Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **69**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF POSITIVE WEAK SOLUTIONS FOR A CLASS OF SINGULAR ELLIPTIC EQUATIONS

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ABSTRACT. In this note, we are concerned with positive solutions for a class of singular elliptic equations. Under some conditions, we obtain weak solutions for the equations by elliptic regularization method and sub-super solution method.

1. INTRODUCTION

In this note, we are concerned with following singular elliptic problem:

$$z'' + \tfrac{\beta}{r}z' - \tfrac{\gamma}{z}|z'|^2 + \lambda(r) = 0,\ z > 0, r \in (0,1), \tag{1}$$

subject to the Dirichlet boundary conditions:

$$z(0) = z(1) = 0, \tag{2}$$

where $\beta > 0, \gamma > \beta + 1$ are constants, $c < \lambda(r) \in L^\infty(0,1)$ for some positive constant c.

In [1], the authors studied the problem

$$z'' + \frac{N-1}{r}z' - \gamma\frac{|z'|^2}{z} - 1 = 0,\ r \in (0,1),$$
$$z(1) = 0, \quad z'(0) = 0.$$

Here, $N \geq 2$ is the dimension of $\mathbb{R}^N$ space. Applying ordinary differential equation techniques, they obtained a decreasing positive solution which, subsequently, was used in [2] to study some properties of solutions for a class of degenerate parabolic equations (see [3] for further information). In [4], Xia and Yao studied following problem

$$z'' + \tfrac{\beta}{r}z' - \tfrac{\gamma}{z}|z'|^2 + f(r,z) = 0,\ z > 0, r \in (0,1),$$

2000 *Mathematics Subject Classification.* Primary 35K15, 35K20.
Key words and phrases. existence, singular elliptic equation, weak solution.
Supported by Natural Science Foundation of Tianyuan(11026156) and startup program of Shenzhen University.

subject to the following four-point boundary conditions:

$$z(0) = z(1) = 0,$$
$$z'(0) = z'(1) = 0.$$

Here $f(r, z)$ satisfies the following condition:

(H1) $f(r, z) \in C^1([0, 1] \times [0, \infty), [c_0, \infty))$ for sufficiently small $c_0 > 0$, and f is non-increasing with respect to z.

They showed that the above problem has at least one classical solution. For more details of equations dependent of first derivative, see [5-12] and the references therein.

Problem (1) is closely related with some equations. For example, if $\beta = N - 1$, suppose $\lambda(x)$ is a radially symmetric function with respect to $x \in B_1 \subset \mathbb{R}^N (N \geq 2)$, then problem (1) is related with following problem:

$$\begin{cases} -\Delta z + \gamma \frac{|\nabla z|^2}{z} = \lambda(x), & z > 0, x \in B_1 \setminus \{0\}, \\ z = 0, & x \in \partial B_1 \cup \{0\}, \end{cases} \tag{3}$$

where B_1 is the unit ball in $\mathbb{R}^N$. Note that solutions of (1)(2) are radial solutions of problem (3) with $r = |x|$, which may be transformed into following problem with infinite boundaries if we set $\gamma = 1$(or $\gamma = \frac{q+1}{q}, q > 1$), $z = e^{-w}$(or $z = w^{-q}, q > 1, w > 0$):

$$\begin{cases} \Delta w = \lambda(x) g(w), & w > 0, x \in B_1 \setminus \{0\}, \\ w = \infty, & x \in \partial B_1 \cup \{0\}. \end{cases} \tag{4}$$

where, $g(w) = e^w$(or $g(w) = w^q, q > 1$). The last condition means that $w(x) \to \infty$ uniformly as $x \in B_1$, $d(x) = \text{dist}(x, \partial B_1) \to 0$ or $|x| \to 0$. And we call its solution as explosive solution or "large solution". Much attentions have been focused on problems (3)(4) and some related problems, which may have a singularity, we refer readers to [7-12] and the references therein.

For $g(w) = e^w$ or $g(w) = w^q (q > 1)$, problem (4) plays an important role in the theory of Riemann surfaces of constant negative curvatures and automorphic function, arises in the study of high speed diffusion problem, some geometric problems and the electric potential in a glowing hollow metal body. For details of the two classical problems, we refer readers to [7] and the references therein.

In this note, we shall discuss weak solutions of (1)(2), using regularization method and constructing sub-solution and super-solution for problem (1)(2) to obtain the existence result.

