Limit cycles of planar piecewise linear Hamiltonian differential systems with two or three zones

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Abstract. In this paper, we study the existence of limit cycles in continuous and discontinuous planar piecewise linear Hamiltonian differential system with two or three zones separated by straight lines and such that the linear systems that define the piecewise one have isolated singular points, i.e. centers or saddles. In this case, we show that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones, then it has no limit cycles. Now, if the planar piecewise linear Hamiltonian differential system is discontinuous with three zones, then it has at most one limit cycle, and there are examples with one limit cycle. More precisely, without taking into account the position of the singular points in the zones, we present examples with the unique limit cycle for all possible combinations of saddles and centers.

Keywords: limit cycles, piecewise linear differential system, Hamiltonian systems.

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1 Introduction

The first works on piecewise differential systems appeared in the 1930s, see [1]. This class of systems is very important due to numerous applications, for example in control theory, mechanics, electrical circuits, neurobiology, etc (see for instance the book [7]). Recently, this subject has piqued the attention of researchers in qualitative theory of differential equations and numerous studies about this topic have arisen in the literature (see [6,15,19,20,30]).

Piecewise linear differential systems are an interesting class of piecewise differential systems and, unlike the smooth case, have a rich dynamic that is far from being fully understood. In addition to numerous applications in various areas of knowledge. In 1990, Lum and Chua [28] conjectured that a continuous piecewise differential systems in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [9]. The problem becomes more complicated when we have three zones. Conditions for non existence and existence of one, two or three limit cycles have been obtained,

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see [10, 23, 32]. However, the maximum number of limit cycles, as far as we know, is not yet known.

In the discontinuous case, the maximum number of limit cycles is not known even in the simplest case, i.e. for piecewise linear differential systems with two zones separated by a straight line. However, important partial results about this problem have been obtained. In summary, the results about the number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line are given in Table 1.1. The symbol "—" indicates that those cases appear repeated in the table and the empty entries on it correspond to cases not studied in the literature, at least as far as we know.

	F_r	F_v	F_b	S_r	S_r^0	N_v	iN_v	C	C_b
F_r		3	2*	3	2*	3	3	2*	2*
F_v	—	2	2*	2	1*		2	1*	1*
F_b	—	—	1*	2*	1*	2*	2*	1*	1*
S_r	—	—	—	2*	1*	2	2	1*	1*
S_r^0	—	—	—	—	0*	1*	1*	0*	0*
N_v	—	—	—	—	—			1*	1*
iN_v	—	—	—	—	—	—	2	1*	1*
C	—	—	—	—	—	—	—	0*	0*
C_b	—	—	—	—	—	—	—	—	0*

Table 1.1: Lower bounds (Upper bounds*) of the maximum number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line. Here F_r, F_v, F_b, S_r, S_r^0, N_v, iN_v, C and C_b denote real focus, virtual focus, boundary focus, real saddle, real saddle with zero trace, virtual node, improper node, center and boundary center, respectively.

We denote the lower bounds of the entrances from Table 1.1 by the symbols that indicate its position on the table. For example, the lower bound for the case with a real focus F_r and a virtual focus F_v is detonated by $F_r F_v$, i.e. $F_r F_v = 3$. A proof for the lower bound $F_r F_v$ can be found in [22]. A proof for the lower bound $F_r S_r$ can be found in [18]. A proof for the lower bounds $F_r N_v$ and $F_r iN_v$ can be found in [12]. A proof for the lower bound $F_v F_v$ can be found in [11]. A proof for the lower bound $F_v S_r$ can be found in [36]. A proof for the lower bound $F_v iN_v$ can be found in [35]. A proof for the upper bound $S_r S_r$ can be found in [2]. A proof for the lower bounds $S_r N_v$ and $S_r iN_v$ can be found in [26]. A proof for the lower bound $iN_v iN_v$ can be found in [17]. The other cases listed in Table 1.1 can be found in [21]. In the papers [3, 5, 13, 27, 34] we can also find proofs for some lower bounds of Table 1.1.

If the curve between two linear zones is not a straight line it is possible to obtain as many cycles as you want. This fact has been conjectured by Braga and Mello in [4] and firstly proved by Novaes and Ponce in [31]. Exact number of limit cycles, for discontinuous piecewise linear systems with two zones separated by a straight line, were obtained in particular cases. Llibre and Teixeira [24] proved that if the linear systems, that define the piecewise one, has no singular point, then it has at most one limit cycle. Medrado and Torregrossa [29] proved that if the straight line has only crossing sewing points and the piecewise linear system has only a monodronic singular point on it, then the system has at most one limit cycle.

There are a few papers on discontinuous piecewise linear systems with three zones separated by two straight lines (see [8, 25, 38, 39]). In [25], Llibre and Teixeira study the existence of

limit cycles for continuous and discontinuous planar piecewise linear differential system with three zones separated by two parallel straight lines and such that the linear systems involved have a unique singular point which are centers. More precisely, in the continuous case, they prove that the piecewise system has no limit cycles. Now, in the discontinuous case, the piecewise system has at most one limit cycle and there are examples with one. Mello, Llibre and Fonseca, in [8], propose a mix of [24] and [25]. They proved that a piecewise linear Hamiltonian systems with three zones separated by two parallel straight lines without singular points have at most one crossing limit cycle.

In this paper, we contribute along these lines, that is, we are interested in studying the existence and the number of limit cycles of piecewise linear differential systems with two or three zones in the plane with the following hypotheses:

(H1) The separations curves are straight lines, and parallel if there are more than one.

