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# Existence of solution for Kirchhoff model problems with singular nonlinearity

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**Abstract.** We study the fourth order Kirchhoff equation $\Delta^2 u - (a + b\int_\Omega |\nabla u|^2)^\gamma \Delta u = f(u)$ in $\Omega$ with $-\Delta u > 0$ and $u > 0$ in $\Omega$, and $\Delta u = u = 0$ on $\partial\Omega$, where $f(t) = \alpha\frac{1}{t^\theta} + \lambda t^q + \mu t + g(t)$ for $t \geq 0$, $g$ has subcritical growth, $\alpha > 0$, $\lambda > 0$, $\mu \geq 0$, $0 < \theta < 1$, $0 < q < 1$, $\gamma \geq 0$, $a > 0$, $b \geq 0$. We use the Galerkin projection method to show the existence of solution under some boundedness restriction on $\alpha, \lambda, \mu$. In some cases we study the behavior of the norm of the solution $u$ as $\lambda \to 0$ and as $\lambda \to \infty$. Similar issues are addressed for the equation $(a + b\int_\Omega |\nabla u|^2)^\gamma \Delta^2 u - \varrho\Delta u = f(u)$, $\varrho \geq 0$.

**Keywords:** existence of solution, Kirchhoff equation, singular nonlinearity, approximation scheme.

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## 1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$. We solve the following problems.

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_\Omega |\nabla u|^2\right)^\gamma \Delta u = f(u) & \text{in } \Omega \\ -\Delta u > 0, u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} \left(a + b\displaystyle\int_\Omega |\nabla u|^2\right)^\gamma \Delta^2 u - \varrho\Delta u = f(u) & \text{in } \Omega \\ -\Delta u > 0, u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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Equation (1.1) is related to the study of Woinowsky–Krieger [31] in the analysis of buckling and vibrations dynamics of nonlinear beam models. The equation is given by

$$u_{tt} + \tau\, u_{xxxx} - \left(a + b\int_0^L |u_x|^2\right) u_{xx} = f(x,u),$$

where $\tau, a, b$ are physical quantities detailed in the sequel: $\tau = EI/\rho$, $a = H/\rho$ and $b = EA/2\rho L$, where $L$ is the length of the beam in the initial position, $E$ is the modulus of elasticity in tension, $I$ is the cross-sectional moment of inertia, $\rho$ is the mass density, $H$ is the tension in the initial position, $A$ is the cross-sectional area. Here $u(t,x)$ is the deflection of the point $x$ of the beam at time $t$ subjected to a force $f$. More on wave equations in this field can be seen in [6, 10, 14, 21, 32]. In this respect, McKenna–Walter [23, 24] studied oscillations of a hanged bridge as it is conveyed by the equation

$$u_{tt} + u_{xxxx} + \kappa u^+ = f(x,u),$$

where $\kappa > 0$ belongs to a specific range.

Equation (1.1) is also associated to Berger's [5] plate model equation

$$u_{tt} + \Delta^2 u + \left(a + b\int_\Omega |\nabla u|^2\right) \Delta u = f(x,u,u_t)$$

that describes the vertical wave vibration of a thin plate. It takes into account horizontal forces and material resistance represented by $a$ and $b$. Vertical loads $f$ forces the membrane up and down, and may depend on the displacement $u$ and speed $u_t$. Consult also Chueshov–Lasiecka [9] to appreciate the context of the continuum mechanics where such model is inserted.

Equation (1.2) is a fourth order generalization of the Kirchhoff's [16] wave equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\right) \Delta u = f(x,u),$$

that describes changes in length $u$ when a string is transversely fingered with force $f$, and where $a$ and $b$ stand for horizontal tensions magnitudes. This can be viewed as an extension of D'Alembert's wave equation for free vibration strings that gives a more accurate description of vibrations of an elastic string, see for instance [4]. Results dealing with variational methods applied to the stationary equation can be viewed in [11, 18].

Recent works related to (1.1) and (1.2) dealing with variational methods are [2, 7, 8, 12, 17, 22, 25, 29, 33]. The list of papers in this subject is vast, we describe a fill of them below.

A similar equation to (1.1) was studied in [2], namely

$$\begin{cases} \Delta^2 u - \lambda_0 \left(a + b\int_\Omega |\nabla u|^2\right) \Delta u = f(x,u) & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda_0 > 0$ is a parameter. Among other suitable hypotheses, $f$ is $o(|u|)$ at zero, has subcritical growth and satisfies the so-called Ambrosetti–Rabinowitz condition. By means of the mountain pass theorem, it was shown that there exists a $\bar\lambda > 0$ such that the problem has a nontrivial solution for $0 < \lambda_0 < \bar\lambda$.

