Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **26**, 1–34; https://doi.org/10.14232/ejqtde.2022.1.26 www.math.u-szeged.hu/ejqtde/

Solvability of thirty-six three-dimensional systems of difference equations of hyperbolic-cotangent type

Stevo Stević

Mathematical Institute of the Serbian Academy of Sciences,
Knez Mihailova 36/III, 11000 Beograd, Serbia

Department of Medical Research, China Medical University Hospital, China Medical University,
Taichung 40402, Taiwan, Republic of China

Received 4 October 2021, appeared 28 May 2022
Communicated by Gennaro Infante

Abstract. We present thirty-six classes of three-dimensional systems of difference equations of the hyperbolic-cotangent type which are solvable in closed form.

Keywords: system of difference equations, solvable systems, closed-form formulas.

2020 Mathematics Subject Classification: 39A45.

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ be the sets of natural, integer and complex numbers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Difference equations and systems have been studied for a long time. Some information on old results can be found in the classical books such as [4, 8, 9, 15–18]. Each of the books contain a part devoted to solvability. Solvability seems the first topic which has been seriously studied. The following papers and books contain some of the most important classical results on solvability [3,5–7,12–14]. Quite old presentations of these and other old results can be found in [10,11]. Many difference equations and systems have appeared in some applications. Some classical applications can be found in [8, 9, 17, 29, 44]. A great majority of the equations and systems is very difficult or impossible to solve, because of which it is of some interest to look for their invariants, which might also help in studying of long-term behaviour of their solutions as it was the case, for example, in [20–22,25,30,31,35]. The following papers: [26,28,32–34,36–43] contain some recent results on solvability.

During the '90s Papaschinopoulos and Schinas started studying systems which frequently possessed some kind of symmetry (see, e.g., [19–25, 27, 30, 31]), which was one of the motivations for our investigation of solvability of such systems (see, e.g., [32–34, 36–39, 42, 43]). Product-type difference equations and systems are closely related to linear ones, some of which are solvable. This fact motivated some recent investigations of their solvability (see, e.g., [32]).

✉ Email: sstevic@ptt.rs

The following class of systems (the hyperbolic-cotangent-type class)

$$x_{n+1} = \frac{p_{n-k}q_{n-l} + a}{p_{n-k} + q_{n-l}}, \quad y_{n+1} = \frac{r_{n-k}s_{n-l} + a}{r_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with complex initial values, where $k, l \in \mathbb{N}_0$, $a \in \mathbb{C}$, and p_n, q_n, r_n, s_n are x_n or y_n for all n, has been studied considerably during the last several years. The corresponding scalar equation has been studied for the last two decades (see [28,40] and the related references therein). The fact that the equation can be easily reduced to the case $a = 1$, which has the form of the hyperbolic-cotangent sum formula, has suggested the name of the class of systems.

System (1.1) is closely related to some product-type ones. Depending on the characteristic polynomial associated with a linear difference equation appearing during finding closed-form formulas for solutions to such a system, some of them are theoretically, but some are practically solvable, i.e., solvable in closed form, due to the Abel theorem [1] (each linear homogeneous difference equation with constant coefficients of order less than or equal to four is practically solvable, unlike the case when the order is bigger than four when in some cases the equation is only theoretically solvable). This fact suggests that for practical solvability delays k and l have to take small values. In a series of papers we have studied solvability of the systems of the form in (1.1) with small k and l. In [34,42,43] was investigated the case $k = 0$ and $l = 1$, in [33] the case $k = 1$ and $l = 2$, in [37] the case $k = 0$ and $l = 2$, in [38] the case $k = 2$ and $l = 3$, in [39] the case $k = 0$ and $l = 3$, in [36] the case $k = l \in \mathbb{N}_0$. The case $k = 1$ and $l = 3$ reduces to the case $k = 0$ and $l = 1$, since in the case the system is with interlacing indices (for the notion and some basic fact see, e.g., [41]). In these papers was shown that the corresponding systems are practically solvable, which in some cases is a bit surprising result, e.g., when $k = 2$ and $l = 3$. Namely, it turns out that all the associated polynomials appearing during finding solutions to the corresponding systems are solvable by radicals.

It is a natural problem to study solvability of the corresponding thee-dimensional systems of difference equations. Hence, in this paper we study solvability of some of the three-dimensional systems of difference equations of the form

$$x_{n+1} = \frac{p_n q_n + a}{p_n + q_n}, \quad y_{n+1} = \frac{r_n s_n + a}{r_n + s_n}, \quad z_{n+1} = \frac{t_n g_n + a}{t_n + g_n}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $a, p_0, q_0, r_0, s_0, t_0, g_0 \in \mathbb{C}$, and p_n, q_n, r_n, s_n, t_n, g_n are one of the sequences x_n, y_n, z_n.

2 Systems studied in the paper

In this section we transform the system (1.2) into another and give a list of its special cases which are studied in the paper. Before we do this we first note that if $a = 0$, then the system (1.2) is essentially linear with constant coefficients (see [34]; special cases frequently appear in problem books [2]), so the case is not much interesting. Hence, the case will not be treated here and we may assume that $a \neq 0$.

First note that from (1.2) we easily obtain

$$\begin{aligned} x_{n+1} \pm \sqrt{a} &= \frac{(p_n \pm \sqrt{a})(q_n \pm \sqrt{a})}{p_n + q_n}, \\ y_{n+1} \pm \sqrt{a} &= \frac{(r_n \pm \sqrt{a})(s_n \pm \sqrt{a})}{r_n + s_n}, \\ z_{n+1} \pm \sqrt{a} &= \frac{(t_n \pm \sqrt{a})(g_n \pm \sqrt{a})}{t_n + g_n}, \end{aligned}$$

for $n \in \mathbb{N}_0$, and consequently

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \frac{p_n+\sqrt{a}}{p_n-\sqrt{a}} \cdot \frac{q_n+\sqrt{a}}{q_n-\sqrt{a}}, \\ \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} &= \frac{r_n+\sqrt{a}}{r_n-\sqrt{a}} \cdot \frac{s_n+\sqrt{a}}{s_n-\sqrt{a}}, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \frac{t_n+\sqrt{a}}{t_n-\sqrt{a}} \cdot \frac{g_n+\sqrt{a}}{g_n-\sqrt{a}}, \end{aligned} \tag{2.1}$$

for $n \in \mathbb{N}_0$.

Since each of the sequences p_n, q_n, r_n, s_n, t_n, g_n is one of the sequences x_n, y_n, z_n, there are a lot of systems of difference equations of the form in (2.1). They all are not different, since some of them are equivalent to each other.

