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$C^{1,\gamma}$ regularity for fully nonlinear elliptic equations on a convex polyhedron

Duan Wu and **Pengcheng Niu** ✉

School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, 710129, China

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Abstract. In this note, we prove the boundary and global $C^{1,\gamma}$ regularity for viscosity solutions of fully nonlinear uniformly elliptic equations on a convex polyhedron by perturbation and iteration techniques.

Keywords: $C^{1,\gamma}$ regularity, fully nonlinear elliptic equation, viscosity solution.

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1 Introduction

The purpose of this note is to investigate the $C^{1,\gamma}$ regularity up to the boundary for viscosity solutions of the following fully nonlinear elliptic equation

$$\begin{cases} F(D^2u, x) = f & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is a convex polyhedron, F is assumed to be uniformly elliptic (see (1.4)).

With respect to the boundary regularity of solutions of linear elliptic equations, Li and Wang [7,8] proved the boundary differentiability on convex domains and demonstrated that only under the assumption that Ω is convex, no continuity of the gradient of solutions along the boundary can be expected (see the counterexamples in [7]). On the other hand, Lian and Zhang [11, Theorem 1.6] proved the boundary $C^{1,\alpha}$ regularity of solutions of fully nonlinear elliptic equations under the assumption that the boundary $\partial\Omega$ is $C^{1,\alpha}$. In this note, we show the $C^{1,\gamma}$ regularity for fully nonlinear elliptic equations (linear elliptic equations as a special case) by strengthening convex domain into convex polyhedron. And we do not need such high smoothness condition on the boundary as in [11].

Before stating our main results, we give several definitions.

✉ Corresponding author. Email: pengchengniu@nwpu.edu.cn

Definition 1.1. Let $\Omega \subset R^n$ be a bounded set and f be a function defined on Ω. We say that f is $C^{1,\alpha}$ at $x_0 \in \Omega$ denoted by $f \in C^{1,\alpha}(x_0)$ if there exist a linear polynomial L, constants C and $r_0 > 0$ such that

$$|f(x) - L(x)| \leq C|x - x_0|^{1+\alpha}, \quad \forall\, x \in \Omega \cap B_{r_0}(x_0). \tag{1.2}$$

Note that there may exist many L and C (e.g. $\Omega = B_1 \cap R^{n-1}$). We take L_0 with

$$\|L_0\| = \min\left\{\|L\| \mid \exists C \text{ such that (1.2) holds with } L \text{ and } C\right\},$$

where $\|L\| = |L(x_0)| + |DL(x_0)|$. Define

$$Df(x_0) = DL_0(x_0),$$

$$\|f\|_{C^1(x_0)} = \|L_0\|,$$

$$[f]_{C^{1,\alpha}(x_0)} = \min\left\{C \mid \text{(1.2) holds with } L_0 \text{ and } C\right\}$$

and

$$\|f\|_{C^{1,\alpha}(x_0)} = \|f\|_{C^1(x_0)} + [f]_{C^{1,\alpha}(x_0)}.$$

If $f \in C^{1,\alpha}(x)$ for any $x \in \Omega$ with the same r_0 and

$$\|f\|_{C^{1,\alpha}(\bar{\Omega})} := \sup_{x\in\Omega} \|f\|_{C^1(x)} + \sup_{x\in\Omega} [f]_{C^{1,\alpha}(x)} < +\infty,$$

we say $f \in C^{1,\alpha}(\bar{\Omega})$.

Definition 1.2. Let Ω and f be as in Definition 1.1. We call that f is $C^{-1,\alpha}$ at $x_0 \in \Omega$ denoted by $f \in C^{-1,\alpha}(x_0)$ if there exist constants C and $r_0 > 0$ such that

$$\|f\|_{L^n(\bar{\Omega}\cap B_r(x_0))} \leq Cr^\alpha, \quad \forall\, 0 < r < r_0, \tag{1.3}$$

and denote

$$\|f\|_{C^{-1,\alpha}(x_0)} = \min\left\{C \mid \text{(1.3) holds with } C\right\}.$$

If $f \in C^{-1,\alpha}(x)$ for any $x \in \Omega$ with the same r_0 and

$$\|f\|_{C^{-1,\alpha}(\bar{\Omega})} := \sup_{x\in\Omega} \|f\|_{C^{-1,\alpha}(x)} < +\infty,$$

we say $f \in C^{-1,\alpha}(\bar{\Omega})$.