2. Main result and the proof

Definition 2.1. *A function z is called a solution for (1)(2), if $z \in C^{1/2}[0,1]$, $z(r) > 0$ in $(0,1)$, $r^\beta |z'|^2 \in L^1(0,1)$, $z'(0)$ and $z'(1)$ exist, $z(r)$ satisfies (2) and*

$$\int_0^1 \Big(z'\psi' - \beta\frac{z'}{r}\psi + \gamma\frac{|z'|^2}{z}\psi - \lambda(r)\psi\Big)dr = 0,$$

for any $\psi \in C_0^\infty(0,1)$, the space of smooth functions $\chi : (0,1) \to \mathbb{R}$ with compact support in $(0,1)$.

The main result of this note is as follows.

Theorem 2.1. *Under the hypothesis of this note, problem (1)(2) admits at least a solution.*

To prove Theorem 2.1, we use the classical method of regularization. Precisely, we consider

$$z_\delta'' + \tfrac{\beta}{r+\delta} z_\delta' - \tfrac{\gamma}{z_\delta+\delta^2}|z_\delta'|^2 + \lambda(r) = 0, \;\; z_\delta > 0, r \in (0,1), \tag{5}$$

subject to conditions (2).

We call v a sub-(sup-) solution for (5), if $v \geq 0, v \in L^\infty(0,1) \cap W^{1,2}(0,1)$, and for any $0 \leq \psi \in L^\infty(0,1) \cap W_0^{1,2}(0,1)$ there holds

$$\int_0^1 \Big(v'\psi' - \frac{\beta}{r+\delta}v'\psi + \frac{\gamma}{v+\delta^2}|v'|^2\psi - \lambda(r)\psi\Big)dr \leq (\geq) 0.$$

v is called a weak solution for (5)(2), if v is both a sub-solution and a sup-solution for (5) and satisfies (2). By [13](Th 9.1, Chapter 4), problem (5)(2) admits a solution $0 < z_\delta \in W_0^{1,2}(0,1) \cap L^\infty(0,1)$.

Lemma 2.1. *Assume that z_1 and z_2 are sub-solution and sup-solution for (5) respectively, $z_1(0) \le z_2(0), z_1(1) \le z_2(1)$. Then*

$$z_1 \le z_2 \quad a.e.\ in\ (0,1).$$

Proof. For any $0 \le \psi \in L^\infty(0,1) \cap W_0^{1,2}(0,1)$ there holds

$$\begin{aligned} &\int_0^1 \Big(z_2'(\psi' - \frac{\beta}{r+\delta}\psi) + \frac{\gamma}{z_2 + \delta^2}|z_2'|^2\psi - \lambda(r)\psi \Big) dr \ge 0, \\ &\int_0^1 \Big(z_1'(\psi' - \frac{\beta}{r+\delta}\psi) + \frac{\gamma}{z_1 + \delta^2}|z_1'|^2\psi - \lambda(r)\psi \Big) dr \le 0, \end{aligned} \tag{6}$$

Let $f(s) : (0,\infty) \to \mathbf{R}$ be defined by

$$f(s) = \begin{cases} (1-\gamma)^{-1}s^{1-\gamma}, & \gamma \ne 1, \\ \ln s, & \gamma = 1. \end{cases}$$

Set $u_1 = z_1 + \delta^2, u_2 = z_2 + \delta^2$. Since $u_1, u_2 \in L^\infty(0,1) \cap W^{1,2}(0,1)$, $f(s)$ is increasing and $u_2 \ge u_1$ at points $\{0,1\}$, we have $(f(u_1) - f(u_2))_+ \in L^\infty(0,1) \cap W_0^{1,2}(0,1)$. This and $u_1, u_2 \ge \delta^2 > 0$ in $(0,1)$ imply $\psi_{u_j} = (r+\delta)^\beta u_j^{-\gamma}(f(u_1) - f(u_2))_+ \in L^\infty(0,1) \cap W_0^{1,2}(0,1), j = 1,2$. So ψ_{u_2} and ψ_{u_1} can be chosen in (6) as test functions. Hence

$$\begin{aligned} &\int_0^1 (r+\delta)^\beta u_2^{-\gamma} \Big[u_2'(f(u_1) - f(u_2))_+' - \lambda(r)(f(u_1) - f(u_2))_+ \Big] dr \ge 0, \\ &\int_0^1 (r+\delta)^\beta u_1^{-\gamma} \Big[u_1'(f(u_1) - f(u_2))_+' - \lambda(r)(f(u_1) - f(u_2))_+ \Big] dr \le 0, \end{aligned}$$

which imply that

$$\begin{aligned} &\int_0^1 (r+\delta)^\beta \Big[(f'(u_1) - f'(u_2))(f(u_1) - f(u_2))_+' \\ &+\lambda(r)(h(u_1) - h(u_2))(f(u_1) - f(u_2))_+ \Big] dr \le 0, \end{aligned} \tag{7}$$

where $h : (0,\infty) \to \mathbf{R}^-$ is defined by $h(s) = -s^{-\gamma}$.

It is easy to see that

$$\int_0^1 (r+\delta)^\beta (f'(u_1) - f'(u_2)) \cdot (f(u_1) - f(u_2))_+' dr \ge 0,$$

which and (7) yield that

$$\int_0^1 (r+\delta)^\beta \lambda(r)(h(u_1) - h(u_2))(f(u_1) - f(u_2))_+ dr \le 0.$$

But this and $\lambda(r) > 0$ in $(0,1)$ imply that $(h(u_1) - h(u_2))(f(u_1) - f(u_2))_+ = 0$ a.e. in $(0,1)$, i.e., $u_2 \geq u_1$ a.e. in $(0,1)$. The proof is completed.

Let $\omega = \frac{1}{2}(r - r^2)$ be the unique classical solution for problem

$$-z'' = 1, \ r \in (0,1),$$
$$z(0) = z(1) = 0.$$

Lemma 2.2. *Let $\underline{z} = C_0\omega^2$, $z_{1\delta} = C_1(r+\delta)^2$, $z_{2\delta} = C_1(1+\delta-r)^2$, $\overline{z}_\delta = \min\{z_{1\delta}, z_{2\delta}\}$, where C_0 and $C_1 \geq 1$ are some positive constants. Then*

$$\underline{z} \leq z_\delta \leq \overline{z}_\delta \ \textit{a.e. in } (0,1), \ \textit{for all } \delta \in (0,1). \tag{8}$$

Proof. Note that if $\underline{z}$ is a sub-solution and $\overline{z}_{i,\delta}(i = 1,2)$ are both sup-solutions for (5), it follows from Lemma 2.1 that $\underline{z} \leq z_\delta \leq \overline{z}_\delta$. The proof of former conclusion follows similarly from Lemma 2.1 in [4]. Hence Lemma 2.2 is proved.

Lemma 2.3. *For all $\delta \in (0,1)$, we have*

$$\int_0^1 (r+\delta)^\beta |z'_\delta|^2 dr \leq C,$$

where C is a constant independent of δ.

Proof. Multiplying (5) by $(r+\delta)^\beta z_\delta$, integrating over $(0,1)$ and integrating by parts, we have

$$\begin{aligned} &\int_0^1 (r+\delta)^\beta [1 + \gamma \frac{z_\delta}{z_\delta + \delta^2}] |z'_\delta|^2 dr \\ = &\int_0^1 (r+\delta)^\beta \lambda(r) z_\delta dr \leq C. \end{aligned} \tag{9}$$

The last inequality follows from (8) and $0 < \lambda(r) \in L^\infty(0,1)$.

From Lemma 2.3, for any $0 < \sigma < 1$ there holds

$$\int_\sigma^1 |z'_\delta|^2 dr \leq C_\sigma,$$

where C_σ is a constant dependent of σ. Going to a subsequence of z_δ if necessary, denoted by z_{δ_n}, we assert that there exists a nonnegative function $z \in L^\infty(0,1) \cap W^{1,2}(\sigma,1)$ such that, as $\delta = \delta_n \to 0$,

$$\begin{aligned} z_\delta &\to z \quad \text{a.e. in } [0,1], && (10)\\ z'_\delta &\rightharpoonup z' \quad \text{weakly in } L^2(\sigma,1). && (11) \end{aligned}$$

Since $W^{1,2}(\sigma,1) \hookrightarrow C^{1/2}[\sigma,1]$ and z_δ is uniformly bounded with respect to δ, from Arzela-Ascoli theorem and diagonal sequential process, we further claim that, as $\delta = \delta_n \to 0$,

$$z_\delta \to z \text{ uniformly in } [\sigma,1], \tag{12}$$

and $z(1) = 0$.

On the other hand, from (8)(10) we obtain that

$$C_0\omega^2 \le z \le C_1 \min\{r^2, (1-r)^2\} \text{ in } (0,1). \tag{13}$$

This implies that z has Hölder continuity near $r = 0$ and $\lim\limits_{r\to 0} z(r) = 0$. Define $z(0) = 0$, we see that z satisfies (2), $z \in C^{\frac{1}{2}}[0,1]$ and

$$z_\delta \to z \text{ in } [0,1], \tag{14}$$

as $\delta = \delta_n \to 0$.

From Lemma 2.3 and (11), we also have

$$\begin{aligned} (r+\delta)^{\beta/2} z'_\delta &\rightharpoonup r^{\beta/2} z' \quad \text{weakly in } L^2(0,1),\\ r^{\beta/2} z'_\delta &\rightharpoonup r^{\beta/2} z' \quad \text{weakly in } L^2(0,1). \end{aligned} \tag{15}$$

as $\delta = \delta_n \to 0$. From (15) and weak lower semi-continuity of the norm in $L^2(0,1)$, it follows that

$$\int_0^1 r^\beta |z'|^2 dr \le C, \tag{16}$$

where C is independent of δ.

Next we show that z satisfies the integral identity of Definition 2.1.

Lemma 2.4. *For any $\xi \in C_0^\infty(0,1)$, as $\delta = \delta_n \to 0$, we have*

(1) $\displaystyle\int_0^1 r^{\beta+1}\xi |z'_\delta - z'|^2 dr \to 0;$

(2) $\displaystyle\int_0^1 r^{\beta+1}\xi \big||z'_\delta|^2 - |z'|^2\big| dr \to 0;$

(3) $\displaystyle\int_0^1 r^{\beta+1}\xi \Big|\frac{z'_\delta}{r+\delta} - \frac{z}{r}\Big| dr \to 0;$

(4) $\displaystyle\int_0^1 r^{\beta+1}\xi\Big|\frac{|z_\delta'|^2}{z_\delta+\delta^2}-\frac{|z'|^2}{z}\Big|dr\to 0.$

Proof. From (14) and Lemma 2.3, for any fixed $\delta\in(0,1)$, $\varphi_\delta = r^{\beta+1}\xi(z_\delta - z)\in L^\infty(0,1)\cap W_0^{1,2}(0,1)$. Thus we may take φ_δ as a test function in (6) to obtain

$$\begin{aligned}
&\int_0^1 r^{\beta+1}\lambda(r)\xi(z_\delta-z)dr\\
&=\gamma\int_0^1 r^{\beta+1}\xi\frac{|z_\delta'|^2}{z_\delta+\delta^2}(z_\delta-z)dr+\int_0^1 r^{\beta+1}\xi z_\delta'(z_\delta'-z')dr\\
&\ +\int_0^1(\beta+1-\frac{r\beta}{r+\delta})r^\beta\xi z_\delta'(z_\delta-z)dr+\int_0^1 r^{\beta+1}\xi' z_\delta'(z_\delta-z)dr\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

Since $\xi\in C_0^\infty(0,1)$, *supp* $\xi\subset\subset(0,1)$. From which, Lemma 2.3, (8)(12) and $\omega>0$ on $\overline{supp\ \xi}\subset[0,1]$, there hold

$$\begin{aligned}
I_1\ &\le C\int_{supp\ \xi} r^\beta\xi z_\delta^{-1}|z_\delta'|^2|z_\delta-z|dr\\
&\le C\max_{r\in\overline{supp\ \xi}}(\omega^{-2}|z_\delta-z|)\Big(\int_{supp\ \xi} r^\beta|z_\delta'|^2dr\Big)\\
&\to 0,\ (\delta=\delta_n\to 0).
\end{aligned}$$

Now we estimate I_3, I_4. Using the similar method as in I_1, we obtain from Hölder's inequality that

$$\begin{aligned}
I_3\ &\le(\beta+1)\int_0^1 r^\beta\xi|z_\delta'||z_\delta-z|dr\to 0,\\
I_4\ &\le\int_0^1 r^\beta|\xi'||z_\delta'||z_\delta-z|dr\to 0,
\end{aligned}$$

as $\delta=\delta_n\to 0$.

From Lebesgue's dominated convergence theorem, we have

$$\int_0^1 r^{\beta+1}\lambda(r)\xi(z_\delta-z)dr\to 0,\ (\delta=\delta_n\to 0),$$

hence

$$\begin{aligned}
I_2\ &=\int_0^1 r^{\beta+1}\xi z_\delta'(z_\delta'-z')dr\\
&=\int_0^1 r^{\beta+1}\xi|z_\delta'-z'|^2+\int_0^1 r^{\beta+1}\xi z'(z_\delta'-z')dr\\
&=I_{21}+I_{22}\to 0,\ (\delta=\delta_n\to 0).
\end{aligned}$$

From (15) and Hölder's inequality, we have as $\delta=\delta_n\to 0$

$$\begin{aligned} I_{22} &= \int_0^1 r^{\beta+1}\xi z'(z'_\delta - z')dr \\ &\le C\int_0^1 r^{\beta/2}z'\cdot r^{\beta/2}(z'_\delta - z')dr \to 0. \end{aligned}$$

Thus (1) follows.

Now we prove (2). From Hölder's inequality, Lemma 2.3, (16) and conclusion (1), we deduce

$$\begin{aligned} & \int_0^1 r^{\beta+1}\xi\big||z'_\delta|^2-|z'|^2\big|dr \\ \le\ & 2\int_0^1 r^{\beta+1}\xi(|z'_\delta|+|z'|)|z'_\delta - z'|dr \\ \le\ & 2\Big(\int_0^1 r^{\beta+1}\xi(|z'_\delta|+|z'|)^2dr\Big)^{1/2}\cdot\Big(\int_0^1 r^{\beta+1}\xi|z'_\delta-z'|^2dr\Big)^{1/2} \\ \to\ & 0,\ (\delta=\delta_n\to 0), \end{aligned}$$

and (2) follows.

Next we prove (3). Indeed we have

$$\begin{aligned} & \int_0^1 r^{\beta+1}\xi\Big|\frac{z'_\delta}{r+\delta}-\frac{z}{r}\Big|dr \\ \le\ & \int_0^1 \frac{r}{r+\delta}r^\beta\xi|z'_\delta - z'|dr + \int_0^1 r^\beta\xi\Big|\frac{r}{r+\delta}-1\Big||z'|dr \\ =\ & J_1+J_2. \end{aligned}$$

From conclusion (1) and Hölder's inequality, we have

$$J_1 \le C\Big(\int_0^1 r^{\beta+1}\xi|z'_\delta - z'|^2dr\Big)^{1/2}\to 0,\ (\delta=\delta_n\to 0).$$

Since $\frac{r}{r+\delta}\to 1$ a.e. in $(0,1)(\delta=\delta_n\to 0)$, by similar proof of I_3, we have $J_2\to 0$. Hence (3) follows.

Finally we need to prove (4). At first, we obtain

$$\begin{aligned} & \int_0^1 r^{\beta+1}\xi\Big|\frac{|z'_\delta|^2}{z_\delta+\delta^2}-\frac{|z'|^2}{z}\Big|dr \\ =\ & \int_0^1 r^{\beta+1}\xi\frac{\big||z'_\delta|^2-|z'|^2\big|}{z_\delta+\delta^2}dr + \int_0^1 r^{\beta+1}\xi|z'|^2\Big|\frac{1}{z_\delta+\delta^2}-\frac{1}{z}\Big|dr \\ =\ & K_1+K_2. \end{aligned}$$

From conclusion (2), (8), there holds

$$K_1 \leq C \max_{r \in supp\ \xi} (\omega)^{-2} \int_{supp\ \xi} r^{\beta+1} ||z'_\delta|^2 - |z'|^2| dr \to 0,$$

as $\delta = \delta_n \to 0$. From (12) we have $\frac{1}{z_\delta + \delta^2} \to \frac{1}{z}$ uniformly in $[\sigma, 1]$, for any $0 < \sigma < 1/2$. Similarly, we deduce

$$K_2 \leq C \max_{r \in supp\ \xi} \left| \frac{1}{z_\delta + \delta^2} - \frac{1}{z} \right| \cdot \int_{supp\ \xi} r^{\beta+1} \xi |z'|^2 dr \to 0,$$

as $\delta = \delta_n \to 0$. Thus (4) is true.

From Lemma 2.4, we see that z satisfies the integral identity of Definition 2.1. To finish the proof of Theorem 2.1, it remains to prove that $z'(0) = z'(1) = 0$. From (13), we have

$$C_0 \frac{\omega^2}{r} \leq \frac{z(r)}{r} \leq C_1 r, \ r \in (0, \frac{1}{2}),$$

$$C_0 \frac{\omega^2}{1-r} \leq \frac{z(r)}{1-r} \leq C_1 (1-r), \ r \in (\frac{1}{2}, 1).$$

Note that $\omega = \frac{1}{2}(r - r^2)$, we obtain

$$\lim_{r \to 0^+} \frac{z(r)}{r} = \lim_{r \to 1^-} \frac{z(r)}{1-r} = 0,$$

i.e. $z'(0) = z'(1) = 0$.

Acknowledgement. The authors are very grateful for the referee's and the editor's helpful advices.

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(Received June 10, 2011)

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