By using the change of variables

$$u_n = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}, \quad v_n = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}, \quad w_n = \frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

the systems of difference equations in (2.1) are transformed to some product type systems of difference equations. Bearing in mind that the product type systems of difference equations are theoretically solvable and that some of them are practically solvable, it is a natural problem to study practical solvability of (1.2).

Note that from (2.2) we have the following relations

$$x_n = \sqrt{a}\,\frac{u_n+1}{u_n-1}, \quad y_n = \sqrt{a}\,\frac{v_n+1}{v_n-1}, \quad z_n = \sqrt{a}\,\frac{w_n+1}{w_n-1}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

which will be used in the proofs of all the theorems in the paper.

Our aim here is to show practical solvability of the following 36 systems of difference equations by presenting closed-form formulas for their well-defined solutions.

System 1. Case $p_n = q_n = r_n = s_n = t_n = g_n = x_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.4}$$

System 2. Case $p_n = q_n = r_n = s_n = t_n = x_n$, $g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.5}$$

System 3. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.6}$$

System 4. $p_n = q_n = r_n = s_n = t_n = x_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.7}$$

System 5. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.8}$$

System 6. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.9}$$

System 7. Case $p_n = q_n = r_n = t_n = g_n = x_n$, $s_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.10}$$

System 8. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.11}$$

System 9. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.12}$$

System 10. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.13}$$

System 11. Case $p_n = q_n = r_n = x_n$, $s_n = y_n$, $t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.14}$$

System 12. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.15}$$

System 13. Case $p_n = q_n = t_n = g_n = x_n$, $r_n = s_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.16}$$

System 14. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.17}$$

System 15. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.18}$$

System 16. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = y_n$, $g_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.19}$$

System 17. Case $p_n = q_n = x_n$, $r_n = s_n = y_n$, $t_n = g_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.20}$$

System 18. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = y_n$, $g_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.21}$$

System 19. Case $p_n = q_n = r_n = t_n = g_n = x_n$, $s_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.22}$$

System 20. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = z_n$, $g_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.23}$$

System 21. Case $p_n = q_n = r_n = x_n$, $s_n = z_n$, $t_n = g_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.24}$$

System 22. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = g_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.25}$$

System 23. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = g_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.26}$$

System 24. Case $p_n = q_n = r_n = x_n$, $s_n = g_n = z_n$, $t_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.27}$$

System 25. Case $p_n = q_n = t_n = g_n = x_n$, $r_n = s_n = z_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.28}$$

System 26. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = z_n$, $g_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.29}$$

System 27. Case $p_n = q_n = x_n$, $r_n = s_n = z_n$, $t_n = g_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.30}$$

System 28. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.31}$$

System 29. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.32}$$

System 30. Case $p_n = q_n = x_n$, $r_n = s_n = g_n = z_n$, $t_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.33}$$

System 31. Case $p_n = q_n = t_n = g_n = x_n$, $s_n = z_n$, $r_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.34}$$

System 32. Case $p_n = q_n = t_n = x_n$, $s_n = z_n$, $r_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.35}$$

System 33. Case $p_n = q_n = x_n$, $s_n = z_n$, $r_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.36}$$

System 34. Case $p_n = q_n = t_n = x_n$, $s_n = g_n = z_n$, $r_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.37}$$

System 35. Case $p_n = q_n = x_n$, $s_n = t_n = g_n = z_n$, $r_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.38}$$

System 36. Case $p_n = q_n = x_n$, $s_n = g_n = z_n$, $r_n = t_n = y_n$.

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.39}$$

3 Main results

Here we analyse solvability of each of the systems (2.4)–(2.39), and as a consequence of the analysis, for each of them, we state the corresponding result on its solvability. For each system we also use (2.3).

System 1. By using the change of variables (2.2) system (2.4) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.1}$$

From the first equation in (3.1) we easily obtain

$$u_n = u_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.2}$$

By using (3.2) in the second and third equation in (3.1) we get

$$v_n = u_0^{2^n}, \quad w_n = u_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.3}$$

From (2.3), (3.2) and (3.3) we have that the following theorem holds.

Theorem 3.1. *If $a \neq 0$, then the general solution to system (2.4) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$
$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

Remark 3.2. Note that in the formulas in Theorem 3.1 do not participate the initial values y_0 and z_0, which is caused by the form of system (2.4). The situation that some of the initial values x_0, y_0, z_0, do not participate in the corresponding formulas appears also in several other systems considered in this paper. Such systems seem less interesting than the other ones. Nevertheless, we will also consider them.

System 2. By using the change of variables (2.2) system (2.5) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.4}$$

We have that (3.2) and the first equality in (3.3) hold. By using these relations in the third equation in (3.4) we have

$$w_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}} u_0^{2^{n-1}} = u_0^{2^n}, \quad n \geq 2. \tag{3.5}$$

From (2.3), (3.2), (3.3) and (3.5) we have that the following theorem holds.

Theorem 3.3. *If $a \neq 0$, then the general solution to system (2.5) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$
$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2.$$

Remark 3.4. Note that the solutions to systems (2.4) and (2.5) are not the same. Namely, the formula for z_n holds for $n \in \mathbb{N}$, that is, $n \geq 2$, respectively, whereas the values for z_1 can be different.

System 3. By using the change of variables (2.2) system (2.6) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.6}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.3) in the third equation in (3.6) we have

$$w_n = v_{n-1}^2 = (u_0^{2^{n-1}})^2 = u_0^{2^n}, \quad n \geq 2. \tag{3.7}$$

From (2.3), (3.2), (3.3) and (3.7) we have that the following theorem holds.

Theorem 3.5. *If $a \neq 0$, then the general solution to system (2.6) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2.$$

Remark 3.6. Note that the solutions to systems (2.5) and (2.6) are not the same. Namely, the corresponding values for z_1 can be different.

System 4. By using the change of variables (2.2) system (2.7) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.8}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.2) in the third equation in (3.8) we have

$$w_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}}w_{n-1} = w_0\prod_{j=0}^{n-1}u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1}2^j} = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.9}$$

From (2.3), (3.2), (3.3) and (3.9) we have that the following theorem holds.

Theorem 3.7. *If $a \neq 0$, then the general solution to system (2.7) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 5. By using the change of variables (2.2) system (2.8) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.10}$$

We have that (3.2) and the first equality in (3.3) hold, whereas from the third equation in (3.10) we have

$$w_n = w_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

From (2.3), (3.2), (3.3) and (3.11) we have that the following theorem holds.