Remark 1.3. Without loss of generality, we can assume $r_0 = 1$ throughout this paper.

Remark 1.4. If Ω is a Lipschitz domain, the definition of $C^{1,\alpha}(\bar{\Omega})$ in Definition 1.1 is equivalent to the usual classical definition of $C^{1,\alpha}(\bar{\Omega})$ (see [9]).

Definition 1.5 ([13])**.** A bounded set Ω is called a convex polyhedron if it is the intersection of a finite number of closed half-spaces.

For an n-dimensional convex polyhedron Ω, let F_k $(k = 0, 1, \ldots, n-1)$ be its k-dimensional faces. Specially, 0-dimensional faces are vertices and 1-dimensional faces are edges. Then we classify the boundary points of Ω into two categories. For any $x_0 \in \partial\Omega$, if $x_0 \in F_{n-1}$, we call it the first class boundary point and denote $x_0 \in S_1$. If $x_0 \notin F_{n-1}$, we call it the second class boundary point and denote $x_0 \in S_2$.

We call that $F : S^n \times \Omega \to R$ is a fully nonlinear uniformly elliptic operator with ellipticity constants $0 < \lambda \le \Lambda$ if

$$\lambda\|N\| \le F(M+N,x) - F(M,x) \le \Lambda\|N\|, \qquad \forall\, M,N \in S^n,\ N \ge 0, \tag{1.4}$$

where S^n denotes the set of $n \times n$ symmetric matrices; $\|N\|$ is the spectral radius of N and $N \ge 0$ means the nonnegativeness. The standard notions and notations such as Pucci operators $M^+(M,\lambda,\Lambda)$, $M^-(M,\lambda,\Lambda)$ and Pucci class $\bar{S}(\lambda,\Lambda,f)$, $\underline{S}(\lambda,\Lambda,f)$, $S^*(\lambda,\Lambda,f)$ will be used. For the details, one can refer to [1–3].

Now we state our main results.

Theorem 1.6 (boundary $C^{1,\gamma}$ regularity). *Let $0 < \alpha < \alpha_1$ where α_1 is a universal constant (see Lemma 2.1). Suppose that Ω is a convex polyhedron, $x_0 \in \partial\Omega$ and u is a viscosity solution of*

$$\begin{cases} u \in S^*(\lambda,\Lambda,f) & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f \in C^{-1,\alpha}(x_0)$ and $g \in C^{1,\alpha}(x_0)$. Then u is $C^{1,\gamma}$ at x_0, i.e., for any $x_0 \in \partial\Omega$, there exists a linear polynomial L_{x_0} such that

$$|u(x) - L_{x_0}(x)| \le C|x-x_0|^{1+\gamma}\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}\right), |\ \forall\, x \in \bar{\Omega} \tag{1.6}$$

and

$$|Du(x_0)| \le C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}\right), \tag{1.7}$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

Remark 1.7. The viscosity solutions of (1.1) are in the classes $S^*(\lambda,\Lambda,f)$ (see [1, Proposition 2.13]). So all results for functions in the classes $S^*(\lambda,\Lambda,f)$ are valid for solutions of (1.1).

