.6) as $\mu \to +\infty$. Fix $\lambda \ge \Lambda$. For any sequence $\{\mu_n\} \subset [\mu_*, +\infty)$ with $\mu_n \to +\infty$, let $\{u_{\lambda,\mu_n}\} \subset E_\lambda$ satisfy

$$u^\pm_{\lambda,\mu_n} \ne 0, \qquad \mathcal{J}'_{\lambda,\mu_n}(u_{\lambda,\mu_n}) = 0 \quad \text{in } E^*_\lambda, \qquad \mathcal{J}_{\lambda,\mu_n}(u_{\lambda,\mu_n}) = m_{\lambda,\mu_n}.$$

It easily follows that

$$m_{\lambda,\mu_n} = \mathcal{J}_{\lambda,\mu_n}(u_{\lambda,\mu_n}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu_n}(u_{\lambda,\mu_n}), u_{\lambda,\mu_n}\right\rangle \geq \frac{p-2}{2p}\|u_{\lambda,\mu_n}\|_\lambda^2. \tag{2.29}$$

We assert that $\lim_{n\to\infty} m_{\lambda,\mu_n} \to 0$ in the sense of subsequence. Take $\omega \in H_0^1(\Omega)$ such that $\omega^\pm \neq 0$. Due to Remark 2.2, there exist $s_n > 0$ and $t_n > 0$ such that $s_n\omega^+ + t_n\omega^- \in \mathcal{M}_{\infty,\mu_n}$. Then we have

$$\begin{aligned} s_n^2 \int_\Omega |\nabla\omega^+|^2 + |\omega^+|^2 dx &= A_\alpha s_n^{2\cdot 2_\alpha^*} \int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2_\alpha^*}|\omega^+(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy \\ &\quad + A_\alpha(s_nt_n)^{2_\alpha^*}\int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2_\alpha^*}|\omega^-(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy + \mu_n s_n^p\int_\Omega |\omega^+|^p dx, \end{aligned} \tag{2.30}$$

$$\begin{aligned} t_n^2 \int_\Omega |\nabla\omega^-|^2 + |\omega^-|^2 dx &= A_\alpha t_n^{2\cdot 2_\alpha^*} \int_\Omega\int_\Omega \frac{|\omega^-(x)|^{2_\alpha^*}|\omega^-(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy \\ &\quad + A_\alpha(t_ns_n)^{2_\alpha^*}\int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2_\alpha^*}|\omega^-(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy + \mu_n t_n^p\int_\Omega |\omega^-|^p dx. \end{aligned} \tag{2.31}$$

From (2.30) and (2.31), we easily deduce that both $\{s_n\}$ and $\{t_n\}$ are bounded. Thereby, $s_n \to s_0$ and $t_n \to t_0$ up to subsequences. By using (2.30) and (2.31) again, we derive $s_0 = t_0 = 0$. Consequently, Lemmas 2.3 and 2.6 imply

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} m_{\lambda,\mu_n} \leq \limsup_{n\to\infty} m_{\infty,\mu_n} \leq \limsup_{n\to\infty} \mathcal{J}_{\infty,\mu_n}(s_n\omega^+ + t_n\omega^-) \\ &\leq \limsup_{n\to\infty}\left(s_n^2\int_\Omega |\nabla\omega^+|^2 + |\omega^+|^2dx + t_n^2\int_\Omega |\nabla\omega^-|^2 + |\omega^-|^2 dx\right) = 0. \end{aligned}$$

Now, from (2.29) we conclude $u_{\lambda,\mu_n} \xrightarrow{n} 0$ in E_λ. Naturally $u_{\lambda,\mu_n} \xrightarrow{n} 0$ in $H^1(\mathbb{R}^N)$ in the sense of subsequence. Thus, based on the above arguments, we complete the proof of Theorem 1.2. □

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