Theorem 3.8. *If $a \neq 0$, then the general solution to system (2.8) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$
$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 6. By using the change of variables (2.2) system (2.9) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.12}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.3) in the third equation in (3.12) we have

$$w_n = v_{n-1}w_{n-1} = u_0^{2^{n-1}} w_{n-1} = w_1 \prod_{j=1}^{n-1} u_0^{2^j} = v_0 w_0 u_0^{\sum_{j=1}^{n-1} 2^j} = v_0 w_0 u_0^{2^n-2}, \tag{3.13}$$

for $n \in \mathbb{N}$.

From (2.3), (3.2), (3.3) and (3.13) we have that the following theorem holds.

Theorem 3.9. *If $a \neq 0$, then the general solution to system (2.9) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$
$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}.$$

System 7. By using the change of variables (2.2) system (2.10) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.14}$$

We have that (3.2) and the second equality in (3.3) hold. By using (3.2) in the second equation in (3.14) we have

$$v_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}} v_{n-1} = v_0 \prod_{j=0}^{n-1} u_0^{2^j} = v_0 u_0^{\sum_{j=0}^{n-1} 2^j} = v_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.15}$$

From (2.3), (3.2), (3.3) and (3.15) we have that the following theorem holds.

Theorem 3.10. *If $a \neq 0$, then the general solution to system (2.10) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 8. By using the change of variables (2.2) system (2.11) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.16}$$

We have that (3.2) and (3.15) hold. From this and since $v_n = w_n$, $n \in \mathbb{N}$, we have

$$w_n = v_0 u_0^{2^n-1}, \quad n \in \mathbb{N}. \tag{3.17}$$

From (2.3), (3.2), (3.15) and (3.17) we have that the following theorem holds.

Theorem 3.11. *If $a \neq 0$, then the general solution to system (2.11) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}.$$

System 9. By using the change of variables (2.2) system (2.12) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.18}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.18), we have

$$w_n = v_{n-1}^2 = v_0^2 u_0^{2^n-2}, \quad n \in \mathbb{N}. \tag{3.19}$$

From (2.3), (3.2), (3.15) and (3.19) we have that the following theorem holds.

Theorem 3.12. *If $a \neq 0$, then the general solution to system* (2.12) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^2+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^2-1}, \quad n \in \mathbb{N}.$$

System 10. By using the change of variables (2.2) system (2.13) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.20}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.20), we have

$$w_n = u_{n-1}w_{n-1} = w_0 \prod_{j=0}^{n-1} u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1} 2^j} = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.21}$$

From (2.3), (3.2), (3.15) and (3.21) we have that the following theorem holds.

Theorem 3.13. *If $a \neq 0$, then the general solution to system* (2.13) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 11. By using the change of variables (2.2) system (2.14) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.22}$$

We have that (3.2), (3.11), (3.15) hold, form which along with (2.3) it follows that the following theorem holds.

Theorem 3.14. *If $a \neq 0$, then the general solution to system* (2.14) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 12. By using the change of variables (2.2) system (2.15) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.23}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.23), we have

$$\begin{aligned} w_n &= v_{n-1}w_{n-1} = w_0\prod_{j=0}^{n-1} v_j = w_0\prod_{j=0}^{n-1} v_0 u_0^{2^j-1} \\ &= w_0 v_0^n u_0^{\sum_{j=0}^{n-1}(2^j-1)} = w_0 v_0^n u_0^{2^n-n-1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.24}$$

From (2.3), (3.2), (3.15) and (3.24) we have that the following theorem holds.

Theorem 3.15. *If $a \neq 0$, then the general solution to system* (2.15) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 13. By using the change of variables (2.2) system (2.16) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.25}$$

We have that (3.2) and the second equality in (3.3) hold, whereas from the second equation in (3.25) we have

$$v_n = v_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.26}$$

From (2.3), (3.2), (3.3) and (3.26) we have that the following theorem holds.

Theorem 3.16. *If $a \neq 0$, then the general solution to system* (2.16) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 14. By using the change of variables (2.2) system (2.17) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.27}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.27) we have

$$w_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}} v_0^{2^{n-1}}, \quad n \in \mathbb{N}. \tag{3.28}$$

From (2.3), (3.2), (3.26) and (3.28) we have that the following theorem holds.

Theorem 3.17. *If $a \neq 0$, then the general solution to system* (2.17) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}-1}, \quad n \in \mathbb{N}.$$

System 15. By using the change of variables (2.2) system (2.18) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.29}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.29) we have

$$w_n = v_{n-1}^2 = v_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.30}$$

From (2.3), (3.2), (3.26) and (3.30) we have that the following theorem holds.

Theorem 3.18. *If $a \neq 0$, then the general solution to system* (2.18) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 16. By using the change of variables (2.2) system (2.19) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.31}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.31) we have

$$w_n = u_{n-1}w_{n-1} = w_0 \prod_{j=0}^{n-1} u_j = w_0 \prod_{j=0}^{n-1} u_0^{2^j} = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.32}$$

From (2.3), (3.2), (3.26) and (3.32) we have that the following theorem holds.

Theorem 3.19. *If $a \neq 0$, then the general solution to system* (2.19) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 17. By using the change of variables (2.2) system (2.20) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.33}$$

We have that (3.2), (3.11) and (3.26) hold, from which the following theorem follows.

Theorem 3.20. *If $a \neq 0$, then the general solution to system* (2.20) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 18. By using the change of variables (2.2) system (2.21) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.34}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.34) we have

$$w_n = v_{n-1}w_{n-1} = w_0 \prod_{j=0}^{n-1} v_0^{2^j} = w_0 v_0^{\sum_{j=0}^{n-1} 2^j} = w_0 v_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.35}$$

From (2.3), (3.2), (3.26) and (3.35) we have that the following theorem holds.

Theorem 3.21. *If $a \neq 0$, then the general solution to system* (2.21) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 19. By using the change of variables (2.2) system (2.22) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.36}$$

We have that (3.2) and the second relation in (3.3) hold, whereas from the second equation in (3.36) we have

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} u_0^{2^{n-1}} = u_0^{2^n}, \quad n \geq 2. \tag{3.37}$$

From (2.3), (3.2), (3.3) and (3.37) we have that the following theorem holds.