Combining the interior $C^{1,\gamma}$ estimate [1, Theorem 8.3], we have

Theorem 1.8 (global $C^{1,\gamma}$ regularity). *Let α and Ω be as in Theorem 1.6. Suppose that u is a viscosity solution of (1.1) with $f \in C^{-1,\alpha}(\bar{\Omega})$ and $g \in C^{1,\alpha}(\partial\Omega)$. Then there exists $\theta > 0$ depending only on n, λ, Λ and α such that if*

$$\beta_F(x) = \sup_{M \in S\setminus\{0\}} \frac{|F(M,x) - F(M,0)|}{\|M\|} \le \theta, \quad \forall\, x \in \Omega,$$

then $u \in C^{1,\gamma}(\bar{\Omega})$ and

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \le C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(\bar{\Omega})} + \|g\|_{C^{1,\alpha}(\partial\Omega)}\right),$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

The following corollary of Theorem 1.8 is a new result for linear elliptic equations.

Corollary 1.9. *Let u be a viscosity solution of*

$$\begin{cases} -a^{ij}(x)\dfrac{\partial^2 u(x)}{\partial x_i x_j} = f & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where α, Ω, f *and* g *are as in Theorem 1.8. Then there exists* $\theta > 0$ *depending only on* n, λ, Λ *and* α *such that if*

$$\|a^{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \theta,$$

then $u \in C^{1,\gamma}(\bar{\Omega})$ *and*

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq C \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(\bar{\Omega})} + \|g\|_{C^{1,\alpha}(\partial\Omega)} \right), \tag{1.8}$$

where $0 < \gamma < \alpha$ *and* C *depend only on* n, λ, Λ, α *and* Ω.

Remark 1.10. The $C^{1,\gamma}$ estimate (1.8) is also called Cordes–Nirenberg estimate.

Remark 1.11. In this paper, C depending on n, λ, Λ, α and Ω will denote constants which may differ at different occurrences.

The main route of proving Theorem 1.6 is the following. For $x_0 \in S_1$, the $C^{1,\gamma}$ regularity can be obtained as a simple corollary of [11]. For $x_0 \in S_2$, there exist a half ball $B_r(x_0)$ and a cone K such that $\Omega \subset B_r(x_0)$ and $K \subset B_r(x_0) \cap \Omega^c$. This will lead to a higher regularity of u. In addition, if $f, g \equiv 0$, the solutions of (1.5) on the half ball have sufficient regularity (see Lemma 2.1). Noting that cone has the scaling invariance, the boundary $C^{1,\gamma}$ regularity for $x_0 \in S_2$ can be derived by perturbation and iteration techniques which are inspired by [10]. Then the boundary $C^{1,\gamma}$ regularity can be obtained by the technique of patching. Finally, the global $C^{1,\gamma}$ regularity will be deduced by combining the interior $C^{1,\gamma}$ estimate.

In Section 2, we will prove an important estimate (about the $C^{1,\gamma}$ regularity for $x_0 \in S_2$). Theorem 1.6 and Theorem 1.8 will be proved in Section 3. In this note, we use the following notations.

Notation

1. $R^n_+ = \{x \in R^n \mid x_n > 0\}$.
2. $B_r(x_0) = \{x \in R^n \mid ||x - x_0| < r\}$, $B_r = B_r(0)$, $B^+_r(x_0) = B_r(x_0) \cap R^n_+$ and $B^+_r = B^+_r(0)$.
3. $T_r(x_0) = \{(x', 0) \in R^n \mid |x' - x'_0| < r\}$ and $T_r = T_r(0)$.
4. Ω^c: the complement of Ω and $\bar{\Omega}$: the closure of Ω, $\forall \Omega \subset R^n$.
5. $\Omega_r = \Omega \cap B_r$ and $(\partial\Omega)_r = \partial\Omega \cap B_r$.

2 An important estimate

In this section, we introduce some known lemmas. The first concerns the boundary $C^{1,\alpha}$ regularity for solutions with flat boundaries. It was first proved by Krylov [6] for classical solutions and further simplified by Caffarelli (see [4, Theorem 9.31] and [5, Theorem 4.28]), which is applicable to viscosity solutions (see [12]).

