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Dirichlet problems with unbalanced growth and convection

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Abstract. We consider a double phase Dirichlet problem with a gradient dependent reaction term (convection). Using the theory of nonlinear operators of monotone type, we show the existence of a bounded strictly positive solution. Moreover, we show that the set of these solutions is compact in the corresponding generalized Sobolev–Orlicz space.

Keywords: weighted p-Laplacian, eigenvalue, generalized Orlicz space, pseudomonotone operator, double phase.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the existence of a positive solution for the following nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, 1 < q < p < N. \end{cases} \tag{1.1}$$

Given $a \in L^\infty(\Omega) \setminus \{0\}$ with $a(z) \geq 0$ for a.a. $z \in \Omega$ and $r \in (1, \infty)$ by Δ_r^a denotes the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du).$$

When $a(\cdot) \equiv 1$, then we write $\Delta_r^a = \Delta_r$ which is the standard r-Laplace differential operator.

In (1.1) the differential operator is not homogeneous and is related to two-phase integral functional

$$u \to \int_\Omega [a(z)|Du|^p + |Du|^q]\, dz.$$

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The integrand of this functional is the function

$$\mathcal{E}(z,t) = a(z)t^p + t^q \qquad \forall z \in \Omega,\ \forall t \geq 0.$$

We do not assume that the weight $a(\cdot)$ is bounded away from zero (that is, we do not require that $\operatorname{ess\,inf}_\Omega a > 0$) and so $\mathcal{E}(t,\cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \mathcal{E}(z,t) \leq c_0[t^p + t^q] \quad \text{a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

Such integral functionals, were first examined by Marcellini [11] and Zhikov [18] in the context of problems of the calculus of variations and of nonlinear elasticity theory. Until now there is no global regularity theory for unbalanced growth (double phase) boundary value problems analogous to the one for balanced growth problems developed by Lieberman [7]. Only local (interior) regularity results exist, produced primarily by Marcellini [12], Baroni–Colombo–Mingione [1] and Ragusa–Tachikawa [17].

In the reaction (right hand side) of (1.1), we have a Carathéodory function $f(z,x,y)$ (that is, for all $(x,y) \in \mathbb{R} \times \mathbb{R}^N, z \to f(z,x,y)$ is measurable and for a.a. $z \in \Omega, (x,y) \to f(z,x,y)$ is continuous). Since the reaction (source) term is gradient dependent, problem (1.1) is nonvariational. For this reason our approach is topological based on the theory of nonlinear operators of monotone type.

Recently there have been existence and multiplicity results for double phase equations with no gradient dependence (variational problems). We refer to the works of Gasiński–Papageorgiou [3], Gasiński–Winkert [4], Liu–Dai [8], Papageorgiou–Rădulescu–Repovš [14], Papageorgiou–Rădulescu–Zhang [15], Papageorgiou–Vetro–Vetro [16] and the references therein. Double phase problems with gradient dependence (convection), were studied only by Gasiński–Winkert [5] and Liu–Papageorgiou [9] using different conditions on the reaction $f(z,x,y)$.

2 Mathematical background

The unbalanced growth of $\mathcal{E}(z,\cdot)$ leads to a functional framework for problem (1.1) based on generalized Orlicz spaces. A comprehensive account of the theory of these spaces can be found in the book of Harjulehto–Hästö [6].

Let $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. We identify two such functions which differ only on a Lebesgue-null set. Also by $C^{0,1}(\overline{\Omega})$ we denote the space of all functions $u : \overline{\Omega} \to \mathbb{R}$ which are Lipschitz continuous. For the moment we assume that

$$a \in C^{0,1}(\overline{\Omega}),\ a(z) > 0 \qquad \forall z \in \Omega,\ 1 < q < p < N,\ \frac{p}{q} < 1 + \frac{1}{N}. \tag{2.1}$$

The last inequality in (2.1) implies that $p < q^* = \frac{Nq}{N-q}$ and this then leads to useful compact embeddings for some relevant spaces (see Proposition 2.1 below). Also these conditions guarantee the validity of the Poincaré inequality in the appropriate Sobolev–Orlicz space.