(H2) The vector fields which define the piecewise one are linear.

(H3) The vector fields which define the piecewise one are Hamiltonian.

(H4) The vector fields which define the piecewise one have isolated singularities.

Note that, hypotheses (H2), (H3) and (H4) imply that the singular points of the linear systems that define the piecewise differential systems are saddles or centers.

We can classify the systems that satisfy the above hypotheses according to the configuration of their singular points. Thus, denoting the centers by the capital letter C and by S the saddles, in the case of two zones we have systems of the type CC, SC and SS. This is, CC indicates that the singular points of the linear systems that define the piecewise differential system are centers and so on. Following this idea, for three zones, we have the following six class of piecewise linear Hamiltonian systems: CCC, SCC, SCS, CSC, SSS and SSC.

The case with two zones has been study in the literature, i.e. the next theorem is already proved.

Theorem 1.1. *A continuous or discontinuous planar piecewise linear Hamiltonian differential system with two zones separated by a straight line and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has no limit cycles.*

A proof for Theorem 1.1 is contained in the proofs of Theorem 2 and 4 from [21]. Alternative proofs can also be found in other papers. See the proof of Theorem 1 from [27] for the case where one of the linear systems has a center and the other has a center or saddle, and see the proof of Theorem 3.4 from [16] for the case where the linear systems has saddles.

We include a proof of Theorem 1.1 in Section 3 just for the sake of completeness.

Assuming hypotheses (H1)–(H4), the main results in this paper are the follows:

Theorem 1.2. *A continuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has no limit cycles.*

Theorem 1.3. *A discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has at most one limit cycle.*

Theorems 1.2 and 1.3 have been proved for the particular case in which the linear systems that define the piecewise one has only isolated centers, see [25]. Theorem 1.2 has also been proved for the particular case SCS, see the proof of Lemma 11 from [33]. For the other possibilities, as far as we know, the results of Theorems 1.2 and 1.3 are new.

The paper is organized as follows. In Section 2 we introduce the basic definitions and results. In Section 3 we prove Theorems 1.2–1.3. Examples of discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines such that the linear systems that define it have isolated singular points are analyzed in Section 4. That is, we give examples of piecewise linear Hamiltonian systems of type CCC, SCC, SCS, CSC, SSS and SSC with exactly one limit cycle.

2 Preliminary results

In this section, we will present the basic concepts that we need to prove the main results of this paper.

Let $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i = C, L, R$, be the function $h_C(x,y) = x$, $h_L(x,y) = x + 1$ and $h_R(x,y) = x - 1$. By means of rotations, translations and homotheties we can assume without loss of generality that the *switching curve* Σ_C of a piecewise linear system with two zones in the plane is defined as

$$\Sigma_C = h_C^{-1}(0) = \{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

This straight line decomposes the plane in two regions

$$R_L = \{(x,y) \in \mathbb{R}^2 : x < 0\} \quad \text{and} \quad R_R = \{(x,y) \in \mathbb{R}^2 : x > 0\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with two zones is given by

$$\begin{aligned}
X_L(x,y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), & x \le 0, \\
X_R(x,y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), & x > 0.
\end{aligned} \tag{2.1}$$

Note that the Hamiltonian functions that determine the vector field (2.1) are

$$\begin{aligned}
H_L(x,y) &= \frac{b_L}{2}y^2 - \frac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, & x \le 0, \\
H_R(x,y) &= \frac{b_R}{2}y^2 - \frac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, & x > 0.
\end{aligned} \tag{2.2}$$

Assuming the hypothesis (H1), by means of rotations, translations and homotheties we can assume without loss of generality, for the case with three zones, that the switching curves Σ_L and Σ_R are given by

$$\Sigma_L = h_L^{-1}(0) = \{(x,y) \in \mathbb{R}^2 : x = -1\},$$

and

$$\Sigma_R = h_R^{-1}(0) = \{(x,y) \in \mathbb{R}^2 : x = 1\}.$$

This straight lines decomposes the plane in three regions

$$R_L = \{(x,y) \in \mathbb{R}^2 : x < -1\}, \quad R_C = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1\},$$

and

$$R_R = \{(x,y) \in \mathbb{R}^2 : x > 1\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with three zones is give by

$$
\begin{aligned}
X_L(x,y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), & x &\leq -1, \\
X_C(x,y) &= (a_C x + b_C y + \alpha_C, c_C x - a_C y + \beta_C), & -1 &\leq x \leq 1, \\
X_R(x,y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), & x &\geq 1.
\end{aligned}
\tag{2.3}
$$

The Hamiltonian functions that determine the vector field (2.3) are

$$
\begin{aligned}
H_L(x,y) &= \frac{b_L}{2}y^2 - \frac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, & x &\leq -1, \\
H_C(x,y) &= \frac{b_C}{2}y^2 - \frac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, & -1 &\leq x \leq 1, \\
H_R(x,y) &= \frac{b_R}{2}y^2 - \frac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, & x &\geq 1.
\end{aligned}
\tag{2.4}
$$

We will use the vector field X_L and the switching curve Σ_L in the next definitions. However, we can easily adapt the definitions to the vector fields X_C and X_R and the switching curves Σ_C and Σ_R.

The derivative of function h_L in the direction of the vector field X_L, i.e., the expression

$$
X_L h_L(p) = \langle X_L(p), \nabla h_L(p) \rangle,
$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$, characterize the contact between the vector field X_L and the switching curve Σ_L.