Lemma 2.1. *Let* u *be a viscosity solution of*

$$\begin{cases} u \in S(\lambda, \Lambda, 0) & \text{in } B^+_1; \\ u = 0 & \text{on } T_1. \end{cases}$$

Then u is C^{1,α_1} at 0, i.e., there exists a constant a such that

$$|u(x) - ax_n| \leq C_1|x|^{1+\alpha_1}\|u\|_{L^\infty(B_1^+)}, \qquad \forall\, x \in B_{1/2}^+$$

and

$$|a| \leq C_1\|u\|_{L^\infty(B_1^+)},$$

where α_1 and C_1 depend only on n, λ and Λ.

The next Lemma presents the boundary $C^{1,\alpha}$ estimate for solutions of fully nonlinear elliptic equations with the suitable right hand function f and the boundary value g on the curved boundary (see [11, Theorem 1.6]).

Lemma 2.2. *Let $0 < \alpha_2 < \alpha_1$ where α_1 is a universal constant (see Lemma 2.1). Suppose that $\partial\Omega$ is C^{1,α_2} at 0 and u is a viscosity solution of*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & \text{in } \Omega \cap B_1; \\ u = g & \text{on } \partial\Omega \cap B_1, \end{cases}$$

where $f \in C^{-1,\alpha_2}(0)$ and $g \in C^{1,\alpha_2}(0)$. Then u is C^{1,α_2} at 0, i.e., there exists a linear polynomial $\tilde{L}_0$ such that

$$|u(x) - \tilde{L}_0(x)| \leq \tilde{C}|x|^{1+\alpha_2}\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha_2}(0)} + \|g\|_{C^{1,\alpha_2}(0)}\right), \qquad \forall\, x \in \Omega \cap B_{1/2}$$

and

$$|Du(0)| \leq \tilde{C}\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha_2}(0)} + \|g\|_{C^{1,\alpha_2}(0)}\right),$$

where $\tilde{C}$ depends only on n, λ, Λ, α_2 and Ω.

Remark 2.3. The $C^{1,\gamma}$ regularity for the viscosity solutions of (1.5) at $x_0 \in S_1$ is true as a special case of Lemma 2.2.

The next is a Hopf type lemma (see [14, Lemma 2.15]).

Lemma 2.4. *Let $\Gamma \subset \partial B_1^+ \backslash T_1$, and u be a viscosity solution of*

$$\begin{cases} M^-(D^2u, \lambda, \Lambda) = 0 & \text{in } B_1^+; \\ u = x_n & \text{on } \Gamma; \\ u = 0 & \text{on } \partial B_1^+ \backslash \Gamma. \end{cases}$$

Then

$$u(x) \geq c_1 x_n \quad \text{in } B_{1/2}^+,$$

where $c_1 > 0$ depends only on n, λ, Λ and Γ.

It has been known that if Ω occupies a smaller portion in a ball centered at 0 (e.g. $|\Omega \cap B_r|/|B_r|$ is smaller), the regularity of u is higher (roughly speaking). Inspired by this, we have the following result.

Theorem 2.5. *Let α and Ω be as in Theorem 1.6. Suppose that $x_0 \in S_2$ and u is a viscosity solution of (1.5) with $f \in C^{-1,\alpha}(x_0)$ and $g \in C^{1,\alpha}(x_0)$. Then u is $C^{1,\gamma}$ at x_0, i.e., for any $x_0 \in S_2$, there exists a linear polynomial $\bar{L}_{x_0}$ such that*

$$|u(x) - \bar{L}_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}\left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}\right), \qquad \forall\, x \in \bar{\Omega} \quad (2.1)$$

with

$$Du(x_0) = Dg(x_0), \quad (2.2)$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