Then the Lebesgue–Orlicz space $L^{\mathcal{E}}(\Omega)$ is defined by

$$L^{\mathcal{E}}(\Omega) = \{u \in M(\Omega) : \rho_{\mathcal{E}}(u) < \infty\},$$

with $\rho_{\mathcal{E}}(\cdot)$ being the modular function defined by

$$\rho_{\mathcal{E}}(u) = \int_{\Omega} \mathcal{E}(z, |u|)\, dz = \int_{\Omega} [a(z)|u|^p + |u|^q] dz.$$

We equip this space with the so-called "Luxemburg norm" defined by

$$\|u\|_{\mathcal{E}} = \inf\Big[\lambda > 0 : \rho_{\mathcal{E}}\Big(\frac{u}{\lambda}\Big) \leq 1\Big].$$

Normed this way, $L^{\mathcal{E}}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex). Then using $L^{\mathcal{E}}(\Omega)$ we can define the corresponding Sobolev–Orlicz space $W^{1,\mathcal{E}}(\Omega)$ by

$$W^{1,\mathcal{E}}(\Omega) = \{u \in L^{\mathcal{E}}(\Omega) : |Du| \in L^{\mathcal{E}}(\Omega)\}.$$

Here Du denotes the weak gradient of $u(\cdot)$. This space is given the following norm

$$\|u\|_{1,\mathcal{E}} = \|u\|_{\mathcal{E}} + \|Du\|_{\mathcal{E}} \quad \text{for all } u \in W^{1,\mathcal{E}}(\Omega).$$

Here $\|Du\|_{\mathcal{E}} = \||Du|\|_{\mathcal{E}}$. This too is a Banach space which separable and reflexive (in fact uniformly convex). Also set

$$W_0^{1,\mathcal{E}}(\overline{\Omega}) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{E}}},$$

with $C_c^\infty(\Omega) = \{u \in C^\infty(\Omega)$ with compact support$\}$. Conditions (2.1) imply that the Poincaré inequality is valid on $W_0^{1,\mathcal{E}}(\Omega)$ and we can use the following equivalent norm on $W_0^{1,\mathcal{E}}(\Omega)$.

$$\|u\| = \|Du\|_{\mathcal{E}} \quad \text{for all } u \in W_0^{1,\mathcal{E}}(\Omega).$$

For these spaces we have the following useful embeddings.

Proposition 2.1.

(a) $L^{\mathcal{E}}(\Omega) \hookrightarrow L^\tau(\Omega)$ *and* $W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow W_0^{1,\tau}(\Omega)$ *continuously for all* $\tau \in [1, q]$.

(b) $W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow L^\tau(\Omega)$ *continuously for all* $\tau \in [1, q^*]$ *and compactly for all* $\tau \in [1, q^*)$;

(c) $L^p(\Omega) \hookrightarrow L^{\mathcal{E}}(\Omega)$ *continuously.*

There is a close relation between the norm $\|\cdot\|_{\mathcal{E}}$ and the modular function $\rho_{\mathcal{E}}(\cdot)$ on the space $W_0^{1,\mathcal{E}}(\Omega)$.

Proposition 2.2.

(a) $\|u\|_{\mathcal{E}} = \lambda \Leftrightarrow \rho_{\mathcal{E}}(\frac{u}{\lambda}) = 1$;

(b) $\|u\|_{\mathcal{E}} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{\mathcal{E}}(u) < 1$ (*resp.* $= 1, > 1$);

(c) $\|u\|_{\mathcal{E}} \leq 1 \Rightarrow \|u\|_{\mathcal{E}}^p \leq \rho_{\mathcal{E}}(u) \leq \|u\|_{\mathcal{E}}^q$;

(d) $\|u\|_{\mathcal{E}} > 1 \Rightarrow \|u\|_{\mathcal{E}}^q \leq \rho_{\mathcal{E}}(u) \leq \|u\|_{\mathcal{E}}^p$;

(e) $\|u\|_{\mathcal{E}} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho_{\mathcal{E}}(u) \to 0q$(*resp.* $\to +\infty$).