We distinguish the following subsets of Σ_L (the same for Σ_C and Σ_R)

- Crossing set:
$$
\Sigma_L^c = \{p \in \Sigma_L : X_L h_L(p) \cdot X_C h_L(p) > 0\};
$$

- Sliding set:
$$
\Sigma_L^s = \{p \in \Sigma_L : X_L h_L(p) > 0, X_C h_L(p) < 0\};
$$

- Escaping set:
$$
\Sigma_L^e = \{p \in \Sigma_L : X_L h_L(p) < 0, X_C h_L(p) > 0\}.
$$

In a piecewise vector field with two or three zones in the plane, the limit cycles can be of two types: sliding limit cycles or crossing limit cycles; the first one contain some segment of sliding or escaping sets, and the second one does not contain any segments of sliding or escaping sets. In this paper, we only study the crossing limit cycles. In what follows, when we talk about limit cycles, we are talking about crossing limit cycles.

Piecewise vector field (2.1) is called continuous if

$$
X_L(p) = X_R(p), \quad \forall p \in \Sigma_C.
$$

Otherwise, it is called discontinuous. Similarly, piecewise vector field (2.3) is called continuous if

$$
\begin{aligned}
X_L(p) &= X_C(p), \quad \forall p \in \Sigma_L \quad \text{and} \\
X_C(q) &= X_R(q), \quad \forall q \in \Sigma_R.
\end{aligned}
$$

Otherwise, it is called discontinuous.

3   Proof of Theorems 1.1–1.3

This section is devoted to present the proof of main results.

Proof of Theorem 1.1. Consider a discontinuous piecewise linear Hamiltonian vector field with two zones separated by a straight line, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector field (2.1), with $a_i^2 + b_i c_i \neq 0$, for $i = L, R$. If the piecewise linear vector field has a periodic orbit, then it intersects the straight line $x = 0$ at two points, $(0, y_0)$ and $(0, y_1)$, with $y_1 < y_0$, satisfying

$$H_R(0, y_1) = H_R(0, y_0),$$
$$H_L(0, y_0) = H_L(0, y_1),$$

where H_L and H_R are given by (2.2). More precisely, we have the equations

$$-\frac{1}{2}(y_0 - y_1)(b_R(y_0 + y_1) + 2\alpha_R) = 0,$$
$$\frac{1}{2}(y_0 - y_1)(b_L(y_0 + y_1) + 2\alpha_L) = 0.$$

As $y_1 < y_0$, if $b_R = 0$ and $\alpha_R \neq 0$ or $b_L = 0$ and $\alpha_L \neq 0$ the above system has no solutions. If $b_R = \alpha_R = 0$ and $b_L \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_L y_1 + 2\alpha_L)/b_L$, with arbitrary y_1. If $b_L = \alpha_L = 0$ and $b_R \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_R y_1 + 2\alpha_R)/b_R$, with arbitrary y_1. If $b_L b_R \neq 0$, then the above system has a solution (y_0, y_1) with $y_1 < y_0$ only when $b_L = b_R = b$ and $\alpha_L = \alpha_R = \alpha$. Moreover, $y_0 = (-by_1 + 2\alpha)/b$ with arbitrary y_1. If $b_R = b_L = \alpha_R = \alpha_L = 0$, then the system has infinitely many solutions. Therefore, the piecewise linear vector field (2.1) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle.

Note that the continuous case is a constraint of the discontinuous one. In fact, the continuous condition is given by

$$X_R(0, y) = X_L(0, y), \quad \forall y \in \mathbb{R}.$$

which implies

$$a_R = a_L = a, \quad b_R = b_L = b, \quad \alpha_R = \alpha_L = \alpha \quad \text{and} \quad \beta_R = \beta_L = \beta. \qquad \square$$

Proof of Theorem 1.2. Consider a continuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector fields (2.3), with $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, and due to continuity

$$X_R(1, y) = X_C(1, y) \quad \text{and} \quad X_C(-1, y) = X_L(-1, y), \quad \forall y \in \mathbb{R}.$$

These equalities imply that

$$a_R = a_C = a_L = a, \quad b_R = b_C = b_L = b, \quad \alpha_R = \alpha_C = \alpha_L = \alpha$$

and

$$\beta_R - \beta_C - c_C + c_R = \beta_L - \beta_C - c_L + c_C = 0.$$

By Theorem 1.1, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit, then it intersects the straight

lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$
\begin{aligned}
H_R(1, y_1) &= H_R(1, y_0), \\
H_C(1, y_0) &= H_C(-1, y_3), \\
H_L(-1, y_3) &= H_L(-1, y_2), \\
H_C(-1, y_2) &= H_C(1, y_1),
\end{aligned}
\tag{3.1}
$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$
-\frac{1}{2}(y_0 - y_1)(b(y_0 + y_1) + 2(a + \alpha)) = 0,
\tag{3.2}
$$

$$
a(y_0 + y_3) + \frac{1}{2}(y_0 - y_3)(b(y_0 + y_3) + 2\alpha) - 2\beta_C = 0,
\tag{3.3}
$$

$$
-\frac{1}{2}(y_2 - y_3)(b(y_2 + y_3) - 2(a - \alpha)) = 0,
\tag{3.4}
$$

$$
-a(y_1 + y_2) - \frac{1}{2}(y_1 - y_2)(b(y_1 + y_2) + 2\alpha) + 2\beta_C = 0.
\tag{3.5}
$$

