Estimation of the hyper-order of entire solutions of complex linear ordinary differential equations whose coefficients are entire functions

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Abstract. We investigate the growth of solutions of the differential equation $f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = 0$, where $A_0(z), ..., A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. We estimate the hyper-order with respect to the conditions of $A_0(z), ..., A_{n-1}(z)$ if $f \not\equiv 0$ has infinite order.

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1 Introduction and statement of results

We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (see ([5])). Let $\sigma(f)$ denote the order of the growth of an entire function f as defined in ([5]) :

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r, f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic of f (see [5]), and $M(r, f) = \max_{|z|=r} |f(z)|$.

Definition 1. ([1], [2], [8]) Let f be a meromorphic function. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r}. \tag{1.1}$$

Note. Clearly, if $f(z)$ is entire, then

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.2}$$

We define the linear measure of a set $H \subset [0,+\infty[$ by $m(H) = \int_H dt$ and the logarithmic measure of a set $F \subset [1,+\infty[$ by $m_l(F) = \int_F \frac{dt}{t}$. The upper and the lower densities of H are defined by

$$\overline{dens} H = \overline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}, \quad \underline{dens}\ H = \underline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}.$$

Recently in [1], [2], [3] the concept of hyper-order was used to further investigate the growth of infinite order solutions of complex differential equations.

The following results have been obtained for the second order equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.3}$$

where $A(z)$, $B(z) \not\equiv 0$ are entire functions.

Theorem A.([1]) *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z\in H\} > 0$, and let $A(z)$ and $B(z)$ be entire functions such that for some constants α, $\beta > 0$,*

$$|A(z)| \le \exp\left\{o(1)|z|^\beta\right\} \tag{1.4}$$

and

$$|B(z)| \ge \exp\left\{(1+o(1))\alpha|z|^\beta\right\} \tag{1.5}$$

as $z\to\infty$ for $z\in H$. Then every solution $f\not\equiv 0$ of equation (1.3) satisfies $\sigma(f) = +\infty$ and $\sigma_2(f)\ge\beta$.

Theorem B.([2]) *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z\in H\} > 0$, and let $A(z)$ and $B(z)$ be entire functions, with $\sigma(A)\le\sigma(B) = \sigma < +\infty$ such that for some real constant $C(>0)$ and for any given $\varepsilon > 0$,*

$$|A(z)| \le \exp\left\{o(1)|z|^{\sigma-\varepsilon}\right\} \tag{1.6}$$

and

$$|B(z)| \geq \exp\left\{(1+o(1))\, C\, |z|^{\sigma-\varepsilon}\right\} \tag{1.7}$$

as $z \to \infty$ *for* $z \in H$*. Then every solution* $f \not\equiv 0$ *of equation* (1.3) *satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) = \sigma(B)$*.*

For $n \geq 2$, we consider a linear differential equation of the form

$$f^{(n)} + A_{n-1}(z)\, f^{(n-1)} + ... + A_1(z)\, f' + A_0(z)\, f = 0 \tag{1.8}$$

where $A_0(z), ..., A_{n-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$. It is well-known that all solutions of equation (1.8) are entire functions and if some of the coefficients of (1.8) are transcendental, (1.8) has at least one solution with $\sigma(f) = +\infty$.

The main purpose of this paper is to investigate the growth of infinite order solutions of the linear differential equation (1.8).

Theorem 1. *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$*, and let* $A_0(z), ..., A_{n-1}(z)$ *be entire functions such that for some constants* $0 \leq \beta < \alpha$ *and* $\mu > 0$*, we have*

$$|A_0(z)| \geq e^{\alpha |z|^{\mu}} \tag{1.9}$$

and

$$|A_k(z)| \leq e^{\beta |z|^{\mu}},\ k = 1, ..., n-1 \tag{1.10}$$

as $z \to \infty$ *for* $z \in H$*. Then every solution* $f \not\equiv 0$ *of equation* (1.8) *satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) \geq \mu$*.*

Theorem 2. *Let* H *be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$*, and let* $A_0(z), ..., A_{n-1}(z)$ *be entire functions with* $\max\{\sigma(A_k) : k = 1, ..., n-1\} \leq \sigma(A_0) = \sigma < +\infty$ *such that for some real constants* $0 \leq \beta < \alpha$*, we have*

$$|A_0(z)| \geq e^{\alpha |z|^{\sigma-\varepsilon}} \tag{1.11}$$

and

$$|A_k(z)| \leq e^{\beta |z|^{\sigma-\varepsilon}}, \quad k = 1, ..., n-1 \tag{1.12}$$

as $z \to \infty$ *for* $z \in H$*. Then every solution* $f \not\equiv 0$ *of equation* (1.8) *satisfies* $\sigma(f) = +\infty$ *and* $\sigma_2(f) = \sigma(A_0)$*.*

2 Preliminary Lemmas

Our proofs depend mainly upon the following Lemmas.