Let $V : W_0^{1,\mathcal{E}}(\Omega) \to W_0^{1,\mathcal{E}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_{\Omega} (a(z)|Du|^{p-2}Du + |Du|^{p-2}Du, Dh)_{\mathbb{R}^N} dz, \quad \text{for all } u, h \in W_0^{1,\mathcal{E}}(\Omega).$$

This operator has the following properties (see [8]).

Proposition 2.3. *The operator $V(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, "if $u_n \xrightarrow{w} u$ in $W_0^{1,\mathcal{E}}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,\mathcal{E}}(\Omega)$."*

For $x \in \mathbb{R}$, we set $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. Then if $u \in M(\Omega)$, we define $u^+(z) = u(z)^+$ and $u^-(z) = u(z)^-$ for all $z \in \Omega$. We know that $u = u^+ - u^-, |u| = u^+ + u^-$ and if $u \in W_0^{1,\mathcal{E}}(\Omega)$, then $u^\pm \in W_0^{1,\mathcal{E}}(\Omega)$.

3 Some auxiliary results

In this section we prove some auxiliary results concerning the weighted p-Laplacian Δ_p^a, which we will need in the analysis of problem (1.1)).

We strengthen the conditions on the weight $a(\cdot)$. By $\widetilde{A}_p$ we denote the p-Muckenhoupt class (see Harjulehto–Hästö [6, p. 114]). The stronger conditions on the weight $a(\cdot)$ are the following:

H_0: $a \in C^{0,1}(\overline{\Omega}) \cap \widetilde{A}_p$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Let $\mathcal{E}_0(z,t) = a(z)t^p$ for all $z \in \Omega$ for all $t \ge 0$. On account of hypotheses H_0 above we have that $W_0^{1,\mathcal{E}_0}(\Omega) \hookrightarrow L^{\mathcal{E}_0}(\Omega)$ compactly (see Liu–Papageorgiou [10]). We will use this fact to produce a smallest eigenvalue for $(-\Delta_p^a, W_0^{1,\mathcal{E}_0}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p^a u(z) = \hat{\lambda} a(z)|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad \mathrm{u}|_{\partial\Omega} = 0. \tag{3.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if the above Dirichlet problem admits a nontrivial solution $\hat{u} \in W_0^{1,\mathcal{E}_0}(\Omega)$ known as an"eigenfunction" corresponding to $\hat{\lambda}$.

Proposition 3.1. *If hypotheses H_0 hold, then problem (3.1) has a smallest eigenvalue $\hat{\lambda}_1^a = \hat{\lambda}_1^a(p) > 0$ and every corresponding eigenfunction $\hat{u} \in W_0^{1,\mathcal{E}_0}(\Omega)$ satisfies $\hat{u}(z) > 0$ or $\hat{u}(z) < 0$ a.a. in Ω (has constant sign).*

Proof. Let $\hat{\lambda}_1^a = \inf\left[\frac{\rho_a(Du)}{\rho_a(u)} : u \in W_0^{1,\mathcal{E}_0}(\Omega), u \ne 0\right]$, where for every $v \in L^{\mathcal{E}_0}(\Omega)$ we define $\rho_a(v) = \int_\Omega a(z)|v|^p dz$. The homogeneity of $\rho_a(\cdot)$ implies that

$$\hat{\lambda}_1^a = \inf\left[\rho_a(Du) : u \in W_0^{1,\mathcal{E}_0}(\Omega), \rho_a(u) = 1\right]. \tag{3.2}$$

Consider a sequence $\{u_n\}_{n\ge\mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ such that

$$\rho_a(Du_n) \downarrow \hat{\lambda}_1^a \quad \text{and} \quad \rho_a(u_n) = 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Evidently $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{and} \quad u_n \to \hat{u} \text{ in } L^{\mathcal{E}_0}(\Omega). \tag{3.4}$$

The function $\rho_a(\cdot)$ is continuous, convex, thus sequentially weakly lower semicontinuous. So, from (3.4) we have

$$\begin{aligned} & \rho_a(D\hat{u}) \le \liminf_{n\to+\infty} \rho_a(Du_n), \quad \rho_a(u_n) \to \rho_a(\hat{u}), \\ \Rightarrow \quad & \rho_a(D\hat{u}) \le \hat{\lambda}_1^a, \quad \rho_a(\hat{u}) = 1 \quad (\text{see } (3.3)), \\ \Rightarrow \quad & \rho_a(D\hat{u}) = \hat{\lambda}_1^a > 0. \end{aligned}$$