As $y_1 < y_0$, $y_2 < y_3$ and $a^2 + bc_i \neq 0$, for $i = L, C, R$, if either $b = 0$ and $a + \alpha \neq 0$ or $b = 0$ and $a + \alpha = 0$ the above system has no solutions. If $b \neq 0$, as $y_1 < y_0$ and $y_2 < y_3$, from equation (3.2) we can obtain y_0 as a function of y_1, i.e.

$$
y_0 = \frac{-by_1 - 2(a + \alpha)}{b}.
\tag{3.6}
$$

Now, from equation (3.4) we can obtain y_2 as a function of y_3, i.e.

$$
y_2 = \frac{-by_3 - 2(\alpha - a)}{b}.
\tag{3.7}
$$

Substituting (3.6) and (3.7) in equations (3.3) and (3.5), respectively, we obtain a solution (y_0, y_1, y_2, y_3) of the system (3.1) satisfying $y_1 < y_0$ and $y_2 < y_3$, given by $(\varphi_1(y_1), y_1, \varphi_2(y_1), \varphi_3(y_1))$, where

$$
\varphi_1(y_1) = \frac{-by_1 - 2(a + \alpha)}{b},
$$

$$
\varphi_2(y_1) = \frac{a - \alpha + \sqrt{b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b},
$$

$$
\varphi_3(y_1) = \frac{a - \alpha + \sqrt{b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b},
$$

with arbitrary y_1. Note that the inequality $b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C \leq 0$ for all $y_1 \in \mathbb{R}$ is not possible. Therefore, the piecewise linear vector field (2.3) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle. $\qquad\square$

Proof of Theorem 1.3. Consider a discontinuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector fields (2.3), with $-a_i^2 - b_i c_i \neq 0$, for $i = L, C, R$. By Theorem 1.1, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit,

then it intersects the straight lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$
\begin{aligned}
H_R(1, y_1) &= H_R(1, y_0), \\
H_C(1, y_0) &= H_C(-1, y_3), \\
H_L(-1, y_3) &= H_L(-1, y_2), \\
H_C(-1, y_2) &= H_C(1, y_1),
\end{aligned}
\tag{3.8}
$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$
\frac{1}{2}(y_1 - y_0)(b_R(y_0 + y_1) + 2(a_R + \alpha_R)) = 0,
\tag{3.9}
$$

$$
\frac{1}{2}(y_0 - y_3)(b_C(y_0 + y_3) + 2\alpha_C) - 2\beta_C + a_C(y_0 + y_3) = 0,
\tag{3.10}
$$

$$
\frac{1}{2}(y_3 - y_2)(b_L(y_2 + y_3) - 2(a_L - \alpha_L)) = 0,
\tag{3.11}
$$

$$
\frac{1}{2}(y_2 - y_1)(b_C(y_1 + y_2) + 2\alpha_C) + 2\beta_C - a_C(y_1 + y_2) = 0.
\tag{3.12}
$$

To determine all the solutions of the above systems, restricted to the conditions $y_1 < y_0$, $y_2 < y_3$ and $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, we distinguish two cases. In the first case we assume that $b_R b_L b_C = 0$. For this cases, system (3.9)–(3.12) has no solutions when

- $b_R = 0$ and $a_R + \alpha_R \neq 0$;

- $b_L = 0$ and $a_L - \alpha_L \neq 0$;

- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$;

- $b_R = a_R + \alpha_R = b_C = \alpha_C - a_C = 0$ and $b_L \neq 0$;

- $b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$ and $b_R \neq 0$;

- $b_C = 0$, $b_R b_L \neq 0$ and $b_R \alpha_C (a_L - \alpha_L) + a_C b_R (\alpha_L - a_L) + b_L (a_R + \alpha_R)(a_C + \alpha_C) + 2 b_L b_R \beta_C \neq 0$;

and it has infinitely many solutions when

- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = 0$ and $\alpha_C - a_C \neq 0$;

- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = 0$ and $b_C \neq 0$;

- $b_R = a_R + \alpha_R = b_C = 0$, $\alpha_C - a_C \neq 0$ and $b_L \neq 0$;

- $b_R = a_R + \alpha_R = 0$ and $b_L b_C \neq 0$;

- $b_L = a_L - \alpha_L = b_C = 0$, $b_R \neq 0$ and $\alpha_C - a_C \neq 0$;

- $b_L = a_L - \alpha_L = 0$ and $b_R b_C \neq 0$;

- $b_C = 0$, $b_R b_L \neq 0$ and $b_R \alpha_C (a_L - \alpha_L) + a_C b_R (\alpha_L - a_L) + b_L (a_R + \alpha_R)(a_C + \alpha_C) + 2 b_L b_R \beta_C = 0$.

In the second case, we assume that $b_L b_C b_R \neq 0$. From equation (3.9), we can obtain y_0 as a function of y_1, i.e.

$$
y_0 = \frac{-b_R y_1 - 2(a_R + \alpha_R)}{b_R}.
\tag{3.13}
$$

Now, from equation (3.11), we can obtain y_2 as a function of y_3, i.e.

$$y_2 = \frac{-b_L y_3 - 2(\alpha_L - a_L)}{b_L}. \tag{3.14}$$

Substituting (3.13) and (3.14) in equations (3.10) and (3.12), respectively, we obtain the equations of two hyperbolas in the $y_1 y_3$ plane, given by

$$
\begin{aligned}
\frac{(y_1 - A)^2}{K} - \frac{(y_3 - B)^2}{K} - C &= 0, \\
\frac{(y_1 - D)^2}{K} - \frac{(y_3 - E)^2}{K} - C &= 0,
\end{aligned}
\tag{3.15}
$$

with

$$K = \frac{2}{b_C}, \quad A = \frac{b_R(a_C + \alpha_C) - 2b_C(a_R + \alpha_R)}{b_C b_R},$$

$$B = \frac{a_C - \alpha_C}{b_C}, \quad C = \frac{2(a_C \alpha_C + b_C \beta_C)}{b_C},$$

$$D = -\frac{(a_C + \alpha_C)}{b_C} \quad \text{and} \quad E = \frac{b_L(\alpha_C - a_C) - 2b_C(\alpha_L - a_L)}{b_C b_L}.$$