Lemma 1. ([4], *p.* 90) *Let f be a transcendental entire function of finite order σ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ denote a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, ..., m$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, \infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E$ and for all $(k, j) \in \Gamma$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2. ([4]) *Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E$, we have*

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq c \left[T(\alpha r, f) r^{\varepsilon} \log T(\alpha r, f) \right]^k, k \in \mathbf{N}. \tag{2.2}$$

Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu(r)$ be the maximum term, i.e $\mu(r) = \max\{|a_n| r^n; n = 0, 1, ...\}$, and let $\nu_f(r)$ be the central index of f, i.e $\nu_f(r) = \max\{m, \mu(r) = |a_m| r^m\}$.

Lemma 3.([2]) *Let $f(z)$ be an entire function of infinite order with the hyper-order $\sigma_2(f) = \sigma$, and let $\nu_f(r)$ be the central index of f. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log\log \nu_f(r)}{\log r} = \sigma. \tag{2.3}$$

Lemma 4.($Wiman - Valiron$, [6], [7]) *Let $f(z)$ be a transcendental entire function and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z|$ outside a set E of r of finite logarithmic measure, we have*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^k (1 + o(1)), \ (k \text{ is an integer}, r \notin E) \tag{2.4}$$

where $\nu_f(r)$ is the central index of f.

3 Proof of Theorem 1

Suppose that $f \not\equiv 0$ is a solution of equation (1.8) with $\sigma(f) < \infty$. By (1.8) we can write

$$\frac{1}{A_0(z)} \frac{f^{(n)}}{f} + \frac{A_{n-1}(z)}{A_0(z)} \frac{f^{(n-1)}}{f} + ... + \frac{A_1(z)}{A_0(z)} \frac{f^{/}}{f} + 1 = 0 \tag{3.1}$$

or

$$\frac{1}{A_0(z)} \frac{f^{(n)}}{f} + \sum_{k=1}^{n-1} \frac{A_k(z)}{A_0(z)} \frac{f^{(k)}}{f} = -1. \tag{3.2}$$

Then, by Lemma 1, there exists a set $E_1 \subset [0, \infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E_1$ and for all $k = 1, 2, ...n$, we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq |z|^{k\,c}, \quad k = 1, ..., n\ ; \quad c = \sigma - 1 + \varepsilon. \tag{3.3}$$

Also, by the hypothesis of Theorem1, there exists a set E_2 with $\overline{dens}\,\{|z| : z \in E_2\} > 0$ such that for all z satisfying $z \in E_2$, we have

$$|A_0(z)| \geq e^{\alpha |z|^{\mu}} \tag{3.4}$$

and

$$|A_k(z)| \leq e^{\beta |z|^{\mu}}, \quad k = 1, ..., n-1 \tag{3.5}$$

as $z \to \infty$. Hence from (3.3), (3.4) and (3.5) it follows that for all z satisfying $z \in E_2$ and $|z| \notin E_1$, we have

$$\left| \frac{A_k(z)}{A_0(z)} \right| \left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \frac{1}{e^{(\alpha-\beta)|z|^{\mu}}} |z|^{k\,c}, \quad k = 1, ..., n-1\,; \quad c = \sigma - 1 + \varepsilon \tag{3.6}$$

and

$$\left| \frac{1}{A_0(z)} \right| \left| \frac{f^{(n)}(z)}{f(z)} \right| \leq \frac{1}{e^{\alpha |z|^{\mu}}} |z|^{n\,c}, \ c = \sigma - 1 + \varepsilon \tag{3.7}$$

as $z \to \infty$. Thus there exists a set $H \subset [0,\infty)$ with a positive upper density such that (3.6) , (3.7) hold. Since

$$\lim_{\substack{z\to\infty \\ z\in H}} \frac{1}{e^{(\alpha-\beta)|z|^{\mu}}} |z|^{k\,c} = 0, \quad k = 1, ..., n-1$$

and

$$\lim_{\substack{z\to\infty \\ z\in H}} \frac{1}{e^{\alpha|z|^{\mu}}} |z|^{n\,c} = 0,$$

it follows that

$$\lim_{\substack{z\to\infty \\ z\in H}} \left|\frac{A_k(z)}{A_0(z)}\right| \left|\frac{f^{(k)}(z)}{f(z)}\right| = 0, \quad k = 1, ..., n-1$$

and

$$\lim_{\substack{z\to\infty \\ z\in H}} \left|\frac{1}{A_0(z)}\right| \left|\frac{f^{(n)}(z)}{f(z)}\right| = 0.$$

By making $z \to \infty$ for $z \in H$ in the relation (3.2) , we get a contradiction. Then every solution $f \not\equiv 0$ of equation (1.8) has infinite order.