Theorem 3.22. *If $a \neq 0$, then the general solution to system (2.22) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2,$$
$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 20. By using the change of variables (2.2) system (2.23) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.38}$$

We have that (3.2) holds. From the second and third relation in (3.38) we have

$$v_n = u_{n-1}w_{n-1} = u_{n-1}u_{n-2}v_{n-2} = u_0^{3\cdot 2^{n-2}}v_{n-2}, \quad n \geq 2,$$

from which we obtain

$$v_{2n} = u_0^{3\cdot 2^{2n-2}}v_{2n-2} = v_0\prod_{j=1}^{n}u_0^{3\cdot 2^{2j-2}} = v_0 u_0^{3\sum_{j=1}^{n}4^{j-1}} = v_0 u_0^{4^n-1}, \quad n \in \mathbb{N}_0, \tag{3.39}$$

and

$$v_{2n+1} = u_0^{3\cdot 2^{2n-1}}v_{2n-1} = v_1\prod_{j=1}^{n}u_0^{3\cdot 2^{2j-1}} = u_0 w_0 u_0^{6\sum_{j=1}^{n}4^{j-1}} = w_0 u_0^{2^{2n+1}-1}, \tag{3.40}$$

for $n \in \mathbb{N}_0$.

Further, by (3.2), (3.39) and (3.40), we have

$$w_{2n} = u_{2n-1}v_{2n-1} = u_0^{2^{2n-1}}w_0 u_0^{2^{2n-1}-1} = w_0 u_0^{4^n-1}, \quad n \in \mathbb{N}_0, \tag{3.41}$$

and

$$w_{2n+1} = u_{2n}v_{2n} = u_0^{2^{2n}}v_0 u_0^{4^n-1} = v_0 u_0^{2^{2n+1}-1}, \quad n \in \mathbb{N}_0. \tag{3.42}$$

From (2.3), (3.2), (3.39)–(3.42) we have that the following theorem holds.

Theorem 3.23. *If $a \neq 0$, then the general solution to system (2.23) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 21. By using the change of variables (2.2) system (2.24) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.43}$$

We have that (3.2) holds. From the second and third relation in (3.43) we have

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} v_{n-2}^2, \quad n \geq 2, \tag{3.44}$$

from which we obtain

$$\begin{aligned} v_{2n} &= u_0^{2^{2n-1}} v_{2n-2}^2 = u_0^{2^{2n-1}} (u_0^{2^{2n-3}} v_{2n-4}^2)^2 = u_0^{2^{2n-1}+2^{2n-2}} v_{2n-4}^{2^2} \\ &= u_0^{2^{2n-1}+2^{2n-2}} (u_0^{2^{2n-5}} v_{2n-6}^2)^{2^2} = u_0^{2^{2n-1}+2^{2n-2}+2^{2n-3}} v_{2n-6}^{2^3}. \end{aligned}$$

Assume that we have proved

$$v_{2n} = u_0^{2^{2n-1}+2^{2n-2}+\cdots+2^{n+1}+2^n} v_0^{2^n} = u_0^{2^n(2^n-1)} v_0^{2^n}, \tag{3.45}$$

for an $n \in \mathbb{N}$.

Then by using (3.44) and (3.45) we have

$$v_{2n+2} = u_0^{2^{2n+1}} v_{2n}^2 = u_0^{2^{2n+1}} (u_0^{2^n(2^n-1)} v_0^{2^n})^2 = u_0^{2^{n+1}(2^{n+1}-1)} v_0^{2^{n+1}},$$

from which along with the method of induction it follows that formula (3.45) holds for every $n \in \mathbb{N}$. In fact, a simple calculation shows that it also holds for $n = 0$.

Further, we have

$$v_{2n+1} = u_0^{2^{2n}} v_{2n-1}^2 = u_0^{2^{2n}} (u_0^{2^{2n-2}} v_{2n-3}^2)^2 = u_0^{2^{2n}+2^{2n-1}} v_{2n-3}^{2^2}, \quad n \geq 2.$$

Assume that we have proved

$$\begin{aligned} v_{2n+1} &= u_0^{2^{2n}+2^{2n-1}+\cdots+2^{n+2}+2^{n+1}} v_1^{2^n} = u_0^{2^{n+1}(2^n-1)}(u_0w_0)^{2^n} \\ &= u_0^{2^n(2^{n+1}-1)} w_0^{2^n}, \end{aligned} \tag{3.46}$$

for an $n \in \mathbb{N}_0$.

Then by using (3.44) and (3.46) we have

$$v_{2n+3} = u_0^{2^{2n+2}} v_{2n+1}^2 = u_0^{2^{2n+2}} (u_0^{2^n(2^{n+1}-1)} w_0^{2^n})^2 = u_0^{2^{n+1}(2^{n+2}-1)} w_0^{2^{n+1}},$$

from which along with the method of induction it follows that formula (3.46) holds for every $n \in \mathbb{N}_0$.

By using (3.45) and (3.46) into the third equation in (3.43) we get

$$w_{2n} = v_{2n-1}^2 = (u_0^{2^{n-1}(2^n-1)} w_0^{2^{n-1}})^2 = u_0^{2^n(2^n-1)} w_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.47}$$

and

$$w_{2n+1} = v_{2n}^2 = (u_0^{2^n(2^n-1)} v_0^{2^n})^2 = u_0^{2^{n+1}(2^n-1)} v_0^{2^{n+1}}, \quad n \in \mathbb{N}_0. \tag{3.48}$$

From (2.3), (3.2), (3.45)–(3.48) we have that the following theorem holds.

Theorem 3.24. *If $a \neq 0$, then the general solution to system (2.24) is*

$$\begin{aligned} x_n &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ y_{2n} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ y_{2n+1} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ z_{2n} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ z_{2n+1} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n+1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n+1}}-1}, \quad n \in \mathbb{N}_0. \end{aligned}$$

System 22. By using the change of variables (2.2) system (2.25) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.49}$$

We have that (3.2) holds and that $v_n = w_n$, $n \in \mathbb{N}$. Hence

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} v_{n-1} = v_1 \prod_{j=1}^{n-1} u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1} 2^j} = w_0 u_0^{2^n-1} \tag{3.50}$$

for $n \in \mathbb{N}$, and consequently

$$w_n = w_0 u_0^{2^n - 1}, \quad n \in \mathbb{N}. \tag{3.51}$$

In fact, a simple calculation shows that (3.51) also holds for $n = 0$.

From (2.3), (3.2), (3.50) and (3.51) we have that the following theorem holds.

Theorem 3.25. *If $a \neq 0$, then the general solution to system (2.25) is*

$$x_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) - 1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) - 1}, \quad n \in \mathbb{N}_0.$$

System 23. By using the change of variables (2.2) system (2.26) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.52}$$

We have that (3.2) and (3.11) hold, from which it follows that

$$v_n = u_{n-1} w_{n-1} = u_0^{2^{n-1}} w_0^{2^{n-1}} = (u_0 w_0)^{2^{n-1}}, \quad n \in \mathbb{N}. \tag{3.53}$$

From (2.3), (3.2), (3.11) and (3.53) we have that the following theorem holds.