The Schrödinger–Kirchhoff equation

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right) \Delta u + V(x)u = f(x,u) + h(x) \text{ in } \mathbb{R}^N \tag{1.4}$$

was studied in [33]. When $h \geq 0$, by the mountain pass theorem, there is a nontrivial solution. For that matter the potential $V$ satisfies some suitable hypotheses and $f$ is $o(|u|)$ at zero, has subcritical growth and satisfies the so-called Ambrosetti–Rabinowitz condition. In case $h = 0$ and $f$ has some symmetric properties, there are infinitely many high–energy solutions which are obtained by the symmetric mountain pass theorem. Moreover, there are infinitely many radial solutions.

The equation with critical growth

$$\Delta^2 u - M\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u + u = \lambda_0 f(u) + |u|^{\frac{8}{N-4}}u \quad \text{in } \mathbb{R}^N \tag{1.5}$$

was studied in [7] for $N \geq 5$, where $M : [0,\infty) \to [0,\infty)$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with $M(t) \geq m_0 > 0$, $f$ is $o(|u|)$ at zero, has subcritical growth, $f(t)/t$ is increasing and satisfies the so-called Ambrosetti–Rabinowitz condition. Using minimax critical point theorems, the authors show that there is a $\bar{\lambda} > 0$ such that for $\lambda_0 > \bar{\lambda}$ there is a nontrivial solution.

The critical problem with indefinite potentials was considered in [12], namely

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = \lambda_0 a_0(x)|u|^{q_0-2}u + b_0(x)|u|^{p_0-2}u & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Under suitable assumptions on the potentials $a_0$ and $b_0$, there is $\bar{\lambda} > 0$ such that if $1 < q_0 < 2 < p_0 \leq 2N/(N-4)$, $N \geq 5$, then there exists a nontrivial nonnegative solution for $0 < \lambda_0 < \bar{\lambda}$. A second solution exists for $\lambda_0$ small if $1 < q_0 < 2$, $4 < p_0 \leq 2N/(N-4)$ and $N = 5, 6, 7$. The first solution is obtained as the limit of a minimizing sequence by making use of Ekeland's variational principle and the second solution is found by means of the mountain pass theorem.

Using a similar strategy of the Galerkin method compared to the present paper, the following singular fourth order Kirchhoff equation with Hardy potential was studied in [3]. There $\Omega$ is a bounded domain with $0 \in \Omega$, $h$ and $k$ are positive continuous functions, $M : [0,\infty) \to [0,\infty)$ a continuous function such that $M(t) \geq m_0 > 0$ and $\bar{\mu} = \frac{(N(N-4))^2}{16}$ is the best constant of the Hardy inequality. The problem

$$\begin{cases} \Delta^2 u - \lambda_0 M\left(\displaystyle\int_\Omega |\nabla u|^2\right)\Delta u = \mu_0 \dfrac{1}{|x|^4}u + \dfrac{h(x)}{u^\theta} + k(x)u^q & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

has a positive solution for $\lambda_0 > \mu_0/\bar{\mu}m_0$ and $0 < \mu_0 < \bar{\mu}$.

In contrast to some of the above papers, we prescribe mild assumptions on $f$, since we do not need the so-called Ambrosetti–Rabinowitz condition nor specific behavior of $f$ near zero. Instead, we adopt an approximation scheme inspired in [27,28].

Define

$$f(t) = \alpha\frac{1}{t^\theta} + \lambda t^q + \mu t + g(t) \quad \text{for } t \geq 0 \tag{1.8}$$

where

$$\alpha > 0, \quad \lambda > 0, \quad \mu \geq 0, \quad 0 < \theta < 1, \quad 0 < q < 1. \tag{1.9}$$

The constants in the differential operators respect the following rules:

$$\gamma \geq 0, \quad a > 0, \quad b \geq 0, \quad \varrho \geq 0. \tag{1.10}$$

The function

$$g : \mathbb{R} \to \mathbb{R} \quad \text{is continuous} \tag{1.11}$$

and satisfies

$$|g(t)| \le c_1 |t|^p \quad \text{for } t \in \mathbb{R} \text{ and } 1 \le p < 2N/(N-4) \quad (\text{or } 1 \le p < \infty \text{ if } N = 1,2,3,4), \tag{1.12}$$

where $c_1$ is a constant.