From (3.2) and the Lagrange multiplier rule (see [13, p. 422]), we have

$$-\Delta_p^a \hat{u} = \hat{\lambda}_1^a a(z)|\hat{u}|^{p-2}\hat{u} \quad \text{in } \Omega, \quad \hat{\mathrm{u}}|_{\partial\Omega} = 0. \tag{3.5}$$

Suppose $\hat{u}^+ \neq 0$, since $\hat{u}^+ \in W_0^{1,\mathcal{E}_0}(\Omega)$, acting on (3.4) with $\hat{u}^+$, we obtain

$$\begin{aligned} &\rho_a(D\hat{u}^+) = \hat{\lambda}_1^a \rho_a(\hat{u}^+), \\ \Rightarrow \quad &\hat{u}^+ \text{ is an eigenfunction for } \quad \hat{\lambda}_1^a > 0. \end{aligned}$$

From Colasuonno–Squassina [2, Section 3.3], we have that

$$u^+ \in W_0^{1,\mathcal{E}_0}(\Omega) \cap L^\infty(\Omega).$$

Invoking Proposition 2.4 of Papageorgiou–Vetro–Vetro [16], we infer that

$$\begin{aligned} &\hat{u}^+(z) > 0 \quad \text{for } a.a.\ z \in \Omega, \\ \Rightarrow \quad &\hat{u} = \hat{u}^+. \end{aligned}$$

Similarly if $u^- \neq 0$. □

This proposition leads to the following estimate which is useful in case we have nonuniform nonresonance.

Proposition 3.2. *If hypotheses H_0 hold, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\lambda}_1^a$ for a.a. $z \in \Omega$ and*

$$\eta \not\equiv \hat{\lambda}_1^a,$$

then there exists $c_1 > 0$ such that

$$c_1 \|u\|_{1,\mathcal{E}_0}^p \leq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz \quad \textit{for all } u \in W_0^{1,\mathcal{E}_0}(\Omega).$$

Proof. We argue by contradiction. So, suppose that the conclusion of the proposition is not true. We can find $\{u_n\}_{n\geq\mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ such that

$$\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz < \frac{1}{n}\|u_n\|_{1,\mathcal{E}_0}^p \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the p-homogeneity of this inequality, we can say that

$$\left\{ \begin{aligned} &\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz < \frac{1}{n}, \\ &\|u_n\|_{1,\mathcal{E}_0} = 1 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \right\} \tag{3.6}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{\mathcal{E}_0}(\Omega). \tag{3.7}$$

If $u = 0$, then

$$\begin{aligned} &\rho_a(Du_n) \to 0, \\ \Rightarrow \quad &u_n \to 0 \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{(see Proposition 2.2)}, \end{aligned}$$

a contradiction, since $\|u_n\|_{1,\mathcal{E}_0} = 1$, for all $n \in \mathbb{N}$ (see (3.6)).

If $u \neq 0$, then from (3.6) and (3.7)

$$\rho_a(Du) \leq \int_\Omega \eta(z)a(z)|u|^p dz, \tag{3.8}$$

$$\begin{aligned}
&\Rightarrow \quad \rho_a(Du) = \hat{\lambda}_1^a \rho_a(u), \quad \text{(see (3.1))}, \\
&\Rightarrow \quad |u(z)| > 0 \quad \text{for } a.a.\ z \in \Omega, \quad \text{(see Proposition 3.1)}, \\
&\Rightarrow \quad \rho_a(Du) < \hat{\lambda}_1^a \rho_a(u) \quad \text{(see (3.8))},
\end{aligned}$$

which contradicts (3.1).

Therefore we conclude that there exists $c_1 > 0$ such that

$$c_1 \|u\|^p_{1,\mathcal{E}_0} \leq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz \quad \text{for all } u \in W_0^{1,\mathcal{E}_0}(\Omega). \qquad \square$$

4 Positive solution

In this section, using the theory of pseudomonotone operators (see Papageorgiou–Rădulescu–Repovš [13, Section 2.10]), we prove the existence of a positive solution for problem (1.1).