Theorem 3.26. *If $a \neq 0$, then the general solution to system (2.26) is*

$$x_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a} \frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0.$$

System 24. By using the change of variables (2.2) system (2.27) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.54}$$

We have that (3.2) holds. From this, the second and third equation in (3.54) we get

$$w_n = v_{n-1} w_{n-1} = w_{n-1} w_{n-2} u_{n-2} = w_{n-1} w_{n-2} u_0^{2^{n-2}}, \quad n \geq 2.$$

By using the change of variables

$$w_n = \zeta_n u_0^{\alpha_n}, \quad n \in \mathbb{N}_0, \tag{3.55}$$

the last equation becomes

$$\zeta_n = \zeta_{n-1}\zeta_{n-2}u_0^{\alpha_{n-1}+\alpha_{n-2}+2^{n-2}-\alpha_n}, \quad n \geq 2. \tag{3.56}$$

Since $w_0 = w_0$ and $w_1 = v_0 w_0$, and they do not contain u_0, we may take

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_1 = 0. \tag{3.57}$$

Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be the solution to the difference equation

$$\alpha_n = \alpha_{n-1} + \alpha_{n-2} + 2^{n-2}, \quad n \geq 2, \tag{3.58}$$

satisfying the initial conditions in (3.57).

We find a particular solution to equation (3.58) in the form $\alpha_n^p = c2^n$, $n \in \mathbb{N}_0$, where c is a constant ([17]). By employing it in (3.58) we have that it must be

$$c2^n = c(2^{n-1} + 2^{n-2}) + 2^{n-2}, \quad n \in \mathbb{N}$$

from which it follows that $c = 1$. Hence, the general solution to (3.58) has the form

$$\alpha_n = c_1\lambda_1^n + c_2\lambda_2^n + 2^n, \quad n \in \mathbb{N}, \tag{3.59}$$

where λ_1 and λ_2 are the roots of the polynomial $P_2(\lambda) = \lambda^2 - \lambda - 1$.

From (3.57) and (3.59) we have

$$c_1 + c_2 = -1 \quad \text{and} \quad c_1\lambda_1 + c_2\lambda_2 = -2$$

from which it follows that

$$c_1 = \frac{1}{\lambda_2 - \lambda_1}\begin{vmatrix} -1 & 1 \\ -2 & \lambda_2 \end{vmatrix} = \frac{2-\lambda_2}{\lambda_2-\lambda_1} \quad \text{and} \quad c_2 = \frac{1}{\lambda_2 - \lambda_1}\begin{vmatrix} 1 & -1 \\ \lambda_1 & -2 \end{vmatrix} = \frac{\lambda_1 - 2}{\lambda_2 - \lambda_1},$$

from which along with (3.59) we have

$$\alpha_n = \frac{(2-\lambda_2)\lambda_1^n + (\lambda_1 - 2)\lambda_2^n}{\lambda_2 - \lambda_1} + 2^n, \quad n \in \mathbb{N}_0. \tag{3.60}$$

For such a chosen sequence α_n, we have that $(\zeta_n)_{n\in\mathbb{N}_0}$ satisfies the equation

$$\zeta_n = \zeta_{n-1}\zeta_{n-2}, \quad n \geq 2, \tag{3.61}$$

with the initial conditions

$$\zeta_0 = w_0 \quad \text{and} \quad \zeta_1 = v_0 w_0. \tag{3.62}$$

Let $a_1 = b_1 = 1$. Then we have

$$\zeta_n = \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1} = (\zeta_{n-2}\zeta_{n-3})^{a_1}\zeta_{n-2}^{b_1} = \zeta_{n-2}^{a_1+b_1}\zeta_{n-3}^{a_1} = \zeta_{n-2}^{a_2}\zeta_{n-3}^{b_2},$$

where $a_2 = a_1 + b_1$ and $b_2 = a_1$. By using a simple inductive argument we obtain

$$\zeta_n = \zeta_{n-k}^{a_k}\zeta_{n-k-1}^{b_k}$$

for $1 \le k \le n-1$, and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = a_{k-1}. \tag{3.63}$$

The relations in (3.63) hold for every $k \in \mathbb{Z}$.

Hence for $k = n-1$ is obtained

$$\zeta_n = \zeta_1^{a_{n-1}} \zeta_0^{b_{n-1}} = \zeta_1^{a_{n-1}} \zeta_0^{a_{n-2}}, \quad n \in \mathbb{N}, \tag{3.64}$$

and also

$$a_n = a_{n-1} + a_{n-2}, \quad n \ge 3.$$

From this and since $a_1 = 1$ and $a_2 = 2$, we have $a_n = f_{n+1}$ and $b_n = f_n$, where f_n is the Fibonacci sequence ([44]).

From (3.62), (3.63) and (3.64) we have

$$\zeta_n = (v_0 w_0)^{f_n} w_0^{f_{n-1}} = v_0^{f_n} w_0^{f_n + f_{n-1}} = v_0^{f_n} w_0^{f_{n+1}},$$

from which together with (3.55) we obtain

$$w_n = v_0^{f_n} w_0^{f_{n+1}} u_0^{\alpha_n}, \quad n \in \mathbb{N}_0. \tag{3.65}$$

By using (3.2) and (3.65) in the second equation in (3.54) we get

$$v_n = u_{n-1} w_{n-1} = u_0^{\alpha_{n-1} + 2^{n-1}} v_0^{f_{n-1}} w_0^{f_n}, \quad n \in \mathbb{N}. \tag{3.66}$$

A simple calculation shows that (3.66) holds also for $n = 0$.

From (2.3), (3.2), (3.65), (3.66) we have that the following theorem holds.

Theorem 3.27. *If $a \neq 0$, then the general solution to system (2.27) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n} - 1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n+1}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n+1}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ is given by (3.60).

System 25. By using the change of variables (2.2) system (2.28) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.67}$$

We have that (3.2) and the second relation in (3.3) hold. Hence

$$v_n = w_{n-1}^2 = u_0^{2^n}, \quad n \ge 2. \tag{3.68}$$

From (2.3), (3.2), (3.3), (3.68) we have that the following theorem holds.

Theorem 3.28. *If $a \neq 0$, then the general solution to system (2.28) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 26. By using the change of variables (2.2) system (2.29) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.69}$$

This system is obtained from system (3.43) by interchanging letters v and w. Hence, from (3.45)–(3.48) we have

$$v_{2n} = u_0^{2^n(2^n-1)} v_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.70}$$

$$v_{2n+1} = u_0^{2^{n+1}(2^n-1)} w_0^{2^{n+1}}, \quad n \in \mathbb{N}_0, \tag{3.71}$$

$$w_{2n} = u_0^{2^n(2^n-1)} w_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.72}$$

$$w_{2n+1} = u_0^{2^n(2^{n+1}-1)} v_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.73}$$

From (2.3), (3.2), (3.70)–(3.73) we have that the following theorem holds.