Note that the system (3.15) is equivalent to the system

$$
\begin{aligned}
y_1^2 - 2Ay_1 + A^2 - y_3^2 + 2By_3 - B^2 - KC &= 0, \\
2(A - D)y_1 + 2(E - B)y_3 + D^2 - E^2 + B^2 - A^2 &= 0.
\end{aligned}
\tag{3.16}
$$

The system above eventually could have infinitely many solutions (y_1, y_3), for instance when $A = D$ and $B = E$. In this case, the piecewise linear vector field (2.3) has a continuum of periodic orbits, and consequently, it has no limit cycle. Suppose that system (3.16) has finitely many solutions. According to Bezout's Theorem, if a system of polynomial equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials, that for system (3.16) is two. Therefore, the two hyperbolas above intersect at most two points. Note that, by (3.9)–(3.12), if (y_0, y_1, y_2, y_3) is solution of the system (3.8) then (y_1, y_0, y_3, y_2) is also a solution. However, for $y_1 < y_0$ and $y_2 < y_3$ we have at most a single solution. Therefore, the piecewise linear vector field (2.3) can have at most one limit cycle. □

4 Examples

In this section, we will give some examples of discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines with one limit cycle, such that the linear systems that define it have isolated singular points. That is, we given examples of piecewise linear Hamiltonian systems of type CCC, SCC, SCS, CSC, SSS and SSC with exactly one limit cycle. In [25], the authors presented an example of a discontinuous piecewise linear differential system of type CCC with exactly one limit cycle. Here we will show another example for this case.

Example 4.1 (Case CCC). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ and $\alpha_R = \beta_R = -4$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are $\pm 2i$, $\pm 2i$ and $\pm 2i$, respectively, i.e. we have three centers. Therefore, a candidate to limit cycle of vector field (2.3), in this case, correspond to the solution of system (3.8), i.e.

$$(y_1 - y_0)(y_1 + y_0) = 0,$$

$$\frac{1}{3}(y_0(2 + 3y_0) - y_3(2 + 3y_3) - 4) = 0,$$

$$\frac{1}{2}(y_3 - y_2)(8(y_2 + y_3) - 5) = 0,$$

$$\frac{1}{3}(4 - y_1(2 + 3y_1) + y_2(2 + 3y_2)) = 0.$$

After some computations, the unique solution (y_0, y_1, y_2, y_3) of the above system, satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, is given by

$$\left(\frac{31}{48}\sqrt{\frac{1259}{235}}, -\frac{31}{48}\sqrt{\frac{1259}{235}}, \frac{5}{16} - \frac{1}{3}\sqrt{\frac{1259}{235}}, \frac{5}{16} + \frac{1}{3}\sqrt{\frac{1259}{235}} \right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points because

$$\langle X_L(-1, y_2), (1, 0) \rangle \cdot \langle X_C(-1, y_2), (1, 0) \rangle \approx 1.5518 > 0,$$
$$\langle X_L(-1, y_3), (1, 0) \rangle \cdot \langle X_C(-1, y_3), (1, 0) \rangle \approx 17.4969 > 0,$$
$$\langle X_C(1, y_0), (1, 0) \rangle \cdot \langle X_R(1, y_0), (1, 0) \rangle \approx 10.9315 > 0,$$
$$\langle X_C(1, y_1), (1, 0) \rangle \cdot \langle X_R(1, y_1), (1, 0) \rangle \approx 6.9452 > 0.$$

The orbit $(x_R(t), y_R(t))$ of X_R, such that $(x_R(0), y_R(0)) = (1, y_0)$, is given by

$$x_R(t) = 7\cos(2t) + \frac{31}{48}\sqrt{\frac{1259}{235}}\sin(2t) - 6,$$

$$y_R(t) = \left(\frac{31}{48}\sqrt{\frac{1259}{235}} - 14 \right)\cos(2t) - \left(7 + \frac{31}{24}\sqrt{\frac{1259}{235}} \right)\sin(2t) + 14.$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C, such that $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$, is given by

$$x_{C_1}(t) = \frac{2}{3}\cos(2t) + \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}} \right)\sin(2t) + \frac{1}{3},$$

$$y_{C_1}(t) = \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}} \right)\cos(2t) - \frac{1}{3}(1 + 4\cos(t)\sin(t)).$$

The orbit $(x_L(t), y_L(t))$ of X_L, such that $(x_L(0), y_L(0)) = (-1, y_2)$, is given by

$$x_L(t) = -8\cos(2t) - \frac{4}{3}\sqrt{\frac{1259}{235}}\sin(2t) + 7,$$

$$y_L(t) = \left(4 - \frac{1}{3}\sqrt{\frac{1259}{235}} \right)\cos(2t) + \left(4 + \frac{4}{3}\sqrt{\frac{1259}{235}} \right)\cos(t)\sin(t) - \frac{59}{16}.$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, such that $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$, is given by

$$x_{C_2}(t) = -\frac{4}{3}\cos(2t) + \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\sin(2t) + \frac{1}{3},$$

$$y_{C_2}(t) = \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\cos(2t) + \frac{1}{3}(8\cos(t)\sin(t) - 1).$$