We impose the following conditions on the reaction $f(z,x,y)$. In what follows, by $\hat{\lambda}_1(q) > 0$, we denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$ (that is $\hat{\lambda}_1(q) = \hat{\lambda}_1^a(q)$ with $a \equiv 1$).

H_1: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) $|f(z,x,y)| \leq \hat{a}(z)[1 + |x|^{p-1}] + \mu|y|^{q-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$ with $\hat{a} \in L^\infty(\Omega)$ and $\mu < \hat{\lambda}_1(q)$;

(ii) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$\eta(z) \leq \hat{\lambda}_1^a \text{ for } a.a.\ z \in \Omega, \eta \not\equiv \hat{\lambda}_1^a,$$

and for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$f(z,x,y) \leq [\eta(z) + \varepsilon]a(z)x^{p-1} + \mu|y|^q \text{ for } a.a.\ z \in \Omega,\ all\ x \geq M_\varepsilon;$$

(iii) there exists $\vartheta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\begin{aligned}
&\vartheta(z) \geq \hat{\lambda}_1(q) \text{ for } a.a.\ z \in \Omega, \vartheta \not\equiv \hat{\lambda}_1(q), \\
&f(z,x,y) \geq \vartheta(z)x^{q-1} \text{ for } a.a.\ z \in \Omega, \text{ all } 0 \leq x \leq \delta, \text{ all } y \in \mathbb{R}^N \\
&f(z,x,y) \geq -c_2 x^{r-1} \text{ for } a.a.\ z \in \Omega, \text{ all } x \geq \delta, \text{ all } y \in \mathbb{R}^N, \text{some } c_2 > 0,\ p < r < p^*.
\end{aligned}$$

Remark 4.1. Hypothesis H_1(ii) implies that

$$\limsup_{n \to +\infty} \frac{f(z,x,y)}{a(z)x^{p-1}} \leq \eta(z)$$

uniformly for a.a.$z \in \Omega$ and all $y \in \mathbb{R}^N$ on a bounded set. Similarly, hypothesis H_1(iii) implies that

$$\liminf_{x \to 0^+} \frac{f(z,x,y)}{x^{q-1}} \geq \vartheta(z)$$

uniformly for a.a.$z \in \Omega$ and all $y \in \mathbb{R}^N$.

Example 4.2. The following function satisfies all the above hypotheses

$$f(z,x,y) = \begin{cases} \vartheta(x^+)^{q-1} + \Big[\mu|y|^{q-1} + (\eta a(z) - \vartheta)\Big](x^+)^{s-1}, & \text{if } x \leq 1, \\ \eta a(z) x^{p-1} + \mu|y|^{q-1}, & \text{if } 1 < x, \end{cases}$$

with $\mu < \hat{\lambda}_1(q) < \vartheta$, $\eta < \hat{\lambda}_1^a$, $1 < q < s$.

On account of hypotheses H_1(i),(ii), we have

$$f(z,x,y) \geq \vartheta(z)x^{q-1} - c_3 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ all } y \in \mathbb{R}^N, \text{ some } c_3 > 0. \tag{4.1}$$

Based on this unilateral growth condition, we consider the following auxiliary double phase Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \vartheta(z)u(z)^{p-1} - c_3 u(z)^{r-1} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ 1 < q < p < N, \ r > p. \end{array} \right\} \tag{4.2}$$

From Liu–Papageorgiou [9, Proposition 3.1], we have the following result for problem (4.2).

Proposition 4.3. *If hypotheses H_0 hold, then problem* (4.2) *has a unique positive solution $\bar{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega)$ and $\bar{u}(z) > 0$ for a.a. $z \in \Omega$.*

Using the solution $\bar{u}$ we introduce the Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$g(z,x,y) = \begin{cases} f(z,\bar{u}(z),y), & \text{if } x \leq \bar{u}(z), \\ f(z,x,y), & \text{if } \bar{u}(z) < x. \end{cases} \tag{4.3}$$

