Electronic Journal of Qualitative Theory of Differential Equations
2022, No. **33**, 1–10; https://doi.org/10.14232/ejqtde.2022.1.33 www.math.u-szeged.hu/ejqtde/

Periodic and bounded solutions of functional differential equations with small delays

Michal Fečkan ✉ and **Július Pačuta**

Comenius University, Faculty of Mathematics, Physics and Informatics,
Mlynska dolina, 842 48, Bratislava, Slovakia

Received 12 August 2021, appeared 15 July 2022
Communicated by Josef Diblík

Abstract. We study existence and local uniqueness of periodic solutions of nonlinear functional differential equations of first order with small delays. Bifurcations of periodic and bounded solutions of particular periodically forced second-order equations with small delays are investigated as well.

Keywords: periodic and bounded solutions, small delay, topological degree, bifurcation.

2020 Mathematics Subject Classification: 34K13, 34K18.

1 Introduction

In this paper, we study existence and local uniqueness of periodic solutions of nonlinear delay first-order equation

$$\dot{x}(t) = f(x(t-\varepsilon), t), \quad t \in \mathbb{R} \tag{1.1}$$

where ε is a positive parameter, $f : \mathbb{R}^2 \to \mathbb{R}$ and f satisfies assumptions fully specified in Theorems 3.1 and 3.2 below. We use classical methods such as Leray–Schauder degree and a priori estimates to prove that for sufficiently small parameters ε and under certain assumptions to right-hand-side function f, there is a locally unique periodic solution that depends continuously on ε. Similar methods were used e.g., in [7,8] where more complicated neutral differential equations were studied. Bifurcation theory is applied for perturbed second order case of (1.1) to get existence and non-existence results for periodic and bounded solutions with examples in Section 4. Related results to this paper are derived in [1]. We refer the reader to [2] for more papers dealing with the effects of small delays on the dynamical behaviors of systems compared with differential equations without delays.

✉ Corresponding author. Email: michal.feckan@fmph.uniba.sk

2 Preliminaries

As we already mentioned our paper deals with periodic solutions of equation (1.1). Such equations are usually equipped with initial condition $x(t) = \varphi(t)$ for $t \in [-\varepsilon, 0]$ where function φ is given. To avoid defining an initial condition for periodic solutions we introduce the following new problem. We see that a function x is a T-periodic solution of (1.1) and $\varepsilon \in (0, T)$ if and only if it is a solution of the problem

$$\begin{aligned} \dot{x}(t) &= f\left(x\left(t-\varepsilon\right), t\right), & t &\in [0, T], \\ x(t) &= x(T+t), & t &\in [-\varepsilon, 0]. \end{aligned} \tag{2.1}$$

The corresponding problem (when $\varepsilon = 0$) is then

$$\begin{aligned} \dot{x}(t) &= f\left(x\left(t\right), t\right), \quad t \in [0, T], \\ x(0) &= x(T). \end{aligned} \tag{2.2}$$

Here we introduce some notation we will use in the rest of our paper. Let X be the space of continuous, T-periodic functions defined on $\mathbb{R}$ equipped with the maximum norm $\|x\|_\infty := \max_{t \in \mathbb{R}} |x(t)|$. We define the closed ball in X as

$$B_r(y) =: \{x \in X;\ \|x - y\|_\infty \le r\}$$

and let $I : X \to X$ be the identical operator. For the Leray–Schauder degree of function f on domain Ω at point 0, we will use the standard notation $\deg(f, \Omega, 0)$. Properties of Leray–Schauder degree can be found in e.g., in [4].

3 Existence results

Theorem 3.1. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x,t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t. Let there exist $\delta, \eta, K, L > 0$ such that $-L + \eta < K - \eta$ and either*

$$f(x,t) \ge \delta \text{ for } (x,t) \in [-L-\eta, -L+\eta] \times [0,T] \\ \text{and} \quad f(x,t) \le -\delta \text{ for } (x,t) \in [K-\eta, K+\eta] \times [0,T], \tag{3.1}$$

or

$$f(x,t) \le -\delta \text{ for } (x,t) \in [-L-\eta, -L+\eta] \times [0,T] \\ \text{and} \quad f(x,t) \ge \delta \text{ for } (x,t) \in [K-\eta, K+\eta] \times [0,T] \tag{3.2}$$

is satisfied. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, there exists a solution of problem (2.1) that is bounded by K and $-L$.