The fly time of the orbit $(x_R(t), y_R(t))$, from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$, is

$$t_R = \frac{1}{2}\arctan\left(\frac{20832\sqrt{295865}}{25320661}\right).$$

The fly time of the orbit $(x_{C_1}(t), y_{C_1}(t))$, from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$, is

$$t_{C_1} = \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{96(10810 + \sqrt{295865})}{496001}\right).$$

The fly time of the orbit $(x_L(t), y_L(t))$, from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$, is

$$t_L = \frac{1}{2}\arctan\left(\frac{12\sqrt{295865}}{7201}\right).$$

Finally, the fly time of the orbit $(x_{C_2}(t), y_{C_2}(t))$, from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$, is

$$t_{C_2} = -\frac{1}{2}\arctan\left(\frac{96(\sqrt{295865} - 10810)}{496001}\right).$$

Using the Mathematica software (see [37]), we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.1 (a). Figure 4.1 (b) was made with the help of P5 software (see [14]), and provides the phase portrait of vector field (2.3) in this case (the symbol ∘ indicates an invisible singular point).

Example 4.2 (Case SCC). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 2/3$, $c_L = 35$, $\beta_L = 214/3$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ and $c_R = -10$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 6, $\pm 2i$ and $\pm 2i$, respectively, i.e. we have one saddle and two centers. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{2\sqrt{5}}{3}, -\frac{2\sqrt{5}}{3}, \frac{1 - \sqrt{5}}{3}, \frac{1 + \sqrt{5}}{3}\right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$;
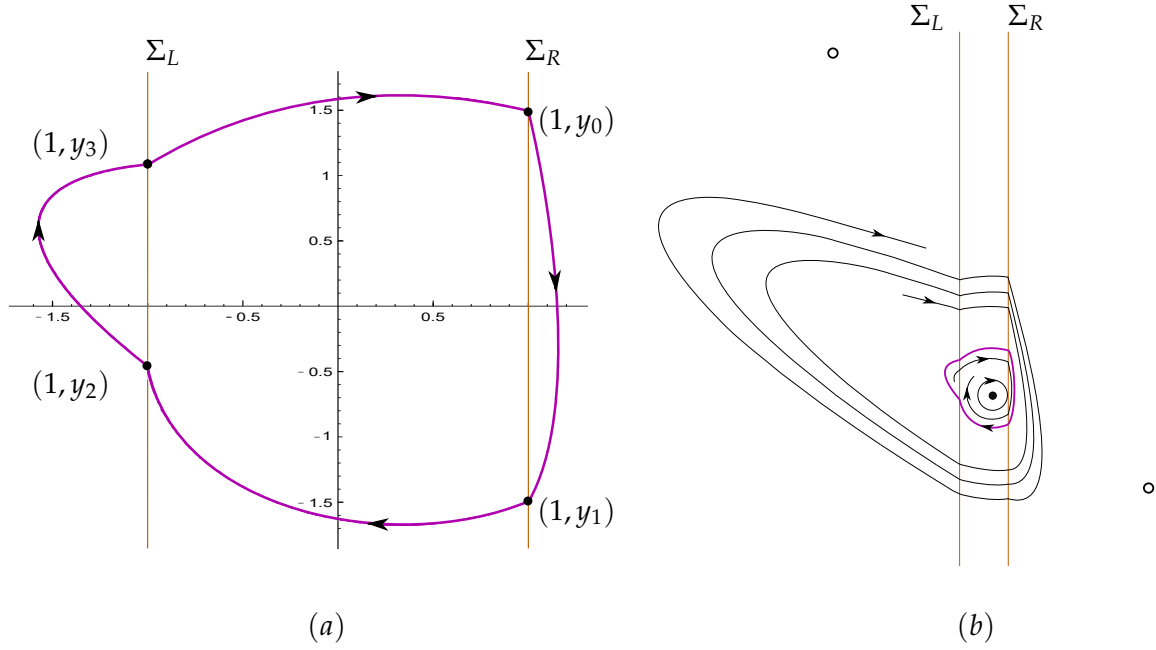
Figure 4.1: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ and $\alpha_R = \beta_R = -4$.

$(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.2 (a). The Figure 4.2 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.

Example 4.3 (Case SCS). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 6, $\pm 2i$ and ± 4, respectively, i.e. we have two saddles and one center. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{18}{5}\sqrt{\frac{2}{7}}, \; -\frac{18}{5}\sqrt{\frac{2}{7}}, \; \frac{2}{5} - 2\sqrt{\frac{2}{7}}, \; \frac{2}{5} + 2\sqrt{\frac{2}{7}} \right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.3 (a). The Figure 4.3 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.
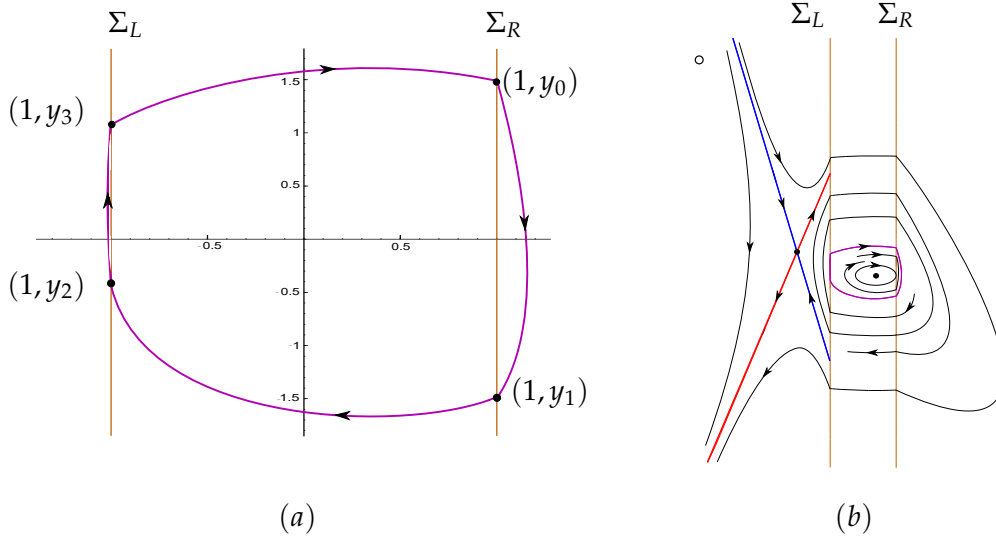
Figure 4.2: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 2/3$, $c_L = 35$, $\beta_L = 214/3$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ and $c_R = -10$.