Now from (1.8) , it follows that

$$|A_0(z)| \leq \left|\frac{f^{(n)}}{f}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f}\right| + ... + |A_1(z)| \left|\frac{f^{/}}{f}\right|. \tag{3.8}$$

Then, by Lemma 2, there exists a set $E_3 \subset [0,+\infty)$ with a finite linear measure such that for all z satysfying $|z| = r \notin E_3$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq r\,[T(2r,f)]^{k+1}, \ k = 1, ..., n. \tag{3.9}$$

Also, by the hypothesis of the Theorem1, there exists a set E_4 with $\overline{dens}\,\{|z| : z \in E_4\} > 0$ such that for all z satisfying $z \in E_4$, we have

$$|A_0(z)| \geq e^{\alpha|z|^{\mu}} \tag{3.10}$$

and

$$|A_k(z)| \leq e^{\beta |z|^{\mu}}, \quad k = 1, ..., n-1 \tag{3.11}$$

as $z \to \infty$. Hence from (3.8), (3.9), (3.10) and (3.11) it follows that for all z satisfying $z \in E_4$ and $|z| \notin E_3$, we have

$$e^{\alpha |z|^{\mu}} \leq |z| \left[T(2|z|, f)\right]^{n+1} \left[1 + (n-1) e^{\beta |z|^{\mu}}\right] \tag{3.12}$$

as $z \to \infty$. Thus there exists a set $H \subset [0, +\infty)$ with positive upper density such that

$$e^{(\alpha-\beta) r^{\mu}(1-o(1))} \leq [T(2r, f)]^{n+1}$$

as $r \to \infty$ in H. Therefore

$$\overline{\lim_{r \to \infty}} \frac{\log \log T(r, f)}{\log r} \geq \mu.$$

This proves Theorem 1.

4 Proof of Theorem 2

Assume that $f \not\equiv 0$ is a solution of equation (1.8). Using the same arguments as in Theorem 1, we get $\sigma(f) = +\infty$.

Now we prove that $\sigma_2(f) = \sigma(A_0) = \sigma$. By Theorem 1, we have $\sigma_2(f) \geq \sigma - \varepsilon$, and since ε is arbitrary, we get $\sigma_2(f) \geq \sigma(A_0) = \sigma$.

On the other hand, by Wiman-Valiron theory, there is a set $E \subset [1, +\infty)$ with logarithmic measure $m_l(E) < \infty$ and we can choose z satisfying $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$, such that (2.4) holds. For any given $\varepsilon > 0$, if r is sufficiently large, we have

$$|A_k(z)| \leq e^{r^{\sigma+\varepsilon}}, k = 0, 1, ..., n-1. \tag{4.1}$$

Substituting (2.4) and (4.1) into(1.8) , we obtain

$$\left(\frac{\nu_f(r)}{|z|}\right)^n |1 + o(1)| \leq e^{r^{\sigma+\varepsilon}} \left(\frac{\nu_f(r)}{|z|}\right)^{n-1} |1 + o(1)| +$$

$$+e^{r^{\sigma+\varepsilon}}\left(\frac{\nu_f(r)}{|z|}\right)^{n-2}|1+o(1)|+...+e^{r^{\sigma+\varepsilon}}\left(\frac{\nu_f(r)}{|z|}\right)|1+o(1)|+e^{r^{\sigma+\varepsilon}} \tag{4.2}$$

where z satisfies $|z|=r\notin[0,1]\cup E$ and $|f(z)|=M(r,f)$. By (4.2), we get

$$\overline{\lim_{r\to\infty}}\frac{\log\log\nu_f(r)}{\log r}\leq\sigma+\varepsilon. \tag{4.3}$$

Since ε is arbitrary, by (4.3) and Lemma 3 we have $\sigma_2(f)\leq\sigma$. This and the fact that $\sigma_2(f)\geq\sigma$ yield $\sigma_2(f)=\sigma$.

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