

Nonconstant positive steady states and pattern formation of a diffusive epidemic model

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Received 11 August 2021, appeared 9 May 2022

Communicated by Péter L. Simon

Abstract. It is our purpose in this paper to make a detailed description for the structure of the set of the nonconstant steady states for the two-dimensional epidemic S-I model with diffusion incorporating demographic and epidemiological processes with zero-flux boundary conditions. We first study the conditions of diffusion-driven instability occurrence, which induces spatial inhomogeneous patterns. The results will extend to the derivative of prey's functional response with prey is positive. Moreover, we establish the local and global structure of nonconstant positive steady state solutions. A priori estimates for steady state solutions will play a key role in the proof. Our results indicate that the diffusion has a great influence on the spread of the epidemic and extend well the finding of spatiotemporal dynamics in the epidemic model.

Keywords: epidemic model, Turing instability, local and global structure, pattern.

2020 Mathematics Subject Classification: 35B35, 35K57, 92D25.

1 Introduction

Since the pioneer work of King [15], Kermack and McKendrick [14], mathematical models have been contributing to improve our understanding of infectious disease dynamics and help us develop preventive measures to control infection spread. Over a period of time, researchers in theoretical and mathematical epidemiology have proposed many epidemic models, and the temporal dynamics of infectious disease transmission described with differential equations has been investigated in either qualitative or numerical analysis [1, 2, 6, 20].

In epidemic models, the incidence rate plays a key role in the spread of an infection [3, 6, 8, 17, 19, 21, 24]. Traditionally, two different types of incidence rate are been frequently used in well-known epidemic models [4, 9]: The density-dependent transmission is the case in which the contact rate between susceptible and infective individuals increases linearly with population size; the frequency-dependent transmission is the case in which the number of contacts is independent of population size [13].

In [5], the susceptible S is a capable of reproducing with logistic law and strong Allee effect and the infected individuals I do not reproduce but they still contribute with S to population

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growth toward the carrying capacity. This assumption is based on [28–30]. If we assume I is capable of reproducing without strong Allee effect, and assume that the disease is not to be transmitted to offspring, newborns of the infected are in the susceptible class. The infected I is removed only by death at rate μ , there is no recovery from the disease. The disease transmission is assumed to be standard incidence term $\frac{\beta SI}{S+I}$, and no vertical transmission, i.e., the number of contacts between infected and susceptible individuals is constant [11]. The transmission coefficient is $\beta > 0$.

From the above assumption, we can establish the following model

$$\begin{aligned}\frac{dS}{dt} &= r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I}, \\ \frac{dI}{dt} &= \frac{\beta SI}{S + I} - \mu I.\end{aligned}\tag{1.1}$$

Here S and I denote the density of the uninfected (susceptible) and infected hosts, respectively. All parameters are nonnegative. Parameter r denotes the maximum birth rate of the hosts; and $0 \leq \rho \leq 1$ describes the reducing reproduction ability of infected hosts: $\rho = 0$ means that infected hosts lose their reproducing ability while $\rho = 1$ indicates that they experience no reduction in reproduction fitness; a measures the per capita density-dependent reduction in birth rate. If $a \neq 0$, then $1/a$ is also called the carrying capacity; if $a = 0$, then the model not consider horizontally transmitted that not reduces fecundity and survival of its host, which in turn is not regulated by density-dependent birth. However, considering the impact of various aspects such as resources and environment on population growth and the model has more practical significance, this paper mainly considers $a \neq 0$.

Suppose that the susceptible (S) and the infections individuals (I) move randomly in the space-described as Brownian random motion [10], and then we propose a simple spatial model corresponding to (1.1) as follows

$$\begin{aligned}\frac{\partial S}{\partial t} - d_1 \Delta S &= r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I}, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} - d_2 \Delta I &= \frac{\beta SI}{S + I} - \mu I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} &= 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x), I(x, 0) &= I_0(x), & x \in \Omega.\end{aligned}\tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, \mathbf{n} is the outward unit normal vector of the boundary $\partial\Omega$ and the homogeneous Neumann boundary condition is being considered. The diffusion coefficients d_1 and d_2 are positive constants, and the initial data $S_0(x), I_0(x)$ are continuous functions. $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian operator in two-dimensional space, which describes the Brownian random motion. The diffusion model provides a useful framework to evaluate some spatially related control measures.

The *Turing instability* refers to “diffusion driven instability”, i.e., the stability of the endemic equilibrium changing from stable for the ordinary differential equations (ODE) dynamics (1.1), to unstable, for the partial differential equations (PDE) dynamics (1.2). And the reason of the occurrence of Turing pattern is the existence of nonconstant positive steady states of model (1.2) as a result of diffusion. And there naturally comes two questions:

(1) How about the existence of nonconstant positive steady states of model (1.2)?

(2) What is the structure of nonconstant positive steady states of model (1.2)?

The main goal of this paper is to solve the two questions above completely. So, we will concentrate on the following steady state problem corresponding to (1.2) is given by

$$\begin{aligned} -d_1\Delta S &= r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I}, & x \in \Omega, \\ -d_2\Delta I &= \frac{\beta SI}{S + I} - \mu I, & x \in \Omega, \\ \frac{\partial S}{\partial \mathbf{n}} &= \frac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{aligned} \quad (1.3)$$

The rest of the paper is organized as follows. In section 2, we perform a priori estimates of positive steady state solutions of (1.3). In section 3, the stability of constant steady state solution and the conditions of Turing instability of model (1.2) are discussed. In section 4, the existence, local and global structure of nonconstant positive solutions of (1.3) are investigated. In the last section, we make some comments on our studies and propose some interesting problems for future studies.

2 A priori estimates

In this section, we investigate the basic estimates of the reaction-diffusion model (1.3) use the following lemma.

Lemma 2.1 ([23]). *Suppose that $g \in C(\overline{\Omega} \times \mathbb{R})$.*

- (1) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies $\Delta w(x) + g(x, w(x)) \geq 0$ $x \in \Omega, \partial_\nu w \leq 0, x \in \partial\Omega$, if $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$;*
- (2) *Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and satisfies $\Delta w(x) + g(x, w(x)) \leq 0, x \in \Omega, \partial_\nu w \geq 0, x \in \partial\Omega$, if $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.*

Lemma 2.2 ([25]). *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n , and let $g \in C(\overline{\Omega} \times \mathbb{R})$.*

- (1) *If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta z + g(x, z) \geq 0 \quad \text{in } \Omega, \quad \partial_n z \leq 0 \quad \text{on } \partial\Omega.$$

and if there is a constant K such that $g(x, z) < 0$ for $z > K$, then $z \leq K$ a.e. in Ω .

- (2) *If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities*

$$\Delta z + g(x, z) \leq 0 \quad \text{in } \Omega, \quad \partial_n z \geq 0 \quad \text{on } \partial\Omega.$$

and if there is a constant K such that $g(x, z) > 0$ for $z < K$, then $z \geq K$ a.e. in Ω .

In order to obtain the existence of nonconstant positive steady states, a priori estimates will play a key role. Our main result in this section is the following.