Let $N_g(u)(\cdot) = g(\cdot, u(\cdot), Du(\cdot))$ for all $u \in W_0^{1,\mathcal{E}}(\Omega)$ (the Nemytski map corresponding to g) and consider the nonlinear operator $K : W_0^{1,\mathcal{E}}(\Omega) \to W_0^{1,\mathcal{E}}(\Omega)^*$ defined by

$$K(u) = V(u) - N_g(u) \quad \text{for all } u \in W_0^{1,\mathcal{E}}(\Omega).$$

Proposition 4.4. *If hypotheses H_0, H_1 hold, then the operator $K(\cdot)$ is pseudomonotone.*

Proof. We consider a sequence $\{u_n\}_{n\geq\mathbb{N}} \subseteq W_0^{1,\mathcal{E}}(\Omega)$ such that

$$\left\{ \begin{array}{l} u_n \xrightarrow{w} u \quad \text{in} \quad W_0^{1,\mathcal{E}}(\Omega) \quad K(u_n) \xrightarrow{w} u^* \text{ in } W_0^{1,\mathcal{E}}(\Omega)^*, \\ \limsup\limits_{n\to\infty} \langle K(u_n), u_n - u \rangle \leq 0. \end{array} \right\} \tag{4.4}$$

Hypotheses H_0 imply $p < q^*$ and so by Proposition 2.1, we have that

$$\begin{aligned} & W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow L^p(\Omega) \text{ compactly}, \\ \Rightarrow \quad & u_n \to u \quad \text{in } L^p(\Omega) \quad \text{(see (4.4))}. \end{aligned} \tag{4.5}$$

We have

$$\begin{aligned} \int_\Omega g(z,u_n,Du_n)(u_n - u)dz = & \int_{\{u_n \leq \bar{u}\}} f(z,\bar{u},Du_n)(u_n - u)dz \\ & + \int_{\{\bar{u} < u_n\}} f(z,u_n,Du_n)(u_n - u)dz \quad \text{(see (4.4))}. \end{aligned} \tag{4.6}$$

On account of hypothesis H_1(i), we have that

$$\{f(\cdot,\bar{u}(\cdot),Du_n(\cdot)\}_{n\geq\mathbb{N}} \subseteq L^{p'}(\Omega)$$
$$\{f(\cdot,u_n(\cdot),Du_n(\cdot)\}_{n\geq\mathbb{N}} \subseteq L^{p'}(\Omega), \quad \left(\frac{1}{p}+\frac{1}{p'}=1\right),$$

are both bounded (recall $q<p$). Therefore, from (4.5) we infer that

$$\int_{\{u_n\leq\bar{u}\}} f(z,\bar{u},Du_n)(u_n-u)dz \to 0, \qquad \int_{\{\bar{u}<u_n\}} f(z,u_n,Du_n)(u_n-u)dz \to 0 \quad \text{as } n\to\infty.$$

From (4.6) it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega g(z,u_n,Du_n)(u_n-u)dz = 0,\\
\Rightarrow\ &\limsup_{n\to\infty}\langle V(u_n),u_n-u\rangle \leq 0 \quad (\text{see } (4.4)),\\
\Rightarrow\ & u_n\to u \quad \text{in} \quad W_0^{1,\mathcal{E}}(\Omega) \quad (\text{see Proposition 2.3}).
\end{aligned}$$

Exploiting the continuity of $K(\cdot)$, we have

$$\begin{aligned}
&K(u_n)\to K(u) \text{ in } W_0^{1,\mathcal{E}}(\Omega)^*,\\
\Rightarrow\ & u^*=K(u) \quad \text{and} \quad \langle K(u_n),u_n\rangle \to \langle K(u),u\rangle \quad (\text{see } (4.4)),\\
\Rightarrow\ & K(\cdot) \quad \text{is generalized pseudomonotone} \quad (\text{see [13, p. 150]}).
\end{aligned}$$

Invoking Proposition 2.10.3, p. 51, of Papageorgiou–Rădulescu–Repovš [13], we conclude that $K(\cdot)$ is pseudomonotone. □

Next we show that $K(\cdot)$ is strongly coercive, that is,

$$\frac{\langle K(u),u\rangle}{\|u\|} \to +\infty \quad \text{as } \|u\|\to\infty.$$

Proposition 4.5. *If hypotheses H_0, H_1 hold, then the operator $K(\cdot)$ is strongly coercive.*