Example 4.4 (Case CSC). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 2$, $c_L = -5/2$, $\beta_L = 5/2$, $a_C = 2/5$, $b_C = 24/5$, $\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ and $\alpha_R = c_R = \beta_R = -8$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are $\pm 2i$, ± 2 and $\pm 4i$, respectively, i.e. we have one saddle and two centers. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{5}{12}\sqrt{\frac{7}{3}}, -\frac{5}{12}\sqrt{\frac{7}{3}}, \frac{1}{4} - \frac{7\sqrt{21}}{36}, \frac{1}{4} + \frac{7\sqrt{21}}{36} \right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.4 (a). The Figure 4.4 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.

Example 4.5 (Case SSS). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = -2/11$, $b_R = 4/11$, $\alpha_R = 1/5$, $c_R = 120/11$ and $\beta_R = -749/55$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 2, ± 2 and ± 2, respectively, i.e. we have three saddles. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$,
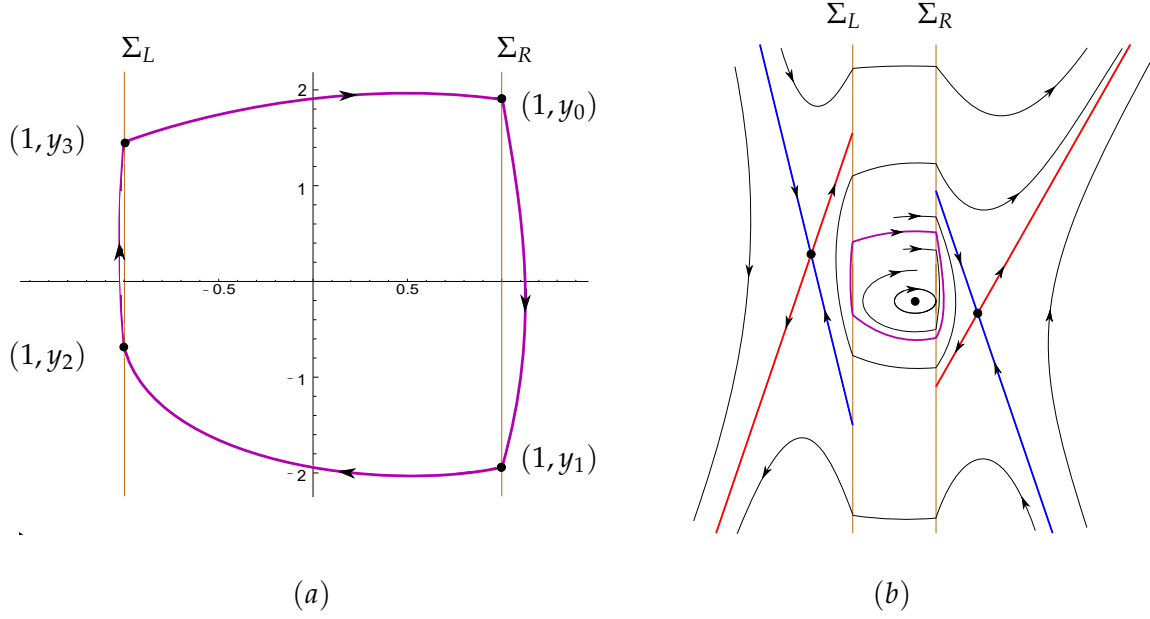
Figure 4.3: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$.
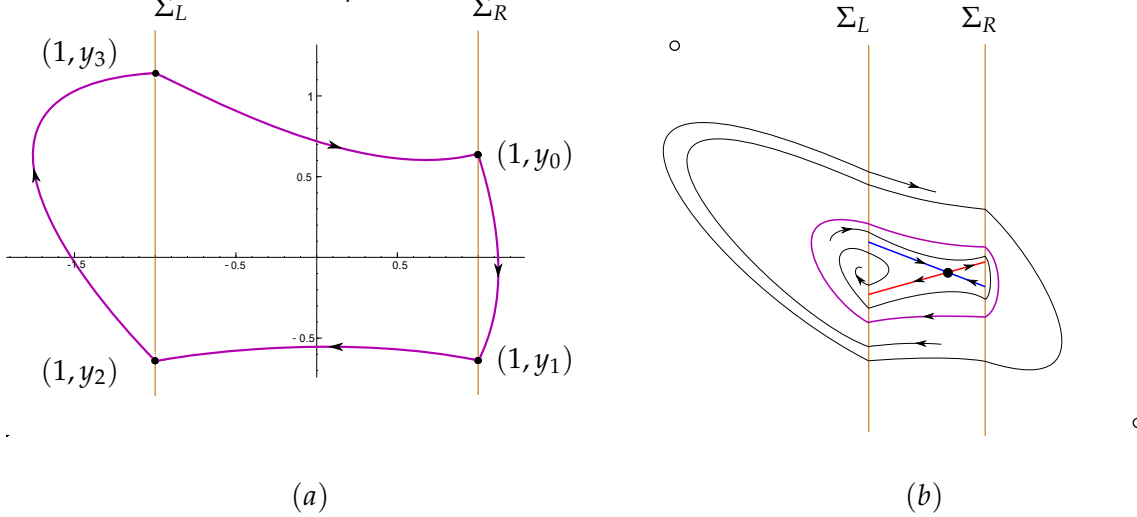

Figure 4.4: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 2$, $c_L = -5/2$, $\beta_L = 5/2$, $a_C = 2/5$, $b_C = 24/5$, $\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ and $\alpha_R = c_R = \beta_R = -8$.

which is given by

$$\left(\frac{43\sqrt{26} - 12}{240}, -\frac{43\sqrt{26} + 12}{240}, -\frac{3}{8}\sqrt{\frac{13}{2}}, \frac{3}{8}\sqrt{\frac{13}{2}} \right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$;

$(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.5 (a). The Figure 4.5 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case..

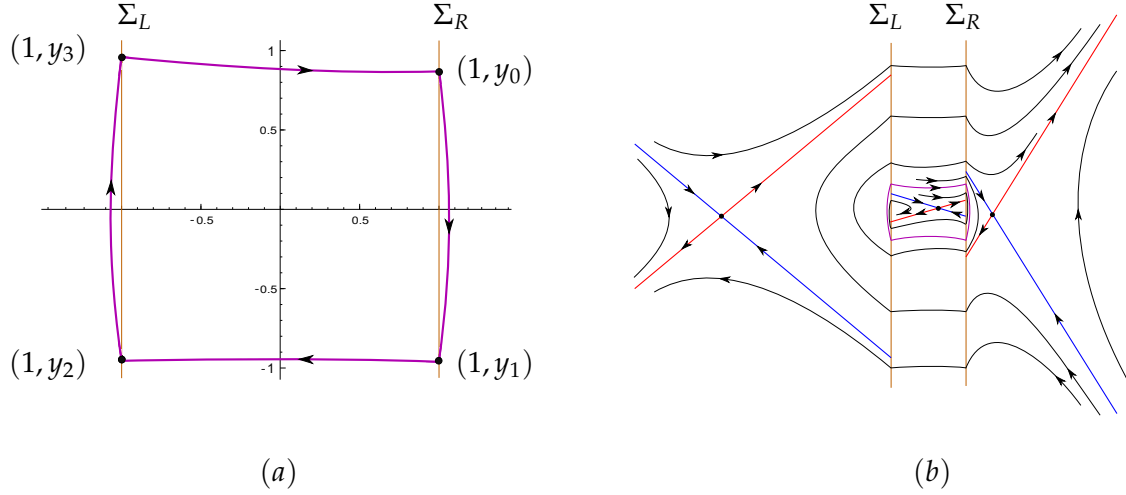(a)   (b)

Figure 4.5: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = -2/11$, $b_R = 4/11$, $\alpha_R = 1/5$, $c_R = 120/11$ and $\beta_R = -749/55$.

Example 4.6 (Case SSC). Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 2, ± 2 and $\pm 4i$, respectively, i.e. we have two saddles and one center. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{17}{8}\sqrt{\frac{43}{470}}, \frac{17}{8}\sqrt{\frac{43}{470}} \right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.6 (a). The Figure 4.6 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.
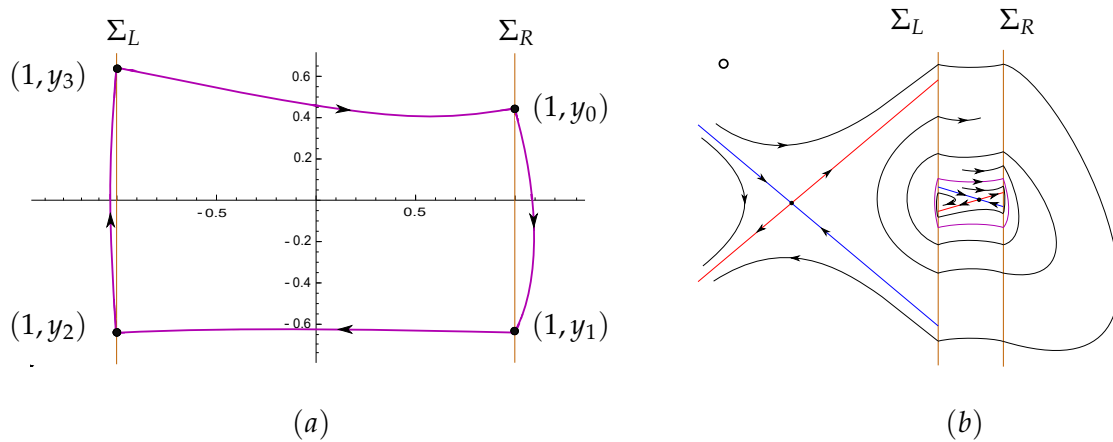
Figure 4.6: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$.

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