Theorem 2.3. *If $d_1 < d_2$, or $d_1 > d_2$ and $d_1(\beta - \mu) < d_2\beta$, then all the non-negative solutions of model (1.3) that start in Ω are bounded with ultimate bound $\Gamma = \frac{1}{a}$ independent of the initial conditions.*

Proof. Model (1.3) can reduce to

$$\begin{aligned} -d_1\Delta S &= r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I}, \\ -d_1\Delta I &= \frac{d_1}{d_2} \frac{\beta SI}{S + I} - \frac{d_1}{d_2} \mu I. \end{aligned} \quad (2.1)$$

Summing up the two equations of (2.1), we have

$$-d_1\Delta(S + I) = r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I} + \frac{d_1}{d_2} \frac{\beta SI}{S + I} - \frac{d_1}{d_2} \mu I. \quad (2.2)$$

(i) If $d_1 < d_2$, then from (2.2) it follows that

$$\begin{aligned} -d_1\Delta(S + I) &\leq r(S + \rho I)[1 - a(S + I)] - \left(1 - \frac{d_1}{d_2}\right) \frac{\beta SI}{S + I} \\ &\leq r(S + \rho I)[1 - a(S + I)]. \end{aligned}$$

(ii) If $d_1 > d_2$ and $d_1(\beta - \mu) < d_2\beta$, then from (2.2) lead to

$$\begin{aligned} -d_1\Delta(S + I) &\leq r(S + \rho I)[1 - a(S + I)] + \left(\left(\frac{d_1}{d_2} - 1\right)\beta - \frac{d_1}{d_2}\mu\right) I \\ &\leq r(S + \rho I)[1 - a(S + I)]. \end{aligned}$$

In addition, by Lemma 2.2, we have $0 < S + I \leq \frac{1}{a}$, and easy to see that $\Gamma = \frac{1}{a}$ independent of the initial conditions, then we can conclude the proof. \square

Theorem 2.4. *If $(S(x), I(x))$ is any positive solution of (1.3) and $\beta > \mu$ holds, then*

$$0 < S(x) < \frac{1}{a}, \quad 0 < I(x) < \frac{\beta - \mu}{\mu a}, \quad x \in \overline{\Omega}.$$

Furthermore, if $M := \frac{r - ra\alpha(1 + \rho) - \beta}{ra} > 0$ holds, then $(S(x), I(x))$ satisfies

$$M < S(x) < \frac{1}{a}, \quad \frac{\beta - \mu}{\mu} M < I(x) < \frac{\beta - \mu}{\mu a}, \quad x \in \overline{\Omega}, \quad (2.3)$$

where $\alpha = \frac{\beta - \mu}{\mu a}$.

Proof. Let (S, I) be a given positive solution of (1.3). First of all, by Theorem 2.3, it is clear that $S(x) < \frac{1}{a}$, for all $x \in \overline{\Omega}$. To obtain the upper bound for I , we let for some $z_0 \in \overline{\Omega}$ such that $I(z_0) = \max I(x)$. By virtue of Lemma 2.1, we have

$$\frac{\beta S(z_0) I(z_0)}{S(z_0) + I(z_0)} \geq \mu I(z_0).$$

Thus

$$I(z_0) \leq \frac{\beta - \mu}{\mu} S(z_0) < \frac{\beta - \mu}{\mu a}.$$

In the following, we proof the lower bound of $(S(x), I(x))$, and $\alpha = \frac{\beta - \mu}{\mu a}$. Since

$$\begin{aligned} -d_1\Delta S &= r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I} \\ &\geq r(S + \rho I)[1 - a(S + I)] - \beta S \\ &\geq S(r - raS - ra\alpha(1 + \rho) - \beta) + r\rho I(1 - aI). \end{aligned}$$

By Theorem 2.3, we have

$$-d_1\Delta S \geq S(r - raS - ra\alpha(1 + \rho) - \beta).$$

Hence, by Lemma 2.2 and strong maximum principle, we can obtain

$$S(x) > \frac{r - ra\alpha(1 + \rho) - \beta}{ra} := M > 0.$$

Similarly, we have

$$I(x) > \frac{\beta - \mu}{\mu} M.$$

This completes our proof. \square

3 Constant steady states and Turing instability

In this section, we mainly discuss the stability of constant steady state solution. For convenience, we denote

$$g_1(S, I) = r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I}, \quad g_2(S, I) = \frac{\beta SI}{S + I} - \mu I. \quad (3.1)$$

Clearly, ODE model (1.1) or PDE model (1.2) has a unique constant steady state $E^* = (S^*, I^*)$ with positive coordinates

$$S^* = \left(1 - \frac{\mu(\beta - \mu)}{r(\mu + \rho(\beta - \mu))}\right) \frac{\mu}{a\beta}, \quad I^* = S^* \frac{\beta - \mu}{\mu}$$

if and only if

$$(P) \quad \mu < \beta < \frac{r\mu}{\mu - r\rho} + \mu \text{ and } \mu > r\rho \text{ hold.}$$

In addition, model (1.1) or model (1.2) has a trivial equilibrium $U_0 = (\frac{1}{a}, 0)$. By the standard linearization method, we can easily prove the following result.

Theorem 3.1. *The trivial equilibrium U_0 is locally asymptotically stable if $\mu > \beta$ and is unstable if $\mu < \beta$.*

Next, we will focus on the stability of E^* for model (1.1) and model (1.2), respectively. By simple calculation, the Jacobian matrix of (1.1) evaluated at E^* is given by

$$J(E^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (3.2)$$

where

$$\begin{aligned} a_{11} &= r[1 - a(S^* + I^*)] - ar(S^* + \rho I^*) - \frac{\beta I^{*2}}{(S^* + I^*)^2}, \\ a_{12} &= r\rho[1 - a(S^* + I^*)] - ar(S^* + \rho I^*) - \frac{\beta S^{*2}}{(S^* + I^*)^2}, \\ a_{21} &= \frac{\beta I^{*2}}{(S^* + I^*)^2} > 0, \quad a_{22} = -\frac{\beta S^* I^*}{(S^* + I^*)^2} < 0. \end{aligned} \quad (3.3)$$

The characteristic equation of $J(E^*)$ is

$$\eta^2 - T\eta + Q = 0,$$

where

$$T = a_{11} + a_{22}, \quad Q = a_{11}a_{22} - a_{21}a_{12}. \quad (3.4)$$

By direct calculation, under the condition **(P)**, we have $T < 0$, $Q > 0$. Thus, we can obtain the following theorem.