Theorem 3.29. *If $a \neq 0$, then the general solution to system (2.29) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n+1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n+1}}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 27. By using the change of variables (2.2) system (2.30) is transformed to

$$u_{n+1}=u_n^2,\quad v_{n+1}=w_n^2,\quad w_{n+1}=v_n^2,\quad n\in\mathbb{N}_0. \tag{3.74}$$

We have that (3.2) holds. From the second and third equation in (3.74) we have

$$v_n=w_{n-1}^2=v_{n-2}^4,\quad n\ge 2,$$

from which it follows that

$$v_{2n}=v_{2n-2}^4=v_0^{4^n},\quad n\in\mathbb{N}_0, \tag{3.75}$$

and

$$v_{2n+1}=v_{2n-1}^4=v_1^{4^n}=w_0^{2\cdot 4^n},\quad n\in\mathbb{N}_0, \tag{3.76}$$

By using (3.75) and (3.76) in the third equation in (3.74) we get

$$w_{2n}=v_{2n-1}^2=(w_0^{2\cdot 4^{n-1}})^2=w_0^{4^n},\quad n\in\mathbb{N}_0, \tag{3.77}$$

and

$$w_{2n+1}=v_{2n}^2=(v_0^{4^n})^2=v_0^{2\cdot 4^n},\quad n\in\mathbb{N}_0. \tag{3.78}$$

From (2.3), (3.2), (3.75)–(3.78) we have that the following theorem holds.

Theorem 3.30. *If $a\ne 0$, then the general solution to system (2.30) is*

$$x_n=\sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1},\quad n\in\mathbb{N}_0,$$

$$y_{2n}=\sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{4^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{4^n}-1},\quad n\in\mathbb{N}_0,$$

$$y_{2n+1}=\sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2\cdot 4^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2\cdot 4^n}-1},\quad n\in\mathbb{N}_0,$$

$$z_{2n}=\sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{4^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{4^n}-1},\quad n\in\mathbb{N}_0,$$

$$z_{2n+1}=\sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2\cdot 4^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2\cdot 4^n}-1},\quad n\in\mathbb{N}_0.$$

System 28. By using the change of variables (2.2) system (2.31) is transformed to

$$u_{n+1}=u_n^2,\quad v_{n+1}=w_n^2,\quad w_{n+1}=u_nw_n,\quad n\in\mathbb{N}_0. \tag{3.79}$$

We have that (3.2) holds. From this and the third equation in (3.79) we have

$$w_n=u_{n-1}w_{n-1}=u_0^{2^{n-1}}w_{n-1}=w_0\prod_{j=0}^{n-1}u_0^{2^j}=w_0u_0^{\sum_{j=0}^{n-1}2^j}=w_0u_0^{2^n-1}, \tag{3.80}$$

for $n\in\mathbb{N}_0$, from which and the second equation in (3.79) it follows that

$$v_n=w_{n-1}^2=(w_0u_0^{2^{n-1}-1})^2=w_0^2u_0^{2^n-2},\quad n\in\mathbb{N}. \tag{3.81}$$

From (2.3), (3.2), (3.80), (3.81) we have that the following theorem holds.

Theorem 3.31. *If $a \neq 0$, then the general solution to system (2.31) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 29. By using the change of variables (2.2) system (2.32) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.82}$$

We have that (3.2) and (3.11) hold, and consequently

$$v_n = w_{n-1}^2 = w_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.83}$$

From (2.3), (3.2), (3.11), (3.83) we have that the following theorem holds.

Theorem 3.32. *If $a \neq 0$, then the general solution to system (2.32) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 30. By using the change of variables (2.2) system (2.33) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.84}$$

We have that (3.2) holds. From the second and third equation in (3.84) we have

$$w_n = v_{n-1}w_{n-1} = w_{n-1}w_{n-2}^2, \quad n \geq 2. \tag{3.85}$$

Let $a_1 = 1$ and $b_1 = 2$. Then we have

$$w_n = w_{n-1}^{a_1}w_{n-2}^{b_1} = (w_{n-2}w_{n-3}^2)^{a_1}w_{n-2}^{b_1} = w_{n-2}^{a_1+b_1}w_{n-3}^{2a_1} = w_{n-2}^{a_2}w_{n-3}^{b_2},$$

where $a_2 = a_1 + b_1$ and $b_2 = 2a_1$. By using a simple inductive argument we obtain

$$w_n = w_{n-k}^{a_k}w_{n-k-1}^{b_k}$$

for $1 \le k \le n-1$, and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = 2a_{k-1}. \tag{3.86}$$

Hence for $k = n-1$ we get

$$w_n = w_1^{a_{n-1}} w_0^{b_{n-1}} = (v_0 w_0)^{a_{n-1}} w_0^{2a_{n-2}} = v_0^{a_{n-1}} w_0^{a_n}, \quad n \in \mathbb{N}_0, \tag{3.87}$$

and

$$a_n = a_{n-1} + 2a_{n-1}, \quad n \ge 2. \tag{3.88}$$

In fact, (3.88) holds for each $n \in \mathbb{Z}$.

The characteristic polynomial associated to equation (3.88) is $\widehat{P}_2(\lambda) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$. Hence, general solution to the equation is

$$a_n = c_1(-1)^n + c_2 2^n.$$

From this and since $a_0 = a_1 = 1$, we have

$$c_1 + c_2 = 1, \quad -c_1 + 2c_2 = 1$$

from which it follows that $c_1 = 1/3$ and $c_2 = 2/3$. Hence

$$a_n = \frac{2^{n+1} + (-1)^n}{3}, \quad n \in \mathbb{N}_0,$$

from which together with (3.87) it follows that

$$w_n = v_0^{\frac{2^n - (-1)^n}{3}} w_0^{\frac{2^{n+1} + (-1)^n}{3}}, \quad n \in \mathbb{N}_0. \tag{3.89}$$

By using (3.89) in the second equation in (3.84) we get

$$v_n = w_{n-1}^2 = v_0^{\frac{2^n + 2(-1)^n}{3}} w_0^{\frac{2^{n+1} - 2(-1)^n}{3}}, \quad n \in \mathbb{N}. \tag{3.90}$$

Direct calculation shows that this formula also holds for $n = 0$.

From (2.3), (3.2), (3.89), (3.90) we have that the following theorem holds.