Proof. For every $u\in W_0^{1,\mathcal{E}}(\Omega)$ with $\|u\|\geq 1$, $\|u\|_{1,\mathcal{E}_0}\geq 1$, we have

$$\begin{aligned}
\langle K(u),u\rangle &= \langle V(u),u\rangle - \int_\Omega g(z,u,Du)udz\\
&= \rho_{\mathcal{E}}(Du) - \int_{\{u\leq\bar{u}\}} f(z,\bar{u},Du)udz - \int_{\{\bar{u}<u\}} f(z,u,Du)udz \quad (\text{see } (4.3)).
\end{aligned} \tag{4.7}$$

We have

$$\int_{\{u\leq\bar{u}\}} f(z,\bar{u},Du)udz \leq c_4\|u\| + \int_{\{u\leq\bar{u}\}} \mu|Du|^{q-1}udz \tag{4.8}$$
for some $c_4>0$ (see hypothesis H_1(i))

$$\int_{\{\bar{u}<u\}} f(z,u,Du)udz \leq c_5 + \int_\Omega [\eta(z)+\varepsilon]a(z)u^p dz + \int_{\{\bar{u}<u\}} \mu|Du|^{q-1}udz \tag{4.9}$$
for some $c_5=c_5(\varepsilon)>0$ (see H_1(ii)).

From (4.8) and (4.9) it follows that

$$\int_\Omega g(z,u,Du)udz \leq c_5 + c_4\|u\| + \frac{\mu}{\hat{\lambda}_1(q)}\|Du\|_q^q + \int_\Omega \eta(z)a(z)u^p dz + \varepsilon\rho_a(u) \tag{4.10}$$
(here we have used Hölder's inequality) .

We return to (4.7) and use (4.10). We obtain

$$\begin{aligned}\langle K(u),u\rangle &\geq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz - \frac{\varepsilon}{\hat{\lambda}_1^a}\|u\|_{1,\mathcal{E}_0}^p \\ &\quad + \left(1-\frac{\mu}{\hat{\lambda}_1(q)}\right)\|Du\|_q^q - c_4\|u\| - c_5 \\ &\qquad \text{(recall that } \|u\|_{1,\mathcal{E}_0}\geq 1 \text{ and see Proposition 2.2)} \\ &\geq \left[c_1 - \frac{\varepsilon}{\hat{\lambda}_1^a}\right]\|u\|_{1,\mathcal{E}_0}^p + c_6\|Du\|_q^q - c_4\|u\| - c_5 \\ &\qquad \text{with } c_6 = 1-\frac{\mu}{\hat{\lambda}_1(q)} > 0 \quad \text{(see Proposition 3.2).}\end{aligned}$$

Choosing $\varepsilon \in (0, c_1\hat{\lambda}_1^a)$, we see that

$$\begin{aligned}\langle K(u),u\rangle &\geq c_7\rho_{\mathcal{E}}(Du) - c_4\|u\| - c_5 \\ &\geq c_7\|u\|^q - c_4\|u\| - c_5 \quad \text{for some } c_7>0 \text{ (recall } \|u\|\geq 1 \text{ and see Proposition 2.2)} \\ &\Rightarrow K(\cdot) \quad \text{is strongly coercive.}\end{aligned}$$

□

Now we are ready to prove the existence of a bounded positive solution for problem (1.1).

Theorem 4.6. *If hypotheses H_0, H_1 hold, then problem* (1.1) *admits a positive solution*

$$\hat{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega).$$

such that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$.

Proof. From Propositions 4.4 and 4.5 we have that the operator $K(\cdot)$ is pseudomonotone and strongly coercive. So, by Theorem 2.10.10, p. 156, of Papageorgiou–Rădulescu–Repovš [13], $K(\cdot)$ is surjective. Hence we can find $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega)$ such that

$$\begin{aligned}& K(\hat{u}) = 0 \quad \text{in} \quad W_0^{1,\mathcal{E}}(\Omega)^*, \\ \Rightarrow \quad & \langle K(\hat{u}), (\bar{u}-\hat{u})^+\rangle = 0 \quad (\text{since } (\bar{u}-\hat{u})^+ \text{ in } W_0^{1,\mathcal{E}}(\Omega))\end{aligned}$$