Theorem 3.2. *Assume condition **(P)** holds, then the constant steady state solution E^* of model (1.1) is locally asymptotically stable.*

Next, we analyse the stability of the endemic equilibrium E^* for the reaction-diffusion model (1.2). From now on, let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$$

be the sequence of eigenvalues for the operator $-\Delta$ subject to the Neumann boundary condition on Ω [7]. By $E(\lambda_i)$, we denote the space of eigenfunctions corresponding to λ_i in $H^1(\Omega)$. Set $\{\phi_{ij} : j = 1, 2, \dots, \dim E(\lambda_i)\}$ be the orthonormal basis of $E(\lambda_i)$, $\mathbf{X} = [H^1(\Omega)]^2$, $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} : \mathbf{c} \in \mathbb{R}^2\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{+\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbf{X}_{ij}.$$

Assume that $a_{11} > 0$ and $d_1\lambda_1 < a_{11}$, then we may define $N^0 = N^0(r, a, \rho, \beta, \mu, \Omega)$ to be the largest positive integer such that

$$d_1\lambda_i < a_{11}, \quad \text{for } i \leq N^0.$$

Obviously, if $d_1\lambda_1 < a_{11}$ is satisfied, then $1 \leq N^0 < \infty$. In this situation, define

$$\tilde{d}_2 := \min_{1 \leq i \leq N^0} d_{2,i}, \quad d_{2,i} = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}d_1\lambda_i}{\lambda_i(a_{11} - d_1\lambda_i)}. \quad (3.5)$$

And naturally we can give the stability of E^* of model (1.2).

Theorem 3.3. *Assume condition **(P)** holds.*

(i) *If $a_{11} < 0$, then E^* is locally asymptotically stable.*

(ii) *If $a_{11} > 0$, then*

(ii-1) *if $d_1\lambda_1 < a_{11}$ and $0 < d_2 < \tilde{d}_2$, then E^* is locally asymptotically stable;*

(ii-2) *if $d_1\lambda_1 < a_{11}$ and $d_2 > \tilde{d}_2$, then E^* is Turing unstable.*

Proof. Consider the linearization operator evaluated at E^* of model (1.2)

$$L = \begin{pmatrix} d_1\Delta + a_{11} & a_{12} \\ a_{21} & d_2\Delta + a_{22} \end{pmatrix}.$$

It is easy to see that the eigenvalues of L are given by those of the following operator L_i

$$L_i = \begin{pmatrix} -d_1\lambda_i + a_{11} & a_{12} \\ a_{21} & -d_2\lambda_i + a_{22} \end{pmatrix},$$

whose characteristic equation is

$$\zeta^2 - \zeta T_i + Q_i = 0, \quad i = 0, 1, 2, \dots, \quad (3.6)$$

where

$$\begin{aligned} T_i &= -(d_1 + d_2)\lambda_i + a_{11} + a_{22}, \\ Q_i &= \lambda_i(d_1\lambda_i - a_{11}) \left\{ d_2 - \frac{d_1 a_{22} \lambda_i - a_{11} a_{22} + a_{12} a_{21}}{\lambda_i(d_1\lambda_i - a_{11})} \right\}. \end{aligned} \quad (3.7)$$

(i) If $a_{11} < 0$, then $T_i < 0$ and $Q_i > 0$, which implies that $\text{Re}\{\zeta_i\} < 0$ for all eigenvalues ζ . Therefore, the constant solution E^* is locally asymptotically stable.

(ii) Since $T < 0, Q > 0$, then $T_i < 0$ and $d_1 a_{22} \lambda_i - a_{11} a_{22} + a_{12} a_{21} < 0$.

(ii-1) If $a_{11} > 0$, $d_1 \lambda_1 < a_{11}$ and $0 < d_2 < \tilde{d}_2$, then $d_1 \lambda_i < a_{11}$ and $d_2 < d_{2,i}$ for all $i \in [1, N^0]$. Thus

$$Q_i = \lambda_i(d_1\lambda_i - a_{11}) \left\{ d_2 - \frac{d_1 a_{22} \lambda_i - a_{11} a_{22} + a_{12} a_{21}}{\lambda_i(d_1\lambda_i - a_{11})} \right\} > 0.$$

One the other hand, if $i > N^0$, then $d_1 \lambda_i > a_{11}$. Thus, we have $Q_i > 0$. The analysis yields the local asymptotic stability of E^* .

(ii-2) If $a_{11} > 0$, $d_1 \lambda_1 < a_{11}$ and $d_2 > \tilde{d}_2$, then we may assume the minimum is attained at $j \in [1, N^0]$. Thus $d_2 > d_{2,j}$, which implies

$$Q_j = \lambda_j(d_1\lambda_j - a_{11}) \left\{ d_2 - \frac{d_1 a_{22} \lambda_j - a_{11} a_{22} + a_{12} a_{21}}{\lambda_j(d_1\lambda_j - a_{11})} \right\} < 0.$$

Hence, E^* is unstable in this case. □

Remark 3.4. From Theorem 3.2 and 3.3, we can know that if $a_{11} > 0$, under mild extra conditions, the stability of the constant equilibrium E^* may change from stable, for the (ODE) dynamics (1.1), to unstable, for the (PDE) dynamics (1.2), whereas those of other constant equilibria are invariant.

Remark 3.5. When we regard Q_i as a quadratic polynomial with respect to λ_i , i.e., $Q_i = d_1 d_2 \lambda_i^2 - (d_1 a_{22} + d_2 a_{11}) \lambda_i + a_{11} a_{22} - a_{12} a_{21}$, using the method of [26], we can also get that the condition of Turing instability: Assume that (P) and $a_{11} > 0$ hold. If

$$\frac{d_2}{d_1} > \frac{-(2a_{12}a_{21} - a_{11}a_{22}) + 2\sqrt{a_{12}a_{21}(a_{12}a_{21} - a_{11}a_{22})}}{a_{11}^2},$$

then Turing instability occurs.

Example 3.6. As an example, we take the parameters in model (1.2) as

$$a = 1, \quad \rho = 0.1, \quad \beta = 1, \quad \mu = 0.35, \quad r = 0.61, \quad d_1 = 0.01.$$

There is a unique positive equilibrium $E^* \approx (0.03546, 0.06586)$, and $a_{11} = 0.1 > 0, \tilde{d}_2 = 0.1073$.

For the ODE model (1.1), easy to verify that $T = -0.1275 < 0, Q = 0.0166 > 0$, then E^* is locally asymptotically stable from Theorem 3.2.

For the PDE model (1.2) on one-dimensional space domain $(0, \pi)$, $d_1 \lambda_1 - a_{11} = -0.09 < 0$. If $0.1 = d_2 < \tilde{d}_2$, then E^* is locally asymptotically stable (see Fig. 3.1), and if $0.25 = d_2 > \tilde{d}_2$, E^* is Turing instability from Theorem 3.3. The model (1.2) exhibits *Turing pattern* (see Fig. 3.2).

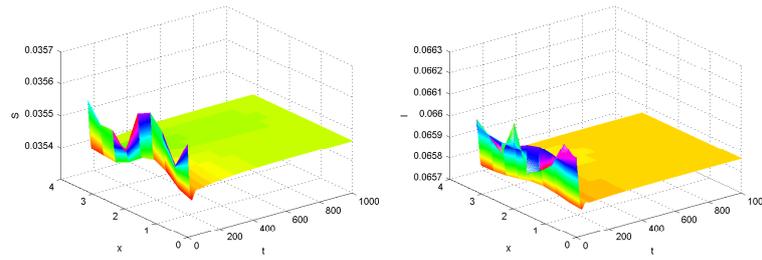


Figure 3.1: Stable behavior with $d_2 = 0.1$ for the model (1.2).

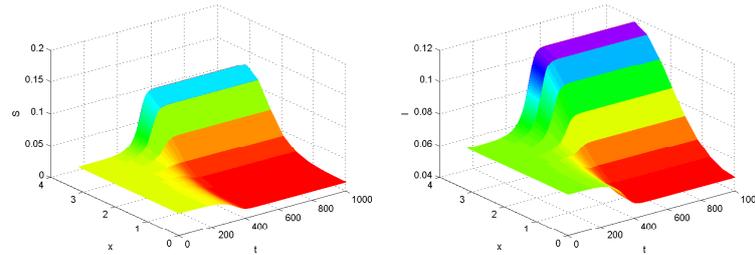


Figure 3.2: Turing instability behavior with $d_2 = 0.25$ for the model (1.2).

For the sake of learning the effect of the diffusion on the Turing pattern of model (1.2) more, as an example, in Fig. 3.3, we demonstrate that the spatial-temporal dynamics to (1.2) are complicated and the pattern formation is extremely sensitive to the variation in diffusion rate d_2 around 0.1073. The transitions between regular and irregular patterning have been well observed in model (1.2).

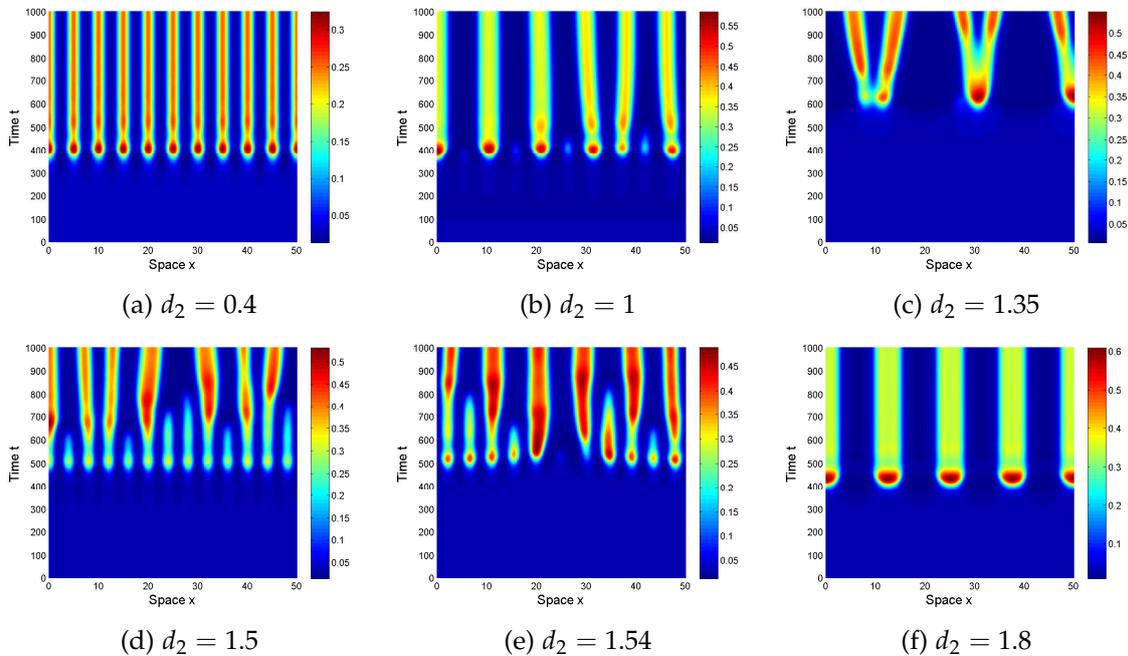


Figure 3.3: Transitions between regular and irregular patterning in model (1.2) with different values of d_2 .

4 Nonconstant positive steady states

In this section, we will focus on the existence and the structure of nonconstant positive solution for the system (1.3).

4.1 Existence of nonconstant positive steady states

In this subsection, we apply priori estimates to yield the existence and nonexistence results of positive nonconstant solutions to (1.3). First, we can easily obtain the nonexistence of nonconstant positive solutions by using the energy method [27], which is relatively simple and we omit the proof here. Also, for notational convenience, we write $\Theta = (r, a, \rho, \beta, \mu)$ in the sequel.

Theorem 4.1. *Under the assumption (P), let D_2 be a fixed positive constant satisfying $D_2 > \frac{\mu}{\lambda_1}$. Then there exists a positive constant $D_1 = D_1(\Theta, D_2)$ such that model (1.3) has no nonconstant positive solution provided that $d_1 \geq D_1$ and $d_2 \geq D_2$.*

With the help of Theorem 4.1, by using the Leray–Schauder degree theory, we discuss the existence of positive nonconstant solutions to (1.3) when the diffusion coefficients d_1 and d_2 vary while the parameters r, a, ρ, β, μ keep fixed.

Rewrite model (1.3) in the form:

$$\begin{cases} -\Delta E = D^{-1}F(E), & x \in \Omega, \\ \frac{\partial E}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \quad (4.1)$$

where $D = \text{diag}(d_1, d_2)$, $E = (S, I)$, $F(E) = (g_1(S, I), g_2(S, I))^T$. Therefore, E solves (4.1) if and only if it satisfies

$$\widehat{f}(d_1, d_2, E) := E - (\mathbf{I} - \Delta)^{-1}\{D^{-1}F(E) + E\} = 0 \quad \text{on } \mathbf{X}, \quad (4.2)$$

where \mathbf{I} is the identity matrix, $(\mathbf{I} - \Delta)^{-1}$ represents the inverse of $\mathbf{I} - \Delta$ with homogeneous Neumann boundary condition.

A straightforward computation reveals

$$D_E \widehat{f}(d_1, d_2, E^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(D^{-1}J + \mathbf{I}).$$

For each \mathbf{X}_i , ζ is an eigenvalue of $D_E \widehat{f}(d_1, d_2, E^*)$ on \mathbf{X}_i if and only if $\zeta(1 + \lambda_i)$ is an eigenvalue of the matrix

$$M_i := \lambda_i I - D^{-1}J = \begin{pmatrix} \lambda_i - d_1^{-1}a_{11} & -d_1^{-1}a_{12} \\ -d_2^{-1}a_{21} & \lambda_i - d_2^{-1}a_{22} \end{pmatrix}.$$

Clearly,

$$\det M_i = d_1^{-1}d_2^{-1}[d_1d_2\lambda_i^2 + (-d_1a_{22} - d_2a_{11})\lambda_i + a_{11}a_{22} - a_{12}a_{21}],$$

and

$$\text{tr} M_i = 2\lambda_i - d_1^{-1}a_{11} - d_2^{-1}a_{22}.$$

Define

$$\widehat{g}(d_1, d_2, \lambda) := d_1d_2\lambda^2 + (-d_1a_{22} - d_2a_{11})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

Thus, $\widehat{g}(d_1, d_2, \lambda_i) = d_1d_2 \det M_i$. If

$$d_1a_{22} + d_2a_{11} > 2\sqrt{d_1d_2(a_{11}a_{22} - a_{12}a_{21})}, \quad (4.3)$$

then $\widehat{g}(d_1, d_2, \lambda) = 0$ has two real roots, that is

$$\lambda_+(d_1, d_2) = \frac{d_1 a_{22} + d_2 a_{11} + \sqrt{(d_1 a_{22} + d_2 a_{11})^2 - 4d_1 d_2 (a_{11} a_{22} - a_{12} a_{21})}}{2d_1 d_2},$$

$$\lambda_-(d_1, d_2) = \frac{d_1 a_{22} + d_2 a_{11} - \sqrt{(d_1 a_{22} + d_2 a_{11})^2 - 4d_1 d_2 (a_{11} a_{22} - a_{12} a_{21})}}{2d_1 d_2}.$$

Let

$$A = A(d_1, d_2) = \{\lambda : \lambda \geq 0, \lambda_-(d_1, d_2) < \lambda < \lambda_+(d_1, d_2)\},$$

$$S_p = \{\lambda_0, \lambda_1, \lambda_2, \dots\},$$

and let $m(\lambda_i)$ be multiplicity of λ_i . In order to calculate the index of $\widehat{f}(d_1, d_2, \cdot)$ at E^* , we need the following lemma.

Lemma 4.2. *Suppose $\widehat{g}(d_1, d_2, \lambda_i) \neq 0$ for all $\lambda_i \in S_p$. Then*

$$\text{index}(\widehat{f}(d_1, d_2, \cdot), E^*) = (-1)^\sigma,$$

where

$$\sigma = \begin{cases} \sum_{\lambda_i \in A \cap S_p} m(\lambda_i), & A \cap S_p \neq \emptyset, \\ 0, & A \cap S_p = \emptyset. \end{cases}$$

In particular, $\sigma = 0$ if $\widehat{g}(d_1, d_2, \lambda_i) > 0$ for all $\lambda_i \geq 0$.

From Lemma 4.2, in order to calculate the index of $\widehat{f}(d_1, d_2, \cdot)$ at E^* , we need to determine the range of λ for which $\widehat{g}(d_1, d_2, \lambda) < 0$.

Theorem 4.3. *Under the conditions of Theorem 2.4 and (P), $a_{11} > 0$ hold. If $\frac{a_{11}}{d_1} \in (\lambda_k, \lambda_{k+1})$ for some $k \geq 1$, and $\sigma_k = \sum_{i=1}^k m(\lambda_i)$ is odd, then there exists a positive constant D^* such that for all $d_2 \geq D^*$, model (1.3) has at least one nonconstant positive solution.*

Proof. Since $a_{11} > 0$, it follows that if d_2 is large enough, then (4.3) holds and $\lambda_+(d_1, d_2) > \lambda_-(d_1, d_2) > 0$. Furthermore,

$$\lim_{d_2 \rightarrow \infty} \lambda_+(d_1, d_2) = \frac{a_{11}}{d_1}, \quad \lim_{d_2 \rightarrow \infty} \lambda_-(d_1, d_2) = 0.$$

As $\frac{a_{11}}{d_1} \in (\lambda_k, \lambda_{k+1})$, there exists $d_0 \gg 1$ such that

$$\lambda_+(d_1, d_2) \in (\lambda_k, \lambda_{k+1}), \quad 0 < \lambda_-(d_1, d_2) < \lambda_1 \quad \forall d_2 \geq d_0. \quad (4.4)$$

From Theorem 4.1, we know that there exists $d > d_0$ such that (1.3) with $d_1 = d$ and $d_2 \geq d$ has no nonconstant positive solution. Let $d > 0$ be large enough such that $\frac{a_{11}}{d_1} < \lambda_1$. Then there exists $D^* > d$ such that

$$0 < \lambda_-(d_1, d_2) < \lambda_+(d_1, d_2) < \lambda_1 \quad \text{for all } d_2 \geq D^*. \quad (4.5)$$

Now we prove that, for any $d_2 \geq D^*$, (1.3) has at least one nonconstant positive solution. By way of contradiction, assume that the assertion is not true for some $D_2^* \geq D^*$. By using

the homotopy argument, we can derive a contradiction in the sequel. Fixing $d_2 = D_2^*$, for $\tau \in [0, 1]$, we define

$$D(\tau) = \begin{pmatrix} \tau d_1 + (1 - \tau)d & 0 \\ 0 & \tau d_2 + (1 - \tau)D^* \end{pmatrix},$$

and consider the following problem

$$\begin{cases} -\Delta E = D^{-1}(\tau)F(E), & x \in \Omega, \\ \frac{\partial E}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \quad (4.6)$$

Thus, E is a positive nonconstant solution of (1.3) if and only if it solves (4.6) with $\tau = 1$. Evidently, E^* is the unique positive constant solution of (4.6). For any $\tau \in [0, 1]$, E is a positive nonconstant solution of (4.6) if and only if

$$h(E, \tau) = E - (\mathbf{I} - \Delta)^{-1}\{D^{-1}(\tau)F(E) + E\} = 0 \quad \text{on } \mathbf{X}. \quad (4.7)$$

From the discussion above, we know that (4.7) has no positive nonconstant solution when $\tau = 0$, and we have assumed that there is no such solution for $\tau = 1$ at $d_2 = D_2^*$. Clearly, $h(E, 1) = \widehat{f}(d_1, d_2, E)$, $h(E, 0) = \widehat{f}(d, D^*, E)$ and

$$\begin{aligned} D_E \widehat{f}(d_1, d_2, E^*) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(D^{-1}J + \mathbf{I}), \\ D_E \widehat{f}(d, D^*, E^*) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\widetilde{D}^{-1}J + \mathbf{I}), \end{aligned}$$

where $\widehat{f}(\cdot, \cdot, \cdot)$ is as given in (4.2) and $\widetilde{D} = \text{diag}(d, D^*)$. From (4.4) and (4.5), we have $A(d_1, d_2) \cap S_p = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $A(d, D^*) \cap S_p = \emptyset$. Since σ_k is odd, Lemma 4.2 yields

$$\begin{aligned} \text{index}(h(\cdot, 1), E^*) &= \text{index}(\widehat{f}(d_1, d_2, \cdot), E^*) = (-1)^{\sigma_k} = -1, \\ \text{index}(h(\cdot, 0), E^*) &= \text{index}(\widehat{f}(d, D^*, \cdot), E^*) = (-1)^0 = 1. \end{aligned}$$

From Theorem 2.4, there exist positive constants $\underline{C} = \underline{C}(d, d_1, D^*, D_2^*, \Theta)$ and $\overline{C} = \overline{C}(d, D^*, \Theta)$ such that the positive solutions of (4.7) satisfy $\underline{C} < S(x), I(x) < \overline{C}$ on $\overline{\Omega}$ for all $\tau \in [0, 1]$.

Define $\Sigma = \{(S, I)^T \in C^1(\overline{\Omega}, \mathbb{R}^2) : \underline{C} < S(x), I(x) < \overline{C}, x \in \overline{\Omega}\}$. Then $h(E, \tau) \neq 0$ for all $E \in \partial\Sigma$ and $\tau \in [0, 1]$. By virtue of the homotopy invariance of the Leray–Schauder degree, we have

$$\deg(h(\cdot, 0), \Sigma, 0) = \deg(h(\cdot, 1), \Sigma, 0). \quad (4.8)$$

Notice that both equations $h(E, 0) = 0$ and $h(E, 1) = 0$ have a unique positive solution E^* in Σ , and we obtain

$$\begin{aligned} \deg(h(\cdot, 0), \Sigma, 0) &= \text{index}(h(\cdot, 0), E^*) = 1, \\ \deg(h(\cdot, 1), \Sigma, 0) &= \text{index}(h(\cdot, 1), E^*) = -1, \end{aligned}$$

which contradicts (4.8). The proof is complete. \square

Remark 4.4. Theorem 4.3 shows that, if the parameters are properly chosen, the existence of nonconstant steady states, i.e., Turing pattern can arise as a result of diffusion.

Next we investigate the structure of nonconstant positive solution for the system (1.3) in the one-dimensional space domain $\Omega = (0, \pi)$. Thus, (1.3) become

$$\begin{aligned} d_1 \Delta S + r(S + \rho I)[1 - a(S + I)] - \frac{\beta SI}{S + I} &= 0, & x \in (0, \pi), \\ d_2 \Delta I + \frac{\beta SI}{S + I} - \mu I &= 0, & x \in (0, \pi), \\ S' = I' &= 0, & x = 0, \pi. \end{aligned} \quad (4.9)$$

For the sake of simplicity, we denote $d_1 = 1$ and $d_2 = d$.

It is well known that the operator $u \rightarrow -\Delta u$ with no-flux boundary conditions has eigenvalues

$$\lambda_0 = 0, \quad \lambda_j = j^2, \quad j = 1, 2, 3, \dots$$

and eigenfunctions

$$\phi_0(x) = \sqrt{\frac{1}{\pi}}, \quad \phi_j(x) = \sqrt{\frac{2}{\pi}} \cos jx, \quad j = 1, 2, 3, \dots$$

We translate (S^*, I^*) to the origin by the translation $(\widehat{S}, \widehat{I}) = (S - S^*, I - I^*)$. For convenience, we still denote \widehat{S}, \widehat{I} by S, I respectively, then we can obtain the following system

$$\begin{aligned} \Delta S + r(S + S^* + \rho(I + I^*)) [1 - a((S + S^*) + (I + I^*))] - \frac{\beta(S + S^*)(I + I^*)}{(S + S^*) + (I + I^*)} &= 0, & x \in (0, \pi), \\ d \Delta I + \frac{\beta(S + S^*)(I + I^*)}{(S + S^*) + (I + I^*)} - \mu(I + I^*) &= 0, & x \in (0, \pi), \\ S' = I' &= 0, & x = 0, \pi. \end{aligned} \quad (4.10)$$

4.2 Local structure of nonconstant positive steady states

In this subsection, we study the local structure of nonconstant positive solutions for the new system (4.10). In brief, by regarding d as the bifurcation parameter, we verify the existence of positive solutions bifurcating from $(d, (0, 0))$. The Crandall–Rabinowitz bifurcation theorem from the simple eigenvalue in [18] will be applied to obtain bifurcations. For the case of double eigenvalues, we shall apply some techniques in [16] and [22] to deal with it.

Let $X = \{(S, I) \in W^{2,p}(0, \pi) \times W^{2,p}(0, \pi) : S' = I' = 0, x = 0, \pi\}$ and $Y = L^p(0, \pi) \times L^p(0, \pi)$. We define the map $F : \mathbb{R}^+ \times X \rightarrow Y$ by

$$F(d, (S, I)) = \begin{pmatrix} \Delta S + r(S + S^* + \rho(I + I^*)) [1 - a((S + S^*) + (I + I^*))] - \frac{\beta(S + S^*)(I + I^*)}{(S + S^*) + (I + I^*)} \\ d \Delta I + \frac{\beta(S + S^*)(I + I^*)}{(S + S^*) + (I + I^*)} - \mu(I + I^*) \end{pmatrix}.$$

Thus, the solutions of (4.10) are exactly zeros of this map $F(d, (S, I))$. Note that $(0, 0)$ is the unique constant solution of (4.10), then we have $F(d, (0, 0)) = 0$. The Fréchet derivative of $F(d, (S, I))$ with respect to (S, I) at $(0, 0)$ can be given by

$$L(d) = F_{(S, I)}(d, (0, 0)) = \begin{pmatrix} \Delta + a_{11} & a_{12} \\ a_{21} & d\Delta + a_{22} \end{pmatrix}.$$

The characteristic equation of $L(d)$ is given by

$$\zeta^2 - \zeta T_i + Q_i = 0, \quad i = 0, 1, 2, \dots, \quad (4.11)$$

where, $T_i = -(1+d)\lambda_i + a_{11} + a_{22}$ and $Q_i = d\lambda_i^2 + (-a_{22} - da_{11})\lambda_i + a_{11}a_{22} - a_{12}a_{21}$.

Throughout this section, we always assume that $\lambda_1 < a_{11}$. Then there exists the largest positive integer $N^0 \geq 1$ such that $\lambda_j < a_{11}$ for $1 \leq j \leq N^0$. Letting $\xi = 0$ in (4.11), we have

$$d = d_j = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}\lambda_j}{\lambda_j(a_{11} - \lambda_j)}, \quad \text{for } 1 \leq j \leq N^0.$$

We shall prove that there exists a nonconstant positive solution of $F(d, (S, I)) = 0$ near $(d_j, (0, 0))$.

Theorem 4.5. *Let $d = d_j$, $\lambda_j = j^2$, for $1 \leq j \leq N^0$. Assume that*

$$r \neq \left\{ \frac{(\beta - \mu)(2\mu^2 - \rho(\beta - \mu)(\beta - 2\mu))}{\rho^2(\beta - \mu)^2 + 2\mu\rho(\beta - \mu) + \mu^2}, \frac{2\mu\beta(\beta - \mu)^2 + \mu^2(\beta - 2\mu)}{\rho^2(\beta - \mu)^2 + 2\mu\rho(\beta - \mu) + \mu^2} \right\}.$$

Suppose that $d_i \neq d_j$ whenever $i \neq j, 1 \leq i, j \leq N^0$. Then $(d_j, (0, 0))$ is a bifurcation point of $F(d, (S, I)) = 0$. Moreover, there is a curve of nonconstant solutions $(d(s), (S(s), I(s)))$ of $F(d, (S, I)) = 0$ for $|s|$ sufficiently small, satisfying $d(0) = d_j, (S(0), I(0)) = (s\phi_j + O(s^2), s\psi_j + O(s^2))$, where $d(s), S(s), I(s)$ are continuously differentiable function with respect to s and $e_j = \frac{\lambda_j - a_{11}}{a_{12}}$.

Proof. By the Crandall–Rabinowitz bifurcation theorem about simple eigenvalues in [18], we see that $(d, (0, 0))$ is a bifurcation point if the following conditions are satisfied:

- (a) the partial derivatives $F_d, F_{(S, I)}$, and $F_{(d, (S, I))}$ exist and are continuous.
- (b) $\dim \ker F_{(S, I)}(d, (0, 0)) = \text{codim} \mathbf{R}(F_{(S, I)}(d, (0, 0))) = 1$.
- (c) Let $\ker F_{(S, I)}(d, (0, 0)) = \text{span}\{\Phi\}$, then $F_{(d, (S, I))}(d, (0, 0))\Phi \notin \mathbf{R}(F_{(S, I)}(d, (0, 0)))$.

Note that

$$L(d_j) = F_{(S, I)}(d, (0, 0)) = \begin{pmatrix} \Delta + a_{11} & a_{12} \\ a_{21} & d_j\Delta + a_{22} \end{pmatrix},$$

and

$$F_d(d, (0, 0)) = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}, \quad F_{(d, (S, I))}(d, (0, 0)) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

It is obvious that the linear operators $F_d, F_{(S, I)}, F_{(d, (S, I))}$ are continuous. So assertion (a) holds.

Suppose $\Phi_j = (\phi, \psi) \in \ker L(d_j)$, and write $\phi = \sum a_j \phi_j, \psi = \sum b_j \phi_j$. Then

$$\sum_{j=0}^{\infty} B_j \begin{pmatrix} a_j \\ b_j \end{pmatrix} \phi_j = 0, \quad \text{where } B_j = \begin{pmatrix} a_{11} - \lambda_j & a_{12} \\ a_{21} & a_{22} - d_j \lambda_j \end{pmatrix}. \quad (4.12)$$

And

$$\det B_j = 0 \quad \Leftrightarrow \quad d = d_j = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}\lambda_j}{\lambda_j(a_{11} - \lambda_j)}$$

implies that

$$\ker L(d_j) = \text{span}\{\Phi_j\}, \quad \Phi_j = \begin{pmatrix} 1 \\ e_j \end{pmatrix} \phi_j,$$

where $e_j = \frac{\lambda_j - a_{11}}{a_{12}}$. The adjoint operator is defined by

$$L^*(d_j) = \begin{pmatrix} \Delta + a_{11} & a_{21} \\ a_{12} & d_j\Delta + a_{22} \end{pmatrix}.$$

In the same way as above we obtain

$$\ker L^*(d_j) = \text{span}\{\Phi_j^*\}, \quad \Phi_j^* = \begin{pmatrix} 1 \\ e_j^* \end{pmatrix} \phi_j,$$

where $e_j^* = \frac{\lambda_j - a_{11}}{a_{21}}$.

Since $\mathbb{R}(L) = \ker(L^*)^\perp$ (\mathbb{R} : image; $^\perp$: complementary set), thus

$$\text{codim}(\mathbb{R}(L(d_j))) = \dim(\ker(L^*(d_j))) = 1.$$

So assertion (b) holds.

Next, we verify assertion (c) holds. Since

$$F_{(d,(S,I))}(d_j, (0,0))\Phi_j = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \Phi_j = \begin{pmatrix} 0 \\ -\lambda_j e_j \phi_j \end{pmatrix}.$$

and

$$\langle F_{(d,(S,I))}(d_j, (0,0))\Phi_j, \Phi_j^* \rangle_Y = \langle -\lambda_j e_j \phi_j, e_j^* \phi_j \rangle = -\lambda_j e_j e_j^* \neq 0.$$

we see that $F_{(d,(S,I))}(d_j, (0,0))\Phi_j \notin \mathbb{R}(L(d_j))$. Hence, the proof is completed. \square

Remark 4.6. Under the assumption of Theorem 4.5, each $(d_j, (S^*, I^*))$ is a bifurcation point with respect to the trivial branch $(d, (S^*, I^*))$. The number of such bifurcation points is infinite.

Remark 4.7. Theorem 4.5 implies that each bifurcation curve Γ_j around $(d_j, (S^*, I^*))$ is of pitchfork type.

4.3 Global structure of nonconstant positive steady states

In this subsection, we study the global structure of the bifurcation solutions form simple eigenvalues. Let J denote the closure of the nonconstant solution set of (4.9), and Γ_j the connected component of $J \cup (d_j, (S^*, I^*))$ to which $(d_j, (S^*, I^*))$ belongs. Theorem 4.5 provides no information on the bifurcating curve Γ_j far from the equilibrium (S^*, I^*) . In order to understand its global structure, a further study is necessary.

Theorem 4.8. *Under the same assumption of Theorem 4.5, the projection of the bifurcation curve Γ_j on the d -axis contains (d_j, ∞) .*

Proof. Rewrite (4.9) as

$$\begin{cases} -\Delta S = a_{11}S + a_{12}I + h_1(S, I), \\ -d\Delta I = a_{21}S + a_{22}I + h_2(S, I), \end{cases}$$

where $h_1(S, I), h_2(S, I)$ are higher-order terms of S and I . The constant steady state (S^*, I^*) of (1.3) is shifted to $(0, 0)$ of this new system. Let

$$G = (-\Delta + a_{11})^{-1}, \quad G_d = (-d\Delta - a_{22})^{-1}, \quad E = (S, I).$$

We next rewrite (4.9) in a form that the standard global bifurcation theory can be more conveniently used. Then

$$K(d)E = (2a_{11}G(S) + a_{12}G(I), a_{21}G_d(S))$$

and

$$H(E) = (G(h_1(S, I)), G_d(h_2(S, I))).$$

Then (4.9) can be interpreted as the equation

$$E = K(d)E + H(E). \quad (4.13)$$

For any fixed $d > 0$, it is noted that $K(d)$ is a compact liner operator on X . $H(E) = o(|E|)$ for E near zero uniformly on closed d sub-intervals of $(0, \infty)$, and is also a compact operator on X .

To apply Rabinowitz's global bifurcation theorem, we first verify that 1 is an eigenvalue of $K(d_j)$ of algebraic multiplicity one. From the argument in the proof of Theorem 4.5 it is seen that $\ker(K(d_j) - \text{Id}) = \ker L = \text{span}\{\Phi_j\}$ (Id : identity operator), so 1 is indeed an eigenvalue of $K(d_j)$, and $\dim \ker(K(d_j) - \text{Id}) = 1$. According to [12], we know that the algebraic multiplicity of the eigenvalue 1 is the dimension of the generalized null space $\cup_{i=1}^{\infty} \ker(K(d_j) - \text{Id})^i$. For our purpose, we need to verify that

$$\ker(K(d_j) - \text{Id}) = \ker(K(d_j) - \text{Id})^2, \quad \text{or} \quad \ker(K(d_j) - \text{Id}) \cap \text{R}(K(d_j) - \text{Id}) = \{0\}.$$

Now, We compute $\ker(K^*(d_j) - \text{Id})$, where $K^*(d_j)$ is the adjoint of $K(d_j)$. Let $(\varphi, \chi) \in \ker(K^*(d_j) - \text{Id})$. Then we have

$$2a_{11}G(\varphi) + a_{21}G_d(\chi) = \varphi, \quad a_{12}G(\varphi) = \chi.$$

By the definition of G and G_d , we obtain

$$-d_j a_{12} \Delta \varphi = f_\varphi \varphi + f_\chi \chi, \quad -\Delta \chi = a_{12} \varphi - a_{11} \chi,$$

where

$$f_\varphi = 2d_j a_{11} a_{12} + a_{12} a_{22}, \quad f_\chi = a_{12} a_{21} - 2a_{11} a_{22} - 2d_j a_{11}^2.$$

Let $\varphi = \sum a_i \phi_i, \chi = \sum b_i \phi_i$. Then

$$\sum_{i=0}^{\infty} B_i^* \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0, \quad \text{where} \quad B_i^* = \begin{pmatrix} -d_j a_{12} \lambda_i + f_\varphi & f_\chi \\ a_{12} & -\lambda_i - a_{11} \end{pmatrix}.$$

By a straightforward calculation one can check that $\det B_i^* = a_{12} \det B_i$, where B_i is given in (4.12). Thus $\det B_i^* = 0$ only for $i = j$, and

$$\ker(K^*(d_j) - \text{Id}) = \text{span}\{\Phi_j'\} \quad \text{where} \quad \Phi_j' = \left(\frac{\lambda_j + a_{11}}{a_{12}}, 1 \right)^\top \phi_j.$$

In addition, we can check that $\int_0^\pi \Phi_j' \Phi_j dx = \frac{2\lambda_j}{a_{12}} \neq 0$, which implies that

$$\Phi_j \notin (\ker(K^*(d_j) - \text{Id}))^\perp = \text{R}((K(d_j) - \text{Id})).$$

Hence, we show $\ker(K(d_j) - \text{Id}) \cap \text{R}(K(d_j) - \text{Id}) = \{0\}$ and the eigenvalue 1 has algebraic multiplicity one.

Suppose that $0 < d \neq d_j$ is in a small neighborhood of d_j , then, for this given d , the liner operator $\text{Id} - K(d) : X \rightarrow X$ is a bijection and 0 is an isolated solution of (4.13). The index of this isolated zero of $\text{Id} - K(d) - H$ is given by

$$\text{index}(\text{Id} - K(d) - H, (d, 0)) = \text{deg}(\text{Id} - K(d), B, 0) = (-1)^P,$$

where B is a sufficiently small ball center at 0, and P is the sum of the algebraic multiplicities of the eigenvalues of $K(d)$ that are greater than one.

For our purpose, it is also necessary to show that this index changes when d crosses d_j , that is, for $\varepsilon > 0$ sufficiently small, we need verify

$$\text{index}(\text{Id} - K(d_j - \varepsilon) - H, (d_j - \varepsilon, 0)) \neq \text{index}(\text{Id} - K(d_j + \varepsilon) - H, (d_j + \varepsilon, 0)). \quad (4.14)$$

Indeed, suppose that ζ is an eigenvalue of $K(d)$ with an eigenfunction $(\tilde{\phi}, \tilde{\psi})$, then

$$\begin{aligned} -\zeta \Delta \tilde{\phi} &= (2 - \zeta)a_{11}\tilde{\phi} + a_{12}\tilde{\psi}, \\ -d\zeta \Delta \tilde{\psi} &= a_{21}\tilde{\phi} + a_{22}\zeta\tilde{\psi}. \end{aligned}$$

Using the Fourier cosine series $\tilde{\phi} = \sum \tilde{a}_i \phi_i$ and $\tilde{\psi} = \sum \tilde{b}_i \phi_i$ leads to

$$\sum_{i=0}^{\infty} \tilde{B}_i \begin{pmatrix} \tilde{a}_i \\ \tilde{b}_i \end{pmatrix} \phi_i = 0, \quad \text{where} \quad \tilde{B}_i = \begin{pmatrix} (2 - \zeta)a_{11} - \lambda_i \zeta & a_{12} \\ a_{21} & (a_{22} - d\lambda_i)\zeta \end{pmatrix}.$$

Thus, the set of eigenvalues of $K(d)$ consists of all ζ' s, which solve the characteristic equation

$$\zeta^2 - \frac{2a_{11}}{a_{11} + \lambda_i} \zeta - \frac{a_{12}a_{21}}{(a_{11} + \lambda_i)(d\lambda_i - a_{22})} = 0, \quad (4.15)$$

where the integer i runs from zero to ∞ . In particular, for $d = d_j$, if $\zeta = 1$ is a root of (4.15), then a simple calculation leads to $d_j = d_i$, and so $j = i$ by the assumption. Therefore, without counting the eigenvalues corresponding to $i \neq j$ in (4.15), $K(d)$ has the same number of eigenvalues greater than 1 for all d close to d_j , and they have the same multiplicities. On the other hand, for $i = j$ in (4.15), we let $\zeta(d), \tilde{\zeta}(d)$ denote the two roots. By a straightforward calculation, we find that

$$\zeta(d_j) = 1 \quad \text{and} \quad \tilde{\zeta}(d_j) = \frac{a_{11} - \lambda_j}{a_{11} + \lambda_j} < 1.$$

When d close to d_j , we obtain $\tilde{\zeta}(d) < 1$. As the constant term $-a_{12}a_{21}/(d\lambda_i - a_{22})$ in (4.15) is a decreasing function of d when $a_{12} < 0$, we know that

$$\zeta(d_j + \varepsilon) > 1, \quad \zeta(d_j - \varepsilon) < 1.$$

Consequently, $K(d_j + \varepsilon)$ has exactly one more eigenvalues that are larger than 1 than $K(d_j - \varepsilon)$ does. Furthermore, by a similar argument above, we can show this eigenvalue has algebraic multiplicity one. So (4.14) holds. And the proof is complete. \square

Remark 4.9. Theorem 4.8 shows that there is a smooth curve Γ_j bifurcating from $(d_j, (S^*, I^*))$, with Γ_j contained in a global branch of the positive solutions of (4.9).

5 Conclusions

In this paper, we study the dynamics of a reaction-diffusion model in the susceptible population. In particular, we are interested in the positive steady states. Diffusion-induced instability of the positive equilibrium E^* is investigated, which produces spatial inhomogeneous patterns (see Theorem 3.3). Since a priori estimates for steady states are necessary in obtaining

the existence of nonconstant positive steady states by applying the global bifurcation theory, establishing a priori bounds for steady states is the key point.

The condition $d_i \neq d_j$ for any integer $i \neq j$ in Theorem 4.5 guarantees $\dim \text{Ker} L(d_j) = 1$, that is 0 is a simple eigenvalue of $L(d_j)$. Hence, we can apply the global bifurcation theory from a simple eigenvalue in this paper. In fact, $j \mapsto d_j$ is not a one-to-one correspondence. On the other hand, d_j is not monotonous function for λ_j . If $d_i = d_j$ for some integer $i \neq j$, then $\dim \text{Ker} L(d_j) > 1$. We hope to discuss this case in the near future.

We also remark that we do not know if it is possible that Γ_j obtained in Theorem 4.8 meets some bifurcation points and then reaches infinity; note that our argument only rules out the possibility that Γ_j meets some bifurcation points without finally reaching infinity. If this case occurs, then some bifurcation branches “collide” each other and the solution undergo a symmetry breaking. Understanding this phenomenon is very important in studying the pattern formation in living organisms.

Acknowledgements

The author would like to thank the associate editor and the anonymous reviewer for their valuable comments and suggestions, which have led to a significant improvement of the whole manuscript. This work is supported by the National Natural Science Foundation of China (Nos. 11761063, 11661051).

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