Proof. For $x_0 \in S_2$, we can assume that $x_0 = 0$, $\Omega \subset R^n_+$ and there exists a cone $K \subset \Omega^c \cap R^n_+$ with 0 being the vertex (by translating and rotating the coordinate system). Further, we assume that $g(0) = 0$ and $Dg(0) = 0$. Otherwise, we can consider $v(x) = u(x) - g(0) - Dg(0)\cdot x$, then the regularity of u follows easily from v. Let $C_g = [g]_{C^{1,\alpha}(0)}$, then

$$|g(x)| \le C_g|x|^{1+\alpha}, \qquad \forall\, x \in (\partial\Omega)_1. \tag{2.3}$$

Let $M = \|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha}(0)} + \|g\|_{C^{1,\alpha}(0)}$. To prove Theorem 2.5, we only need to show that there exists a nonnegative sequence $\{a_k\}$ $(k \ge -1)$ with $a_0 = 0$ such that for all $k \ge 0$,

$$\sup_{\Omega_{\eta^k}}(u - a_k x_n) \le \hat{C}M\eta^{k(1+\alpha)}, \tag{2.4}$$

$$\inf_{\Omega_{\eta^k}}(u + a_k x_n) \ge -\hat{C}M\eta^{k(1+\alpha)} \tag{2.5}$$

and

$$a_k \le (1-c_1)a_{k-1} + \bar{C}\hat{C}M\eta^{(k-1)\alpha}, \tag{2.6}$$

where $\bar{C}$ depends only on n, λ and Λ; $0 < c_1 < 1$ depends only on n, λ, Λ and Ω; $\hat{C}$ and $0 < \eta < 1/4$ depend only on n, λ, Λ and α.

Now we show that (2.4)-(2.6) imply that u is $C^{1,\gamma}$ at 0. Indeed, from (2.6), we have

$$a_k \le \bar{C}\hat{C}M\sum_{i=0}^{k-1}(1-c_1)^{k-1-i}\eta^{i\alpha} \le \bar{C}\hat{C}M\eta^{(k-1)\gamma}\sum_{i=0}^{k-1}\eta^{i(\alpha-\gamma)} \le CM\eta^{k\gamma},$$

provided

$$1 - c_1 \le \eta^\gamma, \quad 0 < \gamma < \alpha.$$

For any $x \in \Omega_1$, there exists $k \ge 0$ such that $\eta^{k+1} \le |x| < \eta^k$. From (2.4), we have

$$u(x) \le \sup_{\Omega_{\eta^k}}(u - a_k x_n) + a_k x_n \le CM\eta^{k(1+\gamma)} \le CM|x|^{1+\gamma}.$$

Similarly, (2.5) and (2.6) imply

$$u(x) \ge -CM|x|^{1+\gamma}.$$

Therefore, u is $C^{1,\gamma}$ at 0 with $Du(0) = Dg(0)$.

We only give the proofs of (2.4) and (2.6); the proof of (2.5) is similar with (2.4) and we omit it. We prove (2.4) and (2.6) by induction. For $k = 0$, by setting $a_{-1} = 0$, they hold clearly. Supposing that they hold for k, we need to prove that they hold for $k+1$.

Let $r = \eta^k/2$ and v_1 solve

$$\begin{cases} M^+(D^2v_1, \lambda, \Lambda) = 0 & \text{in } B_r^+;\\ v_1 = 0 & \text{on } T_r;\\ v_1 = \hat{C}M\eta^{k(1+\alpha)} & \text{on } \partial B_r^+\backslash T_r. \end{cases}$$

By the boundary $C^{1,\alpha}$ estimate for v_1 (see Lemma 2.1) and the maximum principle, there exists $\bar{a} \ge 0$ such that

$$\begin{aligned} \|v_1 - \bar{a}x_n\|_{L^\infty(\Omega_{\eta^{k+1}})} &= \|v_1 - \bar{a}x_n\|_{L^\infty(\Omega_{2\eta r})}\\ &\le C_1\frac{|x|^{1+\alpha_1}}{r^{1+\alpha_1}}\|v_1\|_{L^\infty(B_r^+)}\\ &\le C_1\eta^{\alpha_1-\alpha}\cdot\hat{C}M\eta^{(k+1)(1+\alpha)} \end{aligned} \tag{2.7}$$

and

$$\bar{a} \leq C_1 \hat{C} M \eta^{k\alpha},$$

where α_1 and C_1 depend only on n, λ and Λ.

Let v_2 solve

$$\begin{cases} M^-(D^2 v_2, \lambda, \Lambda) = 0 & \text{in } B_r^+; \\ v_2 = a_k x_n & \text{on } \partial B_r^+ \cap K; \\ v_2 = 0 & \text{on } \partial B_r^+ \backslash K. \end{cases}$$

By Lemma 2.4, there exists $0 < c_1 < 1$ depending only on n, λ, Λ and K such that

$$v_2 \geq c_1 a_k x_n \quad \text{in } B_{2\eta r}^+. \tag{2.8}$$

In addition, by the comparison principle,

$$v_2 \leq a_k x_n \quad \text{in } B_r^+.$$

Letting $w = u - a_k x_n - v_1 + v_2$, it follows that (note that $v_1, v_2 \geq 0$)

$$\begin{cases} w \in \underline{S}(\lambda, \Lambda, -|f|) & \text{in } \Omega \cap B_r^+; \\ w \leq g & \text{on } \partial\Omega \cap B_r^+; \\ w \leq 0 & \text{on } \partial B_r^+ \cap \bar{\Omega}. \end{cases}$$

By the Alexandrov–Bakel'man–Pucci maximum principle, we have

$$\sup_{\Omega_{\eta^{k+1}}} w \leq \sup_{\Omega_r} w \leq C_g \eta^{k(1+\alpha)} + C_2 r \|f\|_{L^n(\Omega_r)} \leq \frac{1 + C_2}{\hat{C}\eta^{1+\alpha}} \cdot \hat{C} M \eta^{(k+1)(1+\alpha)}, \tag{2.9}$$

where C_2 depend only on n, λ and Λ.

Let $\bar{C} := C_1$. Take η small enough such that

$$C_1 \eta^{\alpha_1 - \alpha} \leq \frac{1}{2}.$$

Next, take $\hat{C}$ large enough such that

$$\frac{1 + C_2}{\hat{C}\eta^{1+\alpha}} \leq \frac{1}{2}.$$

Let $a_{k+1} = (1 - c_1) a_k + \bar{a}$. Then (2.6) holds for $k + 1$. Recalling (2.7), (2.8) and (2.9), we have

$$\begin{aligned} u - a_{k+1} x_n &= u - a_k x_n - v_1 + v_2 + v_1 - a x_n + c_1 a_k x_n - v_2 \\ &= w + v_1 - a x_n + c_1 a_k x_n - v_2 \\ &\leq w + v_1 - a x_n \\ &\leq \hat{C} M \eta^{(k+1)(1+\alpha)} \quad \text{in } \Omega_{\eta^{k+1}}. \end{aligned}$$

By induction, the proofs of (2.4) and (2.6) are completed. □

3 Proofs of the main results

Combining Theorem 2.5 and Lemma 2.2, we give the

Proof of Theorem 1.6. We only need to prove that for any $x_0 \in S_1$, there exists a linear polynomial L_{x_0} such that

$$|u(x) - L_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}. \tag{3.1}$$

In fact, for any $x_0 \in S_1$, there exists $y \in S_2$ such that $|y - x_0| = d_{x_0} = d(x_0, S_2)$. We know from Theorem 2.5 that there exist a linear polynomial $\bar{L}_y$ and a constant C such that

$$|u(x) - \bar{L}_y(x)| \leq C|x - y|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}. \tag{3.2}$$

Let $v(x) = u(x) - \bar{L}_y(x)$. There exists a constant $0 < \tau \leq 1$ (depending only on Ω) such that $\Omega \cap B_{\tau d_{x_0}}(x_0)$ is a half ball. That is, $\Omega \cap B_{\tau d_{x_0}}(x_0) = \{x \in R^n \big| \vec{n} \cdot (x - x_0) > 0\} \cap B_{\tau d_{x_0}}(x_0)$, where $\vec{n}$ is the unit inward normal of Ω at x_0. Applying Lemma 2.2 in $\Omega \cap B_{\tau d_{x_0}}(x_0)$ and recalling (3.2), there exists a linear polynomial

$$R_{x_0}(x) = R(x_0) + DR(x_0) \cdot (x - x_0)$$

such that

$$|R(x_0)| = |v(x_0)| \leq C|d_{x_0}|^{1+\gamma},$$
$$|DR(x_0)| \leq C|\tau d_{x_0}|^{\gamma} \leq C|d_{x_0}|^{\gamma}$$

and

$$\begin{aligned} |v(x) - R_{x_0}(x)| &\leq C\frac{|x - x_0|^{1+\gamma}}{|\tau d_{x_0}|^{1+\gamma}} \left(\|v\|_{L^\infty(\Omega \cap B_{\tau d_{x_0}}(x_0))} + |\tau d_{x_0}|^{1+\gamma} (\|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}) \right) \\ &\leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \Omega \cap B_{\tau d_{x_0}/2}(x_0). \end{aligned} \tag{3.3}$$

Define

$$L_{x_0}(x) = \bar{L}_y(x) + R_{x_0}(x).$$

If $|x - x_0| < \tau d_{x_0}/2$, by (3.3), we have

$$|u(x) - L_{x_0}(x)| = |v(x) - R_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}.$$

If $|x - x_0| \geq \tau d_{x_0}/2$, by (3.2), we have

$$\begin{aligned} |u(x) - L_{x_0}(x)| &\leq |u(x) - \bar{L}_y(x)| + |R_{x_0}(x)| \\ &\leq C|x - y|^{1+\gamma} + |R(x_0)| + |DR(x_0)||x - x_0| \\ &\leq C|x - x_0|^{1+\gamma}. \end{aligned}$$

Combining the two cases, we get (3.1). □

The proof of the global $C^{1,\gamma}$ regularity is ended by Theorem 1.6 and the interior $C^{1,\gamma}$ estimate. Now we give the details.

Proof of Theorem 1.8. For any $x_0 \in \Omega$, there exists $y \in \partial\Omega$ such that $|y - x_0| = d_{x_0} = d(x_0, \partial\Omega)$. Then from Theorem 1.6 and Remark 1.7, there exist a linear polynomial L_y and a constant C such that

$$|u(x) - L_y(x)| \le C|x - y|^{1+\gamma}, \quad \forall\, x \in \bar{\Omega}. \tag{3.4}$$

Let $v(x) = u(x) - L_y(x)$. By the interior $C^{1,\alpha}$ estimate in $B_{d_{x_0}}(x_0)$ and (3.4), there exists a linear polynomial

$$Q_{x_0}(x) = Q(x_0) + DQ(x_0) \cdot (x - x_0)$$

such that

$$\begin{aligned} |Q(x_0)| = |v(x_0)| &\le C|d_{x_0}|^{1+\gamma}, \\ |DQ(x_0)| &\le C|d_{x_0}|^{\gamma} \end{aligned}$$

and

$$\begin{aligned} |v(x) - Q_{x_0}(x)| &\le C\frac{|x - x_0|^{1+\gamma}}{|d_{x_0}|^{1+\gamma}} \left(\|v\|_{L^\infty(B_{d_{x_0}}(x_0))} + |d_{x_0}|^{1+\gamma}(\|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)})\right) \\ &\le C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in B_{d_{x_0}/2}(x_0). \end{aligned} \tag{3.5}$$

Define

$$P_{x_0}(x) = L_y(x) + Q_{x_0}(x).$$

If $|x - x_0| < d_{x_0}/2$, by (3.5), we have

$$|u(x) - P_{x_0}(x)| = |v(x) - Q_{x_0}(x)| \le C|x - x_0|^{1+\gamma}.$$

If $|x - x_0| \ge d_{x_0}/2$, by (3.4), we have

$$\begin{aligned} |u(x) - P_{x_0}(x)| &\le |u(x) - L_y(x)| + |Q_{x_0}(x)| \\ &\le C|x - y|^{1+\gamma} + |Q(x_0)| + |DQ(x_0)||x_0 - x| \\ &\le C|x - x_0|^{1+\gamma}. \end{aligned}$$

Combining the two cases, it follows that for any $x_0 \in \Omega$, there exists a linear polynomial P_{x_0} such that

$$|u(x) - P_{x_0}(x)| \le C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}.$$

The proof of Theorem 1.8 is finished. □

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References

[1] L. A. Caffarelli, X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, Vol. 43, American Mathematical Society, Providence, RI, 1995. https://doi.org/10.1090/coll/043; MR1351007; Zbl 0834.35002

[2] L. A. Caffarelli, M. G. Crandall, M. Kocan, A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients, *Comm. Pure Appl. Math.* **49**(1996), No. 4, 365–397. https://doi.org/10.1002/(SICI)1097-0312(199604)49:4<365::AID-CPA3>3.3.CO;2-V; MR1376656; Zbl 0854.35032

[3] M. G. Crandall, H. Ishii, P. L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)* **27**(1992), No. 1, 1–67. https://doi.org/10.1090/S0273-0979-1992-00266-5; MR1118699; Zbl 0755.35015

[4] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. https://doi.org/10.1007/978-3-642-61798-0; MR1814364; Zbl 1042.35002

[5] J. L. Kazdan, *Prescribing the curvature of a Riemannian manifold*, CBMS Regional Conference Series in Mathematics, Vol. 57, American Mathematical Society, Providence, RI, 1985. https://doi.org/10.1090/cbms/057; MR787227; Zbl 0561.53048

[6] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain, *Math. USSR, Izv.* **22**(1984), No. 1, 67–97. https://doi.org/10.1070/IM1984v022n01ABEH001434; MR688919; Zbl 0578.35024

[7] D. S. Li, L. H. Wang, Boundary differentiability of solutions of elliptic equations on convex domains, *Manuscripta Math.* **121**(2006), No. 2, 137–156. https://doi.org/10.1007/s00229-006-0032-8; MR2264018; Zbl 1194.35126

[8] D. S. Li, L. H. Wang, Elliptic equations on convex domains with nonhomogeneous Dirichlet boundary conditions, *J. Differential Equations* **246**(2009), No. 5, 1723–1743. https://doi.org/10.1016/j.jde.2008.12.007; MR2494685; Zbl 1168.35017

[9] Y. Y. Lian, L. H. Wang, K. Zhang, Pointwise regularity for fully nonlinear elliptic equations in general forms, available at arXiv:2012.00324.

[10] Y. Y. Lian, K. Zhang, Boundary Lipschitz regularity and the Hopf lemma for fully nonlinear elliptic equations, available at arXiv:1901.06060.

[11] Y. Y. Lian, K. Zhang, Boundary pointwise $C^{1,\alpha}$ and $C^{2,\alpha}$ regularity for fully nonlinear elliptic equations, *J. Differential Equations* **269**(2020), No. 2, 1172–1191. https://doi.org/10.1016/j.jde.2020.01.006; MR4088470; Zbl 1436.35061

[12] E. Milakis, L. E. Silvestre, Regularity for fully nonlinear elliptic equations with Neumann boundary data, *Comm. Partial Differential Equations* **31**(2006), No. 7–9, 1227–1252. https://doi.org/10.1080/03605300600634999; MR2254613; Zbl 1241.35093

[13] G. C. Shephard, Decomposable convex polyhedra, *Mathematika* **10**(1963), 89–95. https://doi.org/10.1112/S0025579300003995; MR172176; Zbl 0121.39002

[14] D. Wu, Y. Y. Lian, K. Zhang, Pointwise boundary differentiability for fully nonlinear elliptic equations, available at arXiv:2101.00228v2.