$$\begin{aligned}\Rightarrow \langle V(\hat{u}), (\bar{u}-\hat{u})^+\rangle &= \int_\Omega g(z,\hat{u},D\hat{u})(\bar{u}-\hat{u})^+ dz \\ &= \int_\Omega f(z,\hat{u},D\hat{u})(\bar{u}-\hat{u})^+ dz \quad \text{(see (4.3))} \\ &\geq \int_\Omega [\vartheta(z)\bar{u}^{q-1} - c_8\bar{u}^{r-1}](\bar{u}-\hat{u})^+ dz \quad \text{(see (4.1))} \\ &= \langle V(\bar{u}), (\bar{u}-\hat{u})^+\rangle \quad \text{(see Proposition 4.3)} \\ &\Rightarrow \bar{u} \leq \hat{u} \quad \text{(see Proposition 2.3).}\end{aligned}$$

Therefore $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega)$ is a positive solution for problem (1.1). From Theorem 3.1 of Gasiński–Winkert [4], we have that $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega)$. Finally Proposition 2.4 of Papageorgiou–Vetro–Vetro [16] implies that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$. □

Let $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ denote the set of positive solutions of problem (1.1). From Theorem 4.6 we have

$$\emptyset \neq S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega). \tag{4.11}$$

Proposition 4.7. *If hypotheses H_0, H_1 hold, then $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ is nonempty, compact.*

Proof. We already know that $S_+ \neq \emptyset$ (see Theorem 4.6 and (4.11)). Clearly $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ is closed. Let $\{u_n\}_{n\geq \mathbb{N}} \subseteq S_+$. We have

$$\langle V(u_n), h\rangle = \int_\Omega f(z, u_n, Du_n) h dz \quad \text{for all } h \in W_0^{1,\mathcal{E}}(\Omega) \text{ all } n \in \mathbb{N}. \tag{4.12}$$

On account of hypotheses H_1(i)(ii), we have

$$f(z, x, y)x \leq [\eta(z) + \varepsilon]a(z)|x|^p + c_8 + \mu|y|^{q-1} \quad \text{for } a.a.\ z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_8 > 0. \tag{4.13}$$

In (4.12) we use $h = u_n \in W_0^{1,\mathcal{E}}(\Omega)$. Using (4.13) we obtain

$$\begin{aligned}
&\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz - \frac{\varepsilon}{\hat{\lambda}_1^a}\|u_n\|_{1,\mathcal{E}_0}^p + \|Du_n\|_q^q - \mu\|u_n\|_q^q \leq c_8 \quad \text{for all } n \in \mathbb{N},\\
\Rightarrow\ & \left[c_1 - \frac{\varepsilon}{\hat{\lambda}_1^a}\right]\|u_n\|_{1,\mathcal{E}_0}^p + \left[1 - \frac{\mu}{\hat{\lambda}_1(q)}\right]\|Du\|_q^q \leq c_8,\\
\Rightarrow\ & \|u_n\|^p \leq c_9 \quad \text{for some } c_9 > 0, \text{ all } n \in \mathbb{N}\\
& \qquad (\text{choose } \varepsilon \in (0, c_1\hat{\lambda}_1^a) \text{ and recall that } \mu < \hat{\lambda}_1(q))\\
\Rightarrow\ & \{u_n\}_{n\geq\mathbb{N}} \subseteq W_0^{1,\mathcal{E}}(\Omega) \quad \text{is bounded.}
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{E}}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \tag{4.14}$$
$$(\text{recall that } p < q^* \text{ and see Proposition 2.1}).$$

Then (4.14) and hypothesis H_1(i) imply that

$$\begin{aligned}
& \int_\Omega f(z, u_n, Du_n)(u_n - u)dz \to 0,\\
\Rightarrow\ & \lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \quad (\text{see (4.12) with } h = u_n - u)\\
\Rightarrow\ & u_n \to u \quad \text{in } W_0^{1,\mathcal{E}}(\Omega) \quad (\text{see Proposition 4.4})
\end{aligned}$$

Since S_+ is closed, we conclude that it is compact in $W_0^{1,\mathcal{E}}(\Omega)$. □

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