Proof. Denote

$$\Omega = \{x \in X;\ x([0,T]) \subset (-L, K)\}.$$

First, we assume that the condition (3.1) is valid. We define the following operator

$$F : [0,T] \times X \to X, \qquad F(\varepsilon, x)(t) = x(T) + \int_0^t f\left(\widetilde{x}\left(s-\varepsilon\right), s\right)\, \mathrm{d}s \tag{3.3}$$

where $\tilde{x}(t) = x(t)$ if $t \in [0, T]$, and $\tilde{x}(t) = x(T+t)$ if $t \in [-\varepsilon, 0)$. Clearly, the operator F is well defined. Note that F is also continuous and compact due to the local boundedness and continuity of f. In the following, we will use the notation $F_\varepsilon := F(\varepsilon, .)$ whenever $\varepsilon \geq 0$ is fixed.

Our goal is to prove that if $\varepsilon_0 > 0$ is sufficiently small then for the Leray–Schauder degree, there holds

$$\deg(I - F_\varepsilon, \Omega, 0) = \deg(I - F_0, \Omega, 0) = 1$$

for $\varepsilon \in (0, \varepsilon_0]$.

We will prove that $(I - \theta F_0)x \neq 0$ for every $x \in \partial\Omega$ and $\theta \in [0, 1]$. This is clearly true for $\theta = 0$. Assume that there exists some $\theta \in (0, 1)$ and a solution $x \in \partial\Omega$ of problem

$$\begin{aligned} \dot{x}(t) &= \theta f\left(x\left(t\right), t\right) \quad \text{for } t \in [0, T], \\ x(0) &= \theta x(T). \end{aligned} \tag{3.4}$$

This means that x is a fixed point of the operator θF_0. Since $x \in \partial\Omega$, there exists either $t_0 \in [0, T]$ and $x(t_0) = K$, or $t_1 \in [0, T]$ and $x(t_1) = -L$. We will deal with the first case that x attains maximum $K > 0$ at some t_0, since the proof is similar in the second case.

Due to the assumption (3.1), we see that $\dot{x}_0(t_0) \leq -\delta$ and hence x is decreasing in some neighbourhood of t_0. Then necessarily $t_0 = 0$, otherwise x would attain higher values for some $t < t_0$. The solution x satisfies the problem (3.4), hence $x(T) \geq K$. This is not possible, since x is decreasing in some neighbourhood of 0 and decreases whenever x reaches value K due to (3.1). Thus, we proved $\deg(I - \theta F_0, \Omega, 0) = \deg(I, \Omega, 0) = 1$.

Finally, we will prove that $(I - \theta F_0 - (1-\theta)F_\varepsilon)x \neq 0$ for every $x \in \partial\Omega$ and $\theta \in [0, 1]$. This means we have to show that there are no solutions of problem

$$\begin{aligned} \dot{x}(t) &= \theta f\left(x\left(t\right), t\right) + (1-\theta) f\left(x\left(t-\varepsilon\right), t\right), \quad t \in [0, T], \\ x(t) &= x(T+t), \quad t \in [-\varepsilon, 0] \end{aligned} \tag{3.5}$$

that lie on the boundary of Ω. Let there exist a solution x of (3.5) that attains its maximum K at some $t_0 \in [0, T]$ (the case when x attains minimum $-L$ would be treated similarly). This solution can be periodically extended to the whole real line. Moreover, x is Lipschitz continuous with some Lipschitz constant M that is a bound for f on set $\overline{\Omega}$. For some $\varepsilon_0 > 0$ sufficiently small and dependent on x, there holds $x(t_0 - \varepsilon) \in [K - \eta, K]$ for $\varepsilon \in (0, \varepsilon_0]$ and so $\dot{x}(t_0) \leq -\delta$. This is a contradiction, since x is periodic and attains maximum at t_0.

The next step is to remove the dependence of ε_0 on solution x. For every $x \in \overline{\Omega}$, there holds

$$|x(t_0) - x(t_0 - \varepsilon)| \leq M\varepsilon_0 \quad \text{for all } \varepsilon \in [0, \varepsilon_0].$$

If we choose $\varepsilon_0 = \frac{\eta}{M}$ then both values $x(t_0)$ and $x(t_0 - \varepsilon)$ stay in the interval $[K - \eta, K]$ for $\varepsilon \in [0, \varepsilon_0]$. Therefore since x satisies the problem (3.5), we get $\dot{x}(t_0) \leq -\delta$ what contradicts the fact that x attains its maximum at t_0. Thus we proved $\deg(I - F_\varepsilon, \Omega, 0) = \deg(I - F_0, \Omega, 0) = 1$ for $\varepsilon \in (0, \varepsilon_0]$.

Next, we assume that the condition (3.2) is valid. In this case, our goal is to prove the existence of T-periodic solution of problem

$$\begin{aligned} \dot{y}(t) &= -f\left(y\left(t+\varepsilon\right), -t\right), \quad t \in [0, T], \\ y(t) &= y(T+t), \qquad\qquad t \in [0, \varepsilon]\,. \end{aligned} \tag{3.6}$$

for all $\varepsilon > 0$ sufficiently small, since then we just set $s = -t$ and $x(s) := y(-s)$. Then x will be a T-periodic solution of the original problem (2.1).

We define the following operator

$$G : [0,T] \times X \to X, \qquad G(\varepsilon, x)(t) = x(T) - \int_0^t f\left(\widetilde{x}\left(s+\varepsilon\right), -s\right)\,\mathrm{d}s$$

where $\widetilde{x}(t) = x(t)$ if $t \in [0,T]$, and $\widetilde{x}(t) = x(t-T)$ if $t \in (T, T+\varepsilon]$. Note that the right-hand-side function in problem (3.6) satisfies the assumption (3.1). Using the notation $G_\varepsilon := G(\varepsilon, .)$ for $\varepsilon \geq 0$ fixed and using similar arguments as in the previous part of the proof, we come to conclusion

$$\deg(I - G_\varepsilon, \Omega, 0) = \deg(I - G_0, \Omega, 0) = 1$$

for $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small. □

Theorem 3.2. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x,t)$ be a bounded continuous function, T-periodic in variable t and there exist $K, L, \delta > 0$ such that either*

$$\int_0^T f(x,t)\,\mathrm{d}t \geq \delta \quad \text{for } x \geq K \quad \text{and} \quad \int_0^T f(x,t)\,\mathrm{d}t \leq -\delta \quad \text{for } x \leq -L \tag{3.7}$$

or

$$\int_0^T f(x,t)\,\mathrm{d}t \leq -\delta \quad \text{for } x \geq K \quad \text{and} \quad \int_0^T f(x,t)\,\mathrm{d}t \geq \delta \quad \text{for } x \leq -L. \tag{3.8}$$

Then for every $\varepsilon > 0$ there exists a solution of problem (2.1).

Proof. In our proof, we will proceed under the case (3.7) only since the case (3.8) would be dealt with similarly as in the proof of Theorem 3.1 under the assumption (3.2). We define the following operators

$$\begin{aligned} &H : X \to X, \quad T : \mathbb{R} \times X \to \mathbb{R} \times X, \\ &H(x)(t) = \int_0^t f(x(s-\varepsilon), s)\,\mathrm{d}s - \frac{t}{T}\int_0^T f(x(s-\varepsilon), s)\,\mathrm{d}s, \\ &T(r,x)(t) = \left(\int_0^T f(x(s-\varepsilon), s)\,\mathrm{d}s, x(t) - r - H(x)(t)\right). \end{aligned}$$

Observe that $T(r,x) = 0$ for some $r \in \mathbb{R}$ and $x \in X$ if and only if x is a solution of (2.1) and $x(0) = r$.

As in the proof of Theorem 3.1, we will use the Leray–Schauder degree to prove the assertion. Let $M > 0$ be a global bound of right-hand-side function f and let $\alpha = \max\{K, L\} + 2MT + 1$, $\beta = \alpha + 2MT + 1$. We define the domain

$$\Omega := \{(r,x) \in \mathbb{R} \times X;\ |r| < \alpha,\ \|x\|_\infty < \beta\}$$

and the homotopy

$$T_\theta(r,x) = \left(\int_0^T f(r + \theta H(x)(s-\varepsilon), s)\,\mathrm{d}s, x(t) - \theta(r + H(x)(t))\right)$$

where $\theta \in [0,1]$. Our goal is to prove that

$$\deg(T_1, \Omega, 0) = \deg(T_0, \Omega, 0) = 1$$

which means that there exist $(r,x) \in \Omega$ such that $T_1(r,x) = 0$. One can easily prove that $T_1(r,x) = 0$ if and only if $T(r,x) = 0$.

Now, we prove that $T_\theta(r,x) \neq 0$ for $(r,x) \in \partial\Omega$ and $\theta \in [0,1]$. Note that $H(x)(0) = 0$ and $H(x) \in X$ for every $x \in X$. Since

$$|\dot{H}(x)(t)| = \left| f(x(t-\varepsilon),t) - \frac{1}{T}\int_0^T f(x(s-\varepsilon),s)\,\mathrm{d}s \right| \leq 2M$$

due to the boundedness of f, we have $|H(x)(t)| \leq 2MT$ for every $x \in X$ and $t \in \mathbb{R}$. Next, assume by contradiction that there is some $(r,x) \in \partial\Omega$ and $\theta \in [0,1]$ such that $T_\theta(r,x) = 0$. Then it holds

$$|x(t)| = \theta|r + H(x)(t)| \leq \alpha + 2MT < \beta, \quad t \in \mathbb{R}$$

for every $r \in [-\alpha,\alpha]$. Then necessarily $r = \pm\alpha$, otherwise $(r,x) \notin \partial\Omega$. For the case $r = \alpha$, we obtain

$$r + \theta H(x)(s-\varepsilon) \geq \alpha - 2MT \geq K,$$

so due to the assumption (3.7), it holds

$$\int_0^T f(r + \theta H(x)(s-\varepsilon),s)\,\mathrm{d}s \geq \delta.$$

This means that $T_\theta(\alpha,x) \neq 0$ and this is a contradiction. For the case $r = -\alpha$, we obtain a similar estimate

$$r + \theta H(x)(s-\varepsilon) \leq -\alpha + 2MT \leq -L$$

and using (3.7) leads to a contradiction. Thus $\deg(T_1,\Omega,0) = \deg(T_0,\Omega,0)$.

The identity $\deg(T_0,\Omega,0) = 1$ follows from the basic properties of the Leray–Schauder degree. In fact, the domain Ω can be represented as a Cartesian product of interval and a ball in the maximum norm. Hence

$$\deg(T_0,\Omega,0) = \deg((g,I),\Omega,0) = \deg(g,(-\alpha,\alpha),0)$$

where $g = g(r) = \int_0^T f(r,s)\,\mathrm{d}s$. Since $g(-\alpha) < 0 < g(\alpha)$ due to the assumption (3.7), we can define homotopy $g_\theta(r) = \theta g(r) + (1-\theta)r$ and we conclude that $\deg(g,(-\alpha,\alpha),0) = 1$. □

Lemma 3.3. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x,t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t. Denote by M a Lipschitz constant for function f and let $L > M$ be a given constant. Then for any $\varepsilon > 0$ such that*

$$\varepsilon < \frac{L-M}{L} e^{-LT}, \tag{3.9}$$

(periodic) solutions of problem (2.1) do not intersect each other.

Proof. Let x,y be two (periodic) solutions of (2.1) that intersect at some $t_0 \in [0,T]$. Introduce the norm $\|x\|_L := \max_{t\in[t_0,t_0+T]} e^{-L(t-t_0)}|x(t)|$. Let $z =: x - y$ and $t \in [t_0, t_0+T]$. Using standard estimates, the Lipschitz continuity of f (and M denotes the Lipschitz constant), the periodicity of function z and the equality

$$z(t) = \int_{t_0}^t (\dot{x} - \dot{y})(s)\,\mathrm{d}s = \int_{t_0}^t f(x(s-\varepsilon),s) - f(y(s-\varepsilon),s)\,\mathrm{d}s,$$

we obtain the following estimation

$$
\begin{aligned}
e^{-L(t-t_0)}|z(t)| &\le M\int_{t_0}^{t} e^{-L(t-s+\varepsilon)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\\
&\le M\left(\int_{t_0+\varepsilon}^{t} e^{-L(-s+\varepsilon+t)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\right.\\
&\quad\left.+\int_{t_0}^{t_0+\varepsilon} e^{-L(-s+\varepsilon+t)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\right)\\
&\le \frac{M}{L}\left(e^{-L\varepsilon}-e^{-L(t-t_0+\varepsilon)}\right)\|z\|_L\\
&\quad+\int_{t_0}^{t_0+\varepsilon} e^{L(s+T-t-\varepsilon)}e^{-L(s-\varepsilon+T-t_0)}|z(s-\varepsilon+T)|\,\mathrm{d}s\\
&\le \left(\frac{M}{L}+\varepsilon e^{LT}\right)\|z\|_L\,.
\end{aligned}
$$

Hence $z\equiv 0$ due to the assumption (3.9) and this concludes the proof of the lemma. □

Now, we are ready to prove the following Theorem 3.4. The proof relies on Theorem 3.1, however, Theorem 3.4 can be proven also using Theorem 3.2.

Theorem 3.4. *Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $f=f(x,t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t and let x_0 be a (T-periodic) solution of problem (2.2). Assume that there exists a constant $\eta>0$ such that function f is either increasing, or decreasing in variable x for every $t\in[0,T]$ and $x\in[x_0(t)-\eta,x_0(t)+\eta]$. Then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there exists a locally unique solution x_ε of problem (2.1) and x_ε depends continuously on ε.*

Proof. Assume that f is increasing in variable x for every $t\in[0,T]$ and $x\in[x_0(t)-\eta,x_0(t)+\eta]$. We define a new right-hand-side function

$$
g(x,t):=\begin{cases} f(x_0(t)-\eta,t)+x-x_0(t)+\eta, & (x,t)\in(-\infty,x_0(t)-\eta)\times[0,T]\,,\\ f(x,t), & (x,t)\in[x_0(t)-\eta,x_0(t)+\eta]\times[0,T]\,,\\ f(x_0(t)+\eta,t)+x-x_0(t)-\eta, & (x,t)\in(x_0(t)+\eta,\infty)\times[0,T]\,.\end{cases}
$$

Since g is increasing in variable x, the function $x_0\in X$ is the only periodic solution of equation

$$
\dot{x}(t)=g\left(x,t\right),\quad t\in[0,T]. \tag{3.10}
$$

In fact, since x_0 is the periodic solution of (3.10) then necessarily $\int_0^T g(x_0(t),t)\,\mathrm{d}t=0$. Due to the Lipschitz continuity of g, we know that any other solution y does not cross x_0, hence either $y(t)>x_0(t)$ or $y(t)<x_0(t)$ for all $t\in[0,T]$. In both cases due to the strict monotonicity of g, we get $\int_0^T g(y(t),t)\,\mathrm{d}t\neq 0$ so y cannot be periodic.

The new right-hand-side function g satisfies the assumptions of Theorem 3.1 and thus there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there exists at least one (periodic) solution x_ε of problem

$$
\begin{aligned}
\dot{x}(t)&=g\left(x\left(t-\varepsilon\right),t\right), && t\in[0,T],\\
x(t)&=x(T+t), && t\in[-\varepsilon,0]\,.
\end{aligned} \tag{3.11}
$$

Moreover, all such solutions are uniformly bounded independently of ε. We need to verify that for some $\varepsilon_0>0$ sufficiently small, the solution x_ε of (3.11) is also a solution of original problem (2.2). More precisely, we prove that there exists some $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there holds $x_\varepsilon \in B_\eta(x_0)$. Assume that this is not true, i.e. there exists a sequence of positive parameters $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and for every $n \in \mathbb{N}$, it holds $x_{\varepsilon_n} \notin B_\eta(x_0)$. Recall that the solutions x_{ε_n} are fixed points of the operator F defined by (3.3) and due to the compactness of F and the uniform boundedness of functions x_{ε_n}, the set $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is relatively compact in X. Hence some subsequence converges uniformly to a periodic solution of equation (3.10) (since $\varepsilon_n \to 0$) that is not equal to x_0. This contradicts the uniqueness of periodic solution x_0.

Next, we will prove the local uniqueness of periodic solutions of original problem (2.2). More precisely, the solution x_ε is unique in the ball $B_\eta(x_0)$ In fact, let y be a periodic solution that lies in $B_\eta(x_0)$ and is not equal to x_ε. Then necessarily, there exists some $t_0 \in [0,T]$ such that $y(t_0) = x_\varepsilon(t_0)$, otherwise we would come to $\int_0^T f(y(t),t)\,\mathrm{d}t \neq 0$ due to the strict monotonicity of f. We choose some $L > M$ and $\varepsilon_0 > 0$ sufficiently small so the inequality (3.9) is valid for all $\varepsilon \in [0,\varepsilon_0]$. Hence the uniqueness follows from Lemma 3.3. This completes the proof of the local uniqueness of periodic solutions of problem (2.1) for small $\varepsilon > 0$.

The continuous dependence of solution x_ε on parameter ε is a consequence of uniform boundedness of these solutions and compactness of the operator F defined by (3.3). □

Remark 3.5. Lemma 3.3 need not to be true if the inequality (3.9) is not valid. In fact, consider the equation

$$\dot{x}(t) = x\left(t - \frac{3\pi}{2}\right).$$

This equation possesses infinitely many 2π-periodic solutions of form $a\sin(x+b)$ for $a \in \mathbb{R}$ and $b \in [0,\pi)$ and every two of these solutions intersect each other.

4 Bifurcations

We consider the perturbed equation

$$\ddot{x}(t) + g(x(t - \varepsilon\mu_1)) + \varepsilon\mu_2 h(t) = 0 \tag{4.1}$$

where $g, h \in C^3(\mathbb{R},\mathbb{R})$, $\mu_1, \mu_2 \in \mathbb{R}$ and $h(t)$ is T-periodic.

Theorem 4.1. *Assume that there is a T-periodic solution $u(t)$ of equation*

$$\ddot{u} + g(u) = 0 \tag{4.2}$$

such that $v(t) = \dot{u}(t)$ is the only T-periodic solution up to a scalar multiple of

$$\ddot{v} + g'(u(t))v = 0.$$

If the function

$$M(\alpha) = \mu_2 \int_0^T h(t+\alpha)\dot{u}(t)dt - \mu_1 \int_0^T g'(u(t))\dot{u}^2(t)dt$$

has a simple zero α_0, i.e., $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then for any $\varepsilon \neq 0$ small, the equation (4.1) has the unique T-periodic solution $x_\varepsilon(t)$ that satisfies

$$\sup_{t\in\mathbb{R}} |x_\varepsilon(t) - u(t-\alpha_0)| + |\dot{x}_\varepsilon(t) - \dot{u}(t-\alpha_0)| = O(\varepsilon). \tag{4.3}$$

Proof. Note that the equation (4.1) can be written in the form

$$\ddot{x}(t) + g(x(t)) - \varepsilon\mu_1 g'(x(t))\dot{x}(t) + \varepsilon\mu_2 h(t) = O(\varepsilon^2).$$

Hence we can apply the well-known Melnikov theory (see [3,5]) to obtain the result. □

Remark 4.2. 1. Note that $M(\alpha)$ is T-periodic and

$$M(\alpha) = -\mu_1 \int_0^T \ddot{u}^2(t)dt - \mu_2 \int_0^T \dot{h}(t+\alpha)u(t)dt. \tag{4.4}$$

2. The function $u(t)$ is embedded into a 1-parametric family of periodic solutions $u_a(t)$, $a \in (a_1, a_2) \subset \mathbb{R}$ of (4.2) with minimal periods $T(a)$. So $u_{a_0}(t) = u(t)$. If $T'(a_0) \neq 0$ then the assumption of Theorem 4.1 holds.

3. The existence part of Theorem 4.1 holds if

$$\max_{\alpha\in\mathbb{R}} M(\alpha) \min_{\alpha\in\mathbb{R}} M(\alpha) < 0.$$

4. If $M(\alpha) \neq 0, \forall \alpha \in \mathbb{R}$ then there is no bifurcation.

5. If $x(t)$ is a solution of (4.1) then $x(t+kT)$, $k \in \mathbb{Z}$ is also a solution. So we consider in Theorem 4.1 just $\alpha_0 \in [0, T]$.

To illustrate the theory, we consider the following example

$$\ddot{x}(t) + x(t-\varepsilon\mu_1) + x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.5}$$

So (4.2) is the Duffing equation

$$U''(t) + U(t) + U^3(t) = 0$$

possessing a family of periodic solutions

$$u_a(t) = a\,\mathrm{cn}(\sqrt{1+a^2}t)$$

for $a > 0$ with periods $T(a) = \frac{4K(k)}{\sqrt{1+a^2}}$, $k = \frac{a}{\sqrt{2+2a^2}}$. Note $u_a(0) = a$ and $u_a'(0) = 0$. Here cn is the Jacobi elliptic function, $K(k)$ is the complete elliptic function of the first kind and k is the elliptic modulus, see [6]. Moreover, we have

$$T'(a) = \frac{8(E(k)-K(k)) - 4a^2K(k)}{a\sqrt{1+a^2}\,(2+a^2)} < 0,$$

since $E(k) \leq K(k)$, where $E(k)$ is the complete elliptic function of the second kind. So $T(a)$ is decreasing from $T(0) = 2\pi$ to 0, and hence Remark 4.2 2 can be applied. Now $T = \pi$, so we numerically solve $T(a) = \pi$ to get $a_0 \cong 2.03284$ and then (4.4) has the form

$$M(\alpha) = -105.817\mu_1 + 6.17466\mu_2 \sin 2\alpha. \tag{4.6}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.3. *If $|\mu_1| < 0.058352|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.5) has precisely two π-periodic solutions orbitally near $u_{a_0}(t) = 2.03284\,\mathrm{cn}(2.26549t)$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0, \pi)$, namely for roots of (4.6). If $|\mu_1| > 17.1374|\mu_2|$ then (4.5) has no π-periodic solutions orbitally near $u_{a_0}(t)$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$.*

We end this paper with extending the above bifurcation results of periodic solutions to bounded ones.

Theorem 4.4. *Assume that there are x_0 and x_1 such that $g(x_0) = g(x_1) = 0$ and $g'(x_0) < 0$, $g'(x_1) < 0$. Suppose there is a solution $u(t)$ of (4.2) such that $\lim_{t\to-\infty} u(t) = x_0$ and $\lim_{t\to\infty} u(t) = x_1$. If the function*

$$M(\alpha) = -\mu_1 \int_{-\infty}^{\infty} \ddot{u}^2(t)dt + \mu_2 \int_{-\infty}^{\infty} h(t+\alpha)\dot{u}(t)dt \tag{4.7}$$

has a simple zero α_0 then for any $\varepsilon \neq 0$ small, the equation (4.1) has the unique solution $x_\varepsilon(t)$ that satisfies (4.3).

Remark 4.5. The points 3, 4 and 5 of Remark 4.2 remain valid for this case.

To illustrate the theory, we consider

$$\ddot{x}(t) - x(t-\varepsilon\mu_1) + x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.8}$$

So (4.2) is the Duffing equation

$$U''(t) - U(t) + U^3(t) = 0$$

possessing a homoclinic solution

$$u(t) = \sqrt{2}\operatorname{sech} t$$

to $x_0 = x_1 = 0$. Again $h(t) = \cos 2t$. Then the Melnikov function (4.7) is now

$$M(\alpha) = -\frac{28}{15}\mu_1 + 2\sqrt{2}\pi \operatorname{sech} \pi\mu_2 \sin 2\alpha. \tag{4.9}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.6. *If $|\mu_1| < \frac{15\pi \operatorname{sech} \pi}{7\sqrt{2}}|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.8) has precisely two bounded solutions orbitally near $\sqrt{2}\operatorname{sech} t$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0,\pi)$, namely for roots of (4.9). If $|\mu_1| > \frac{15\pi \operatorname{sech} \pi}{7\sqrt{2}}|\mu_2|$ then (4.8) has no bounded solutions orbitally near $\sqrt{2}\operatorname{sech} t$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$. Note $\frac{15\pi \operatorname{sech} \pi}{14\sqrt{2}} \cong 0.41065$.*

Finally, we consider

$$\ddot{x}(t) + x(t-\varepsilon\mu_1) - x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.10}$$

So (4.2) is the Duffing equation

$$U''(t) + U(t) - U^3(t) = 0$$

possessing a heteroclinic solution

$$u(t) = \tanh(t/\sqrt{2}).$$

to $x_0 = -1$ and $x_1 = 1$. Again $h(t) = \cos 2t$. Then the Melnikov function (4.7) is now

$$M(\alpha) = -\frac{4\sqrt{2}}{15}\mu_1 + 2\sqrt{2}\pi \operatorname{csch} \sqrt{2}\pi\mu_2 \cos 2\alpha. \tag{4.11}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.7. *If $|\mu_1| < \frac{15}{2}\pi \operatorname{csch} \sqrt{2}\pi|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.10) has precisely two bounded solutions orbitally near $\tanh(t/\sqrt{2})$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0,\pi)$, namely for roots of (4.11). If $|\mu_1| > \frac{15}{2}\pi \operatorname{csch} \sqrt{2}\pi|\mu_2|$ then (4.10) has no bolunded solutions orbitally near $\tanh(t/\sqrt{2})$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$. Note $\frac{15}{2}\pi \operatorname{csch} \sqrt{2}\pi \cong 0.554347$.*

Acknowledgements

We thank for the referee important comments. M.F. was supported by the Slovak Research and Development Agency under the contract No. APVV-18-0308 and by the Slovak Grant Agency VEGA No. 1/0358/20. J.P. was supported by the Slovak Research and Development Agency under the contract No. APVV-18-0308, in part by grants VEGA No. 1/0358/20 and VEGA No. 1/0339/21.

References

[1] D. D. Bainov, V. Ch. Covachev, Periodic solutions of impulsive systems with a small delay, *J. Phys. A* **27**(1994), No. 16, 5551–5563. MR1295379; Zbl 0837.34069

[2] S. Chen, J. Shen, Smooth inertial manifolds for neutral differential equations with small delays, *J. Dynam. Differential Equations*, published online, 2021. `https://doi.org/10.1007/s10884-021-09993-1`

[3] C. Chicone, *Ordinary differential equations with applications*, Texts in Applied Mathematics, Vol. 34, Springer, New York, 2006. `https://doi.org/10.1007/0-387-35794-7`; MR2224508; Zbl 1120.34001

[4] K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, 1985. `https://doi.org/10.1007/978-3-662-00547-7`; MR787404; Zbl 0559.47040

[5] J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Applied Mathematical Sciences, Vol. 42, Springer-Verlag, New York, 1983. `https://doi.org/10.1007/978-1-4612-1140-2`; MR709768; Zbl 0515.34001

[6] D. F. Lawden, *Elliptic functions and applications*, Applied Mathematical Sciences, Vol. 80, Springer-Verlag, New York, 1989. `https://doi.org/10.1007/978-1-4757-3980-0`; MR1007595; Zbl 0689.33001

[7] B. Liu, L. Huang, Existence and uniqueness of periodic solutions for a kind of first order neutral functional differential equations, *J. Math. Anal. Appl.* **322**(2006), No. 1, 121–132. `https://doi.org/10.1016/jmaa.2005.08.069`; MR2238153; Zbl 1101.34054

[8] S. Lu, W. Ge, On the existence of periodic solutions for neutral differential equation, *Nonlinear Anal.* **54**(2003), No. 7, 1285–1306. `https://doi.org/10.1016/S0362-546X(03)00187-1`; MR1995931; Zbl 1037.34064