Theorem 3.33. *If $a \ne 0$, then the general solution to system (2.33) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}} - 1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}} - 1}, \quad n \in \mathbb{N}_0.$$

System 31. By using the change of variables (2.2) system (2.34) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.91}$$

We have that (3.2) and the second equality in (3.3) hold. By using these formulas we have

$$v_n = w_{n-1}v_{n-1} = v_1 \prod_{j=1}^{n-1} u_0^{2^j} = v_0 w_0 u_0^{\sum_{j=1}^{n-1} 2^j} = v_0 w_0 u_0^{2^n-2}, \quad n \in \mathbb{N}. \tag{3.92}$$

From (2.3), (3.2), (3.3), (3.92) we have that the following theorem holds.

Theorem 3.34. *If $a \neq 0$, then the general solution to system (2.34) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 32. By using the change of variables (2.2) system (2.35) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.93}$$

This system is obtained from system (3.54) by interchanging letters v and w. Hence, from (3.65) and (3.66) we have

$$v_n = v_0^{f_{n+1}} w_0^{f_n} u_0^{\alpha_n}, \quad n \in \mathbb{N}_0, \tag{3.94}$$

$$w_n = u_0^{\alpha_{n-1}+2^{n-1}} v_0^{f_n} w_0^{f_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.95}$$

From (2.3), (3.2), (3.94), (3.95) we have that the following theorem holds.

Theorem 3.35. *If $a \neq 0$, then the general solution to system (2.35) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n+1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n+1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n-1}}-1}, \quad n \in \mathbb{N}_0.$$

where the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ is given by (3.60).

System 33. By using the change of variables (2.2) system (2.36) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.96}$$

This system is obtained from system (3.84) by interchanging letters v and w. Hence, from (3.89) and (3.90) we have

$$v_n = v_0^{\frac{2^{n+1}+(-1)^n}{3}} w_0^{\frac{2^n-(-1)^n}{3}}, \quad n \in \mathbb{N}_0, \tag{3.97}$$

$$w_n = v_0^{\frac{2^{n+1}-2(-1)^n}{3}} w_0^{\frac{2^n+2(-1)^n}{3}}, \quad n \in \mathbb{N}_0. \tag{3.98}$$

From (2.3), (3.2), (3.97), (3.98) we have that the following theorem holds.

Theorem 3.36. *If $a \neq 0$, then the general solution to system (2.36) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}}-1}, \quad n \in \mathbb{N}_0.$$

System 34. By using the change of variables (2.2) system (2.37) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.99}$$

This system is obtained from system (3.23) by interchanging letters v and w. Hence, from (3.15) and (3.24) we have

$$v_n = v_0 w_0^n u_0^{2^n-n-1}, \quad n \in \mathbb{N}_0, \tag{3.100}$$

$$w_n = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.101}$$

From (2.3), (3.2), (3.100), (3.101) we have that the following theorem holds.

Theorem 3.37. *If $a \neq 0$, then the general solution to system (2.37) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 35. By using the change of variables (2.2) system (2.38) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.102}$$

We have that (3.2) and (3.11) hold. By using (3.11) in the second equation in (3.102) we have

$$v_n = v_{n-1} w_{n-1} = v_0 \prod_{j=0}^{n-1} w_0^{2^j} = v_0 w_0^{2^n - 1}, \quad n \in \mathbb{N}_0. \tag{3.103}$$

From (2.3), (3.2), (3.11), (3.103) we have that the following theorem holds.

Theorem 3.38. *If $a \neq 0$, then the general solution to system (2.38) is*

$$x_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a} \frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n-1} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n-1} - 1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a} \frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0.$$

System 36. By using the change of variables (2.2) system (2.39) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.104}$$

We have that (3.2) holds and that $v_n = w_n$, $n \in \mathbb{N}$. By using the last relation in the second equation in (3.104) we have

$$v_n = v_{n-1} w_{n-1} = v_{n-1}^2, \quad n \geq 2. \tag{3.105}$$

Hence

$$v_n = v_1^{2^{n-1}} = (v_0 w_0)^{2^{n-1}}, \quad n \in \mathbb{N}, \tag{3.106}$$

and consequently

$$w_n = (v_0 w_0)^{2^{n-1}}, \quad n \in \mathbb{N}. \tag{3.107}$$

From (2.3), (3.2), (3.106), (3.107) we have that the following theorem holds.

Theorem 3.39. *If $a \neq 0$, then the general solution to system (2.39) is*

$$x_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a} \frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a} \frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \in \mathbb{N}.$$

References

[1] N. H. Abel, Mémoire sur les équations algébriques, ou l'on démontre l'impossibilité de la résolution de l'équation générale du cinquième degré (in French), in: L. Sylow, S. Lie (Eds.), *Œvres complètes de Niels Henrik Abel, I* (2nd ed.), Grondahl & Son, 28–33 (1881) [1824].

[2] M. I. Bashmakov, B. M. Bekker, V. M. Gol'hovoi, *Zadachi po Matematike. Algebra i Analiz* (in Russian), Nauka, Moscow, 1982.

[3] D. Bernoulli, Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur (in Latin), *Commentarii Acad. Petropol. III,* 1728 (1732), 85–100.

[4] G. Boole, *A Treatise on the calculus of finite differences*, Third Edition, Macmillan and Co., London, 1880.

[5] A. de Moivre, *Miscellanea analytica de seriebus et quadraturis* (in Latin), J. Tonson & J. Watts, Londini, 1730.

[6] A. de Moivre, *The doctrine of chances*, 3rd edition, Strand Publishing, London, 1756.

[7] L. Euler, *Introductio in analysin infinitorum, Tomus primus* (in Latin), Lausannae, 1748.

[8] T. Fort, *Finite differences and difference equations in the real domain*, Oxford Univ. Press, London, 1948. MR0024567

[9] C. Jordan, *Calculus of finite differences*, Chelsea Publishing Company, New York, 1956.

[10] S. F. Lacroix, *Traité des differénces et des séries* (in French), Paris, 1800.

[11] S. F. Lacroix, *An elementary treatise on the differential and integral calculus, with an appendix and notes by J. Herschel*, J. Smith, Cambridge, 1816.

[12] J.-L. Lagrange, Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes (in French), *Miscellanea Taurinensia*, t. I, (1759), 33–42 (Lagrange Œuvres, I, 23–36, 1867).

[13] J.-L. Lagrange, *Œuvres* (in French), t. II, Gauthier-Villars, Paris, 1868.

[14] P. S. Laplace, Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards (in French), *Mémoires de l'Académie Royale des Sciences de Paris* 1773, t. VII, (1776) (Laplace Œuvres, VIII, 69–197, 1891).

[15] A. A. Markoff, *Differenzenrechnung* (in German), Teubner, Leipzig, 1896.

[16] L. M. Milne-Thomson, *The calculus of finite differences*, MacMillan and Co., London, 1933. MR0043339

[17] D. S. Mitrinović, J. D. Kečkić, *Metodi izračunavanja konačnih zbirova* [Methods for calculating finite sums] (in Serbian), Naučna Knjiga, Beograd, 1984.

[18] N. E. Nörlund, *Vorlesungen über Differenzenrechnung* (in German), Berlin, Springer, 1924. https://doi.org/10.1007/978-3-642-50824-0

[19] G. Papaschinopoulos, C. J. Schinas, On a system of two nonlinear difference equations, *J. Math. Anal. Appl.* **219**(1998), No. 2, 415–426. https://doi.org/10.1006/jmaa.1997.5829; MR1606350

[20] G. Papaschinopoulos, C. J. Schinas, On the behavior of the solutions of a system of two nonlinear difference equations, *Comm. Appl. Nonlinear Anal.* **5**(1998), No. 2, 47–59. MR1621223

[21] G. Papaschinopoulos, C. J. Schinas, Invariants for systems of two nonlinear difference equations, *Differential Equations Dynam. Systems* **7**(1999), 181–196. MR1860787

[22] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal.* **46**(2001), 967–978. https://doi.org/10.1016/S0362-546X(00)00146-2; MR1866733

[23] G. Papaschinopoulos, C. J. Schinas, Oscillation and asymptotic stability of two systems of difference equations of rational form, *J. Differ. Equations Appl.* **7**(2001), 601–617. https://doi.org/10.1080/10236190108808290; MR1922592

[24] G. Papaschinopoulos, C. J. Schinas, On the system of two difference equations $x_{n+1} = \sum_{i=0}^{k} A_i/y_{n-i}^{p_i}, y_{n+1} = \sum_{i=0}^{k} B_i/x_{n-i}^{q_i}$, *J. Math. Anal. Appl.* **273**(2002), No. 2, 294–309. https://doi.org/10.1016/S0022-247X(02)00223-8; MR1932490

[25] G. Papaschinopoulos, C. J. Schinas, G. Stefanidou, On a k-order system of Lyness-type difference equations, *Adv. Difference Equ.* **2007**, Article ID 31272, 13 pp. https://doi.org/10.1155/2007/31272; MR2322487

[26] G. Papaschinopoulos, G. Stefanidou, Asymptotic behavior of the solutions of a class of rational difference equations, *Inter. J. Difference Equations* **5**(2010), No. 2, 233–249. MR2771327

[27] G. Papaschinopoulos, C. J. Schinas, On the dynamics of two exponential type systems of difference equations, *Comput. Math. Appl.* **64**(2012), No. 7, 2326–2334. https://doi.org/10.1016/j.camwa.2012.04.002; MR2966868

[28] M. H. Rhouma, The Fibonacci sequence modulo π, chaos and some rational recursive equations, *J. Math. Anal. Appl.* **310**(2005), 506–517. https://doi.org/10.1016/j.jmaa.2005.02.038; MR2022941

[29] J. Riordan, *Combinatorial identities*, John Wiley & Sons Inc., New York–London–Sydney, 1968. MR0231725

[30] C. Schinas, Invariants for difference equations and systems of difference equations of rational form, *J. Math. Anal. Appl.* **216**(1997), 164–179. https://doi.org/10.1006/jmaa.1997.5667; MR1487258

[31] C. Schinas, Invariants for some difference equations, *J. Math. Anal. Appl.* **212**(1997), 281–291. https://doi.org/10.1006/jmaa.1997.5499; MR1460198

[32] S. Stević, Solvable product-type system of difference equations whose associated polynomial is of the fourth order, *Electron. J. Qual. Theory Differ. Equ.* **2017**, No. 13, 1–29. https://doi.org/10.14232/ejqtde.2017.1.13; MR3633243

[33] S. Stević, Sixteen practically solvable systems of difference equations, *Adv. Difference Equ.* **2019**, Paper No. 467, 32 pp. https://doi.org/10.1186/s13662-019-2388-6; MR4035930

[34] S. Stević, Solvability of a general class of two-dimensional hyperbolic-cotangent-type systems of difference equations, *Adv. Difference Equ.* **2019**, Paper No. 294, 34 pp. https://doi.org/10.1186/s13662-019-2233-y; MR3984141

[35] S. Stević, Solvability of some classes of nonlinear first-order difference equations by invariants and generalized invariants, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 36, 1–21. https://doi.org/10.14232/ejqtde.2019.1.36; MR3953828

[36] S. Stević, A note on general solutions to a hyperbolic-cotangent class of systems of difference equations, *Adv. Difference Equ.* **2020**, Paper No. 693, 12 pp. https://doi.org/10.1186/s13662-020-03155-1; MR4185156

[37] S. Stević, New class of practically solvable systems of difference equations of hyperbolic-cotangent-type, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 89, 1–25. https://doi.org/10.14232/ejqtde.2020.1.89; MR4208496

[38] S. Stević, General solutions to subclasses of a two-dimensional class of systems of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 12, 1–27. https://doi.org/10.14232/ejqtde.2021.1.1; MR4222726

[39] S. Stević, New classes of hyperbolic-cotangent-type systems of difference equations solvable in closed form, *Math. Methods Appl. Sci.* **44**(2021), 3646–3669. https://doi.org/10.1002/mma.6972; MR4227950

[40] S. Stević, B. Iričanin, W. Kosmala, More on a hyperbolic-cotangent class of difference equations, *Math. Methods Appl. Sci.* **42**(2019), 2974–2992. https://doi.org/10.1002/mma.5541; MR3949546

[41] S. Stević, B. Iričanin, W. Kosmala, Z. Šmarda, Note on the bilinear difference equation with a delay, *Math. Methods Appl. Sci.* **41**(2018), 9349–9360. https://doi.org/10.1002/mma.5293; MR3897790

[42] S. Stević, D. T. Tollu, Solvability and semi-cycle analysis of a class of nonlinear systems of difference equations, *Math. Methods Appl. Sci.* **42**(2019), 3579–3615. https://doi.org/10.1002/mma.5600; MR3961512

[43] S. Stević, D. T. Tollu, Solvability of eight classes of nonlinear systems of difference equations, *Math. Methods Appl. Sci.* **42**(2019), 4065–4112. https://doi.org/10.1002/mma.5625; MR3978642

[44] N. N. Vorobiev, *Fibonacci numbers*, Birkhäuser, Basel, 2002. https://doi.org/10.1007/978-3-0348-8107-4; MR1954396