By a solution of (1.1) and (1.2) we mean a function $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega \Delta u \Delta \phi + \left(a + b \int_\Omega |\nabla u|^2\right)^\gamma \nabla u \nabla \phi - f(u)\phi = 0, \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega)$$

or

$$\left(a + b \int_\Omega |\nabla u|^2\right)^\gamma \Delta u \Delta \phi + \varrho \nabla u \nabla \phi - f(u)\phi = 0, \quad \forall \phi \in H^2(\Omega) \cap H_0^1(\Omega).$$

The underlying idea in the proof of the existence of solution, is to consider the function $f_\varepsilon(t) = \alpha \frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t$ with $0 < \varepsilon < 1$, which is an approximation of $f(t) = \alpha \frac{1}{t^\theta} + \lambda t^q + \mu t$ that avoids the singular term at zero. We use the the spectral Galerkin projection method and transform the original equation into a family of finite dimensional nonlinear equations. In each of them we use Brouwer's theorem to get a solution. Due to the structure of the equations, we are able to obtain uniform estimates and to pass to the limit in the projected finite dimensional equations. We thus obtain a solution $u_\varepsilon$. And some extra reasoning is used to show that $u_\varepsilon$ converges to a nontrivial solution of the original equation as $\varepsilon \to 0$. Since we use the classical strong maximum principle, some arguments do not work if the boundary condition is $u = \frac{\partial u}{\partial \nu} = 0$. A more general boundary condition related to the Kirchhoff–Love model for the vertical vibration of a thin elastic plate is presented in [13, pp. 5–7], motivated to earlier works [15,20], see also [26].

We state the main results.

**Theorem 1.1.** *Assume* (1.8)–(1.10) *and* $g \equiv 0$. *There is* $\mu^* > 0$ *such that for* $0 \le \mu < \mu^*$ *and for every* $\alpha, \lambda > 0$, *equation* (1.1) *has a solution.*

**Theorem 1.2.** *Assume* (1.8)–(1.10) *and* $g \equiv 0$. *There is* $\mu^* > 0$ *such that for* $0 \le \mu < \mu^*$ *and for every* $\alpha, \lambda > 0$, *equation* (1.2) *has a solution.*

**Theorem 1.3.** *Assume* (1.8)–(1.12). *Then there exist* $\alpha^*, \lambda^*, \mu^* > 0$ *such that for every* $0 < \alpha < \alpha^*$, $0 < \lambda < \lambda^*$ *and* $0 \le \mu < \mu^*$ *equation* (1.1) *has a solution.*

**Theorem 1.4.** *Assume* (1.8)–(1.12). *Then there exist* $\alpha^*, \lambda^*, \mu^* > 0$ *such that for every* $0 < \alpha < \alpha^*$, $0 < \lambda < \lambda^*$ *and* $0 \le \mu < \mu^*$ *equation* (1.2) *has a solution.*

**Theorem 1.5.** *Let* $f$ *be such that* $\alpha = \mu = 0$ *and* $g(t) = t^p$ *for* $t \ge 0$ *with* $1 < p < 2N/(N-4)$. *And let* $u_\lambda > 0$ *be the solution obtained in each Theorem 1.3 or 1.4. Then* $\|u_\lambda\|_{H^2 \cap H_0^1} \to 0$ *as* $\lambda \to 0$.

**Theorem 1.6.** *Let* $f(t) = \lambda\big(\frac{1}{t^\theta} + t^q + t\big) + t^p$ *for* $t \ge 0$ *with* $1 < p < 2N/(N-4)$. *And let* $u_\lambda > 0$ *be the solution obtained in each Theorem 1.3 or 1.4. If* $u_\lambda$ *exists for every* $\lambda$ *large, then* $\|u_\lambda\|_{H^2 \cap H_0^1} \to \infty$ *as* $\lambda \to \infty$.

## 2 Preliminaries

The space $H^2(\Omega) \cap H_0^1(\Omega)$ is Hilbert with

$$\text{inner product } (u,v) = \int_\Omega \Delta u \Delta v \text{ and norm } \|u\|_{H^2 \cap H_0^1} = \left( \int_\Omega |\Delta u|^2 \right)^{1/2}.$$

The embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^\sigma(\Omega)$ is continuous if $1 \le \sigma \le 2N/(N-4)$ and compact if $1 \le \sigma < 2N/(N-4)$. The embedding is continuous if $N = 1,2,3,4$ and $1 \le \sigma < \infty$. Also, the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow H_0^1(\Omega)$ is continuous and compact, see [1,12,30]. Moreover $\|u\|_{H_0^1}^2 \le \|u\|_{L^2} \|u\|_{H^2 \cap H_0^1}$, since $\int_\Omega |\nabla u|^2 = \int_\Omega u(-\Delta u)$. The Sobolev embedding constant $C_\sigma$ related to $\|u\|_{L^\sigma} \le C_\sigma \|u\|_{H^2 \cap H_0^1}$ will appear in some computations. The spectrum of $-\Delta$ in $H_0^1(\Omega)$ is given by the numbers $\lambda_i$, $i \in \mathbb{N}$, where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \ldots.$ The corresponding eigenfunctions are $w_i \in H_0^1(\Omega)$, $i \in \mathbb{N}$. The first eigenfunction corresponding to $\lambda_1$ is $w_1 > 0$. For every $i \in \mathbb{N}$ one has

$$\begin{cases} -\Delta w_i = \lambda_i w_i & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

By elliptic regularity $w_i \in C^\infty(\overline{\Omega})$, $i \in \mathbb{N}$. With respect to the biharmonic operator, for every $i \in \mathbb{N}$,

$$\begin{cases} \Delta^2 w_i = \lambda_i^2 w_i & \text{in } \Omega \\ \Delta w_i = w_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In other words, the spectrum of $\Delta^2$ in $H^2(\Omega) \cap H_0^1(\Omega)$ is given by the numbers $\lambda_i^2$, $i \in \mathbb{N}$, where $0 < \lambda_1^2 < \lambda_2^2 \le \lambda_3^2 \le \lambda_4^2 \ldots$ And the corresponding eigenfunctions are also $w_i \in H^2(\Omega) \cap H_0^1(\Omega)$, $i \in \mathbb{N}$. The following orthogonality relations take place

$$\int_\Omega \nabla w_i \nabla w_j = \int_\Omega w_i(-\Delta w_j) = \lambda_j \int_\Omega w_i w_j = 0 \quad \text{if } i \ne j \tag{2.3}$$

and

$$\int_\Omega \Delta w_i \Delta w_j = \int_\Omega w_i(\Delta^2 w_j) = \lambda_j^2 \int_\Omega w_i w_j = 0 \quad \text{if } i \ne j. \tag{2.4}$$

The set of eigenfunctions can be normalized either as $\|w_i\|_{H_0^1} = 1$ or $\|w_i\|_{H^2 \cap H_0^1} = 1$, $i \in \mathbb{N}$. Hence $\mathbb{B} = \{w_1, w_2, \ldots, w_m, \ldots\}$ is an orthonormal basis of $H_0^1(\Omega)$ and of $H^2(\Omega) \cap H_0^1(\Omega)$, according the inner product of each space.

An aside result that will be useful in the proofs is Brouwer's Theorem [19] that says: Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous function such that $(F(\eta), \eta) \ge 0$ for every $\eta \in \mathbb{R}^m$ with $|\eta| = r$ for some $r > 0$. Then, there exists $z_0 \in \mathbb{R}^m$ with $|z_0| \le r$ such that $F(z_0) = 0$.

## 3 Proof of the theorems

We begin proving Theorem 1.1.

*Proof.* Define $f_\varepsilon(t) = \alpha \frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t$ with $0 < \varepsilon < 1$ and let $\mathbb{B} = \{w_1, w_2, \ldots, w_m, \ldots\}$ be an orthonormal basis of $H^2(\Omega) \cap H_0^1(\Omega)$, see (2.3) and (2.4). (Here $w_i, i = 1,2,3,\ldots$ need not to be eigenfuncitons, but we choose a such basis for convenience). Define

$$\mathbb{W}_m = [w_1, w_2, \ldots, w_m],$$

to be the space generated by $\{w_1, w_2, \ldots, w_m\}$. Define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$F(\eta) = (F_1(\eta), F_2(\eta), \ldots, F_m(\eta))$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_m) \in \mathbb{R}^m$,

$$F_j(\eta) = \int_\Omega \Delta u \Delta w_j + \left(a + b \int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega \nabla u \nabla w_j - \int_\Omega f_\varepsilon(|u|) w_j, \quad j = 1, 2, \ldots, m$$

and

$$u = \sum_{i=1}^m \eta_i w_i \quad \in \mathbb{W}_m.$$

Therefore

$$\begin{aligned}(F(\eta), \eta) &= \int_\Omega |\Delta u|^2 + \left(a + b \int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega |\nabla u|^2 - \int_\Omega f_\varepsilon(|u|) u \\ &\geq \|u\|^2_{H^2 \cap H^1_0} - \alpha |\Omega|^\theta C_1^{1-\theta} \|u\|^{1-\theta}_{H^2 \cap H^1_0} - \lambda C_{q+1}^{q+1} \|u\|^{q+1}_{H^2 \cap H^1_0} - \mu C_2^2 \|u\|^2_{H^2 \cap H^1_0}. \end{aligned} \tag{3.1}$$

The function $F$ is continuous because each $F_j$ is continuous by Sobolev embedding and dominated convergence theorem. Here $C_1$, $C_{q+1}$ and $C_2$ are Sobolev embedding constants appearing in $\|u\|_{L^\sigma} \leq C_\sigma \|u\|_{H^2 \cap H^1_0}$, which are independent on $m$ and $\varepsilon$. Hence for $\mu < C_2^{-2}$, there is $R > 0$ such that

$$(F(\eta), \eta) > 0 \quad \text{for} \quad \|u\|_{H^2 \cap H^1_0} = |\eta| = R. \tag{3.2}$$

Brouwer's Theorem asserts that there exists $u_{m,\varepsilon} \in H^2 \cap H^1_0$ with $\|u_{m,\varepsilon}\|_{H^2 \cap H^1_0} \leq R$ satisfying

$$\int_\Omega \Delta u_{m,\varepsilon} \Delta w_j + \left(a + b \int_\Omega |\nabla u_{m,\varepsilon}|^2\right)^\gamma \int_\Omega \nabla u_{m,\varepsilon} \nabla w_j - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|) w_j = 0, \quad j = 1, 2, \ldots, m. \tag{3.3}$$

Hence

$$\int_\Omega \Delta u_{m,\varepsilon} \Delta \zeta_m + \left(a + b \int_\Omega |\nabla u_{m,\varepsilon}|^2\right)^\gamma \int_\Omega \nabla u_{m,\varepsilon} \nabla \zeta_m - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|) \zeta_m = 0, \quad \forall \zeta_m \in \mathbb{W}_m.$$

Let $k \in \mathbb{N}$, then for every $m \geq k$ we obtain

$$\int_\Omega \Delta u_{m,\varepsilon} \Delta \zeta_k + \left(a + b \int_\Omega |\nabla u_{m,\varepsilon}|^2\right)^\gamma \int_\Omega \nabla u_{m,\varepsilon} \nabla \zeta_k - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|) \zeta_k = 0, \quad \forall \zeta_k \in \mathbb{W}_k. \tag{3.4}$$

Since $\|u_{m,\varepsilon}\|_{H^2 \cap H^1_0} \leq R$ and $H^2 \cap H^1_0$ is reflexive, there exists $u_\varepsilon \in H^2 \cap H^1_0$ such that

($i_1$) $u_{m,\varepsilon} \rightharpoonup u_\varepsilon$ weakly in $H^2 \cap H^1_0$ as $m \to \infty$

($i_2$) $u_{m,\varepsilon} \to u_\varepsilon$ in $H^1_0$ as $m \to \infty$

($i_3$) $u_{m,\varepsilon} \to u_\varepsilon$ in $L^\sigma$ for $1 \leq \sigma < 2N/(N-4)$ (or $1 \leq \sigma < \infty$ if $N = 1, 2, 3, 4$) as $m \to \infty$

Letting $m \to \infty$, in the expression (3.4) we get

$$\int_\Omega \Delta u_\varepsilon \Delta \zeta_k + \left(a + b \int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \int_\Omega \nabla u_\varepsilon \nabla \zeta_k - \int_\Omega f_\varepsilon(|u_\varepsilon|) \zeta_k = 0, \quad \forall \zeta_k \in \mathbb{W}_k.$$

Since the space of all subsapces $[\mathbb{W}_m]_{k\in\mathbb{N}}$ is dense in $H^2\cap H_0^1$, then

$$\int_\Omega \Delta u_\varepsilon \Delta\zeta + \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \int_\Omega \nabla u_\varepsilon \nabla\zeta - \int_\Omega f_\varepsilon(|u_\varepsilon|)\zeta = 0, \quad \forall \zeta \in H^2\cap H_0^1. \tag{3.5}$$

Hence $u_\varepsilon$ is a nontrivial weak solution of

$$\begin{cases} \Delta^2 u_\varepsilon - \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \Delta u_\varepsilon = f_\varepsilon(|u_\varepsilon|) & \text{in } \Omega \\ \Delta u_\varepsilon = u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that $-\Delta u_\varepsilon$ satisfy the equation with $f_\varepsilon(|u_\varepsilon|) > 0$, hence the maximum principle applies. Consequently, $-\Delta u_\varepsilon > 0$ and moreover $u_\varepsilon > 0$ in $\Omega$. Thus $u_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 u_\varepsilon - \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \Delta u_\varepsilon = f_\varepsilon(u_\varepsilon) & \text{in } \Omega \\ -\Delta u_\varepsilon > 0, u_\varepsilon > 0 & \text{in } \Omega \\ \Delta u_\varepsilon = u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

As we shall see $u_\varepsilon \geq \delta_0 w_1$ in $\Omega$ for some $\delta_0 > 0$, see (2.1) and (2.2). For that matter denote $-\Delta u_\varepsilon = v$ and rewrite the equation (3.6) in the form

$$-\Delta v + \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma v = f_\varepsilon(u_\varepsilon) \geq \vartheta, \tag{3.7}$$

where $\vartheta > 0$ is a constant which does not depend on $\varepsilon$ such that

$$f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t \geq \alpha\frac{1}{(t+1)^\theta} + \lambda t^q \geq \vartheta \quad \text{for } t \geq 0.$$

Let $V = \delta w_1$ with $\delta > 0$ and notice that $\|u_\varepsilon\|_{H^2\cap H_0^1} \leq \liminf_{m\to\infty} \|u_{m,\varepsilon}\|_{H^2\cap H_0^1} \leq R$, then

$$\begin{aligned} -\Delta V + \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma V &= \delta w_1 \left[\lambda_1 + \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma\right] \\ &\leq \delta w_1 \left[\lambda_1 + \left(a+b\frac{1}{\lambda_1}\|u_\varepsilon\|^2_{H^2\cap H_0^1}\right)^\gamma\right] \\ &\leq \delta w_1 \left[\lambda_1 + \left(a+b\frac{R^2}{\lambda_1}\right)^\gamma\right] \leq \vartheta, \end{aligned}$$

where the last inequality is valid by taking $\delta$ small enough, and it is independent on $\varepsilon$. Owing to (3.7) and remembering that $v = V = 0$ on $\partial\Omega$, we obtain $-\Delta u_\varepsilon = v \geq \delta w_1$ in $\Omega$. By the maximum principle there is $\delta_0 > 0$ such that $u_\varepsilon \geq \delta_0 w_1$ in $\Omega$.

Since $\|u_\varepsilon\|_{H^2\cap H_0^1} \leq R$. By Sobolev embedding and continuing to denote a subsequence $\varepsilon = \varepsilon_n \to 0$, then

$(j_1)$ $u_\varepsilon \rightharpoonup u_0$ weakly in $H^2\cap H_0^1$ as $\varepsilon \to 0$,

$(j_2)$ $u_\varepsilon \to u_0$ in $H_0^1$ as $\varepsilon \to 0$,

$(j_3)$ $u_\varepsilon \to u_0$ in $L^\sigma$ for $1 \leq \sigma < 2N/(N-4)$ (or $1 \leq \sigma < \infty$ if $N = 1,2,3,4$) as $\varepsilon \to 0$,

$(j_4)$ $u_\varepsilon \to u_0$ a.e. in $\Omega$ as $\varepsilon \to 0$,

($j_5$) $|u_\varepsilon| \le h(x)$ a.e. in $\Omega$, for some $h$ in $L^\sigma$, $1 \le \sigma < 2N/(N-4)$ (or $1 \le \sigma < \infty$ if $N = 1,2,3,4$).

We conclude that $u_0 \ge \delta_0 w_1$ in $\Omega$. We rewrite (3.5) below

$$\int_\Omega \Delta u_\varepsilon \Delta\zeta + \left(a + b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \int_\Omega \nabla u_\varepsilon \nabla\zeta - \int_\Omega \left(\alpha \frac{1}{(u_\varepsilon+\varepsilon)^\theta} + \lambda u_\varepsilon^q + \mu u_\varepsilon\right)\zeta = 0, \qquad \forall \zeta \in H^2 \cap H_0^1. \quad (3.8)$$

Using ($j_1$)–($j_5$) and letting $\varepsilon \to 0$ in (3.8) we arrive at

$$\int_\Omega \Delta u_0 \Delta\zeta + \left(a + b\int_\Omega |\nabla u_0|^2\right)^\gamma \int_\Omega \nabla u_0 \nabla\zeta - \int_\Omega \left(\alpha \frac{1}{u_0^\theta}\zeta + \lambda u_0^q + \mu u_0\right)\zeta = 0, \quad \forall \zeta \in H^2 \cap H_0^1. \quad (3.9)$$

The first two integrals of (3.9) are consequences of ($j_1$) and ($j_2$). The integral involving $u_0^q$ follows from ($j_4$), ($j_5$) and dominated convergence theorem. The integral with $\mu$ follows by ($j_3$). It is useful to detail that

$$\int_\Omega \frac{1}{(u_\varepsilon+\varepsilon)^\theta}\zeta \to \int_\Omega \frac{1}{u_0^\theta}\zeta, \quad \forall \zeta \in H^2 \cap H_0^1. \quad (3.10)$$

First notice that $\int_\Omega \frac{1}{u_0^\theta} \le \frac{1}{\delta_0^\theta}\int_\Omega \frac{1}{w_1^\theta} < \infty$. By dominated convergence theorem we can write (3.10) with $\zeta \in C_0^\infty(\Omega)$, and by density we can take $\zeta \in H_0^1$, and finally (3.10) holds for every $\zeta \in H^2 \cap H_0^1$. □

We now prove Theorem 1.2.

*Proof.* We borrow $\mathbb{B}$, $\mathbb{W}_m$ and $F$ defined in the proof of Theorem 1.1. Define

$$F_j(\eta) = \left(a + b\int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega \Delta u \Delta w_j + \varrho\int_\Omega \nabla u \nabla w_j - \int_\Omega f_\varepsilon(|u|)w_j, \quad j = 1,2,\dots,m.$$

Then

$$\begin{aligned}(F(\eta),\eta) &= \left(a + b\int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega |\Delta u|^2 + \varrho\int_\Omega |\nabla u|^2 - \int_\Omega f_\varepsilon(|u|)u \\ &\ge a^\gamma \|u\|^2_{H^2\cap H_0^1} - \alpha|\Omega|^\theta C_1^{1-\theta}\|u\|^{1-\theta}_{H^2\cap H_0^1} - \lambda C_{q+1}^{q+1}\|u\|^{q+1}_{H^2\cap H_0^1} - \mu C_2^2\|u\|^2_{H^2\cap H_0^1}. \end{aligned} \quad (3.11)$$

For $\mu < a^\gamma C_2^{-2}$, there is $R > 0$ verifying (3.2) and $u_{m,\varepsilon} \in H^2 \cap H_0^1$ with $\|u_{m,\varepsilon}\|_{H^2\cap H_0^1} \le R$ and satisfying

$$\left(a + b\int_\Omega |\nabla u_{m,\varepsilon}|^2\right)^\gamma \int_\Omega \Delta u_{m,\varepsilon}\Delta w_j + \varrho\int_\Omega \nabla u_{m,\varepsilon}\nabla w_j - \int_\Omega f_\varepsilon(|u_{m,\varepsilon}|)w_j = 0, \quad j = 1,2,\dots,m.$$

After the same steps of the previous proof and using ($i_1$)–($i_3$) we reach

$$\left(a + b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \int_\Omega \Delta u_\varepsilon \Delta\zeta + \varrho\int_\Omega \nabla u_\varepsilon\nabla\zeta - \int_\Omega f_\varepsilon(|u_\varepsilon|)\zeta = 0, \quad \forall \zeta \in H^2 \cap H_0^1. \quad (3.12)$$

We thus get a nontrivial weak solution $u_\varepsilon$ of

$$\begin{cases} \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \Delta^2 u_\varepsilon - \varrho\Delta u_\varepsilon = f_\varepsilon(|u_\varepsilon|) & \text{in } \Omega \\ \Delta u_\varepsilon = u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

We are in position to apply the maximum principle to the function $-\Delta u_\varepsilon$. Then $-\Delta u_\varepsilon > 0$, thus $u_\varepsilon > 0$ in $\Omega$ and

$$\begin{cases} \left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \Delta^2 u_\varepsilon - \varrho\Delta u_\varepsilon = f_\varepsilon(u_\varepsilon) & \text{in } \Omega \\ -\Delta u_\varepsilon > 0, u_\varepsilon > 0 & \text{in } \Omega \\ \Delta u_\varepsilon = u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

For $V = \delta w_1$ with $\delta > 0$ and using $\|u_\varepsilon\|_{H^2\cap H_0^1} \le R$, then for $\delta$ small enough

$$-\left(a+b\int_\Omega |\nabla u_\varepsilon|^2\right)^\gamma \Delta V + \varrho V \le \delta w_1 \left[\left(a+b\frac{R^2}{\lambda_1}\right)^\gamma \lambda_1 + \varrho\right] \le \vartheta \le f_\varepsilon(u_\varepsilon).$$

Comparing with (3.13) we obtain $-\Delta u_\varepsilon \ge \delta w_1$ and $u_\varepsilon \ge \delta_0 w_1$ in $\Omega$ for $\delta_0 > 0$ small enough. The remaining steps are analogue to the proof of Theorem 1.1. □

Next we describe the main steps of the proof of Theorem 1.3.

*Proof.* Define $f_\varepsilon(t) = \alpha\frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t + g(t)$ with $0 < \varepsilon < 1$. As in the beginning of the proof of Theorem 1.1 we consider $\mathbb{B}$, $\mathbb{W}_m$ and $F$. Estimate (3.1) in this context turns out to be

$$\begin{aligned} (F(\eta),\eta) &= \int_\Omega |\Delta u|^2 + \left(a+b\int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega |\nabla u|^2 - \int_\Omega f_\varepsilon(|u|)u \\ &\ge \|u\|^2_{H^2\cap H_0^1} - \alpha|\Omega|^\theta C_1^{1-\theta}\|u\|^{1-\theta}_{H^2\cap H_0^1} - \lambda C_{q+1}^{q+1}\|u\|^{q+1}_{H^2\cap H_0^1} \\ &\quad - c_1 C_{p+1}^{p+1}\|u\|^{p+1}_{H^2\cap H_0^1} - \mu C_2^2\|u\|^2_{H^2\cap H_0^1}. \end{aligned} \tag{3.14}$$

Hence, there is a constant $K > 0$ such that

$$(F(\eta),\eta) \ge \|u\|^2_{H^2\cap H_0^1} - K\left(\alpha\|u\|^{1-\theta}_{H^2\cap H_0^1} + \lambda\|u\|^{q+1}_{H^2\cap H_0^1} + \|u\|^{p+1}_{H^2\cap H_0^1} + \mu\|u\|^2_{H^2\cap H_0^1}\right). \tag{3.15}$$

Next we will make the choice of $R$, $\alpha^*$, $\mu^*$ and $\lambda^*$. We need $\|u\|_{H^2\cap H_0^1} = R < (2/3K)^{1/(p-1)}$. Thus, let

$$R = \min\{1, [(2/3K)^{1/(p-1)}]/2\}.$$

We require $\alpha < (1/2)^{1+\theta}(2/3K)^{1+\theta/(p-1)}(2/3K)$, then we select $\alpha^*$ with

$$\alpha^* = [(1/2)^{1+\theta}(2/3K)^{1+\theta/(p-1)}(2/3K)]/2.$$

We need $\mu < 2/3K$, thus we take $\mu^* = 1/3K$.

Once $R$ has been chosen, we want $\lambda^*$ such that $R^2 - K\lambda R^{q+1} > 0$, i.e., $\lambda < R^{1-q}/K$ for $\lambda < \lambda^*$. Hence we take

$$\lambda^* = (1/K)\min\{1, (1/2)^{2-q}(2/3K)^{(1-q)/(p-1)}\}.$$

With these these choices of $\alpha^*, \lambda^*, \mu^*$ announced in the statement of the theorem, we have the intervals where $\alpha, \lambda, \mu$ belong to, namely $0 < \alpha < \alpha^*$, $0 < \lambda < \lambda^*$ and $0 \le \mu < \mu^*$.

Thus, let $\Upsilon = R^2 - K\lambda^* R^{q+1} > 0$. Therefore,

$$(F(\eta), \eta) > \Upsilon \quad \text{for } \|u\|_{H^2 \cap H_0^1} = |\eta| = R. \tag{3.16}$$

Brouwer's Theorem asserts that there exists $u_{m,\varepsilon} \in H^2 \cap H_0^1$ with $\|u_{m,\varepsilon}\|_{H^2 \cap H_0^1} \le R$ satisfying (3.3). Notice that there is a constant $\vartheta > 0$, which does not depend on $\varepsilon$ such that

$$f_\varepsilon(t) = \alpha \frac{1}{(t+\varepsilon)^\theta} + \lambda t^q + \mu t + g(t) \ge \alpha \frac{1}{(t+1)^\theta} + \lambda t^q \ge \vartheta \quad \text{for } t \ge 0.$$

The remaining parts of the proof run in the same manner as before, see all steps from (3.3) to (3.10). □

The proof of Theorem 1.4 is similar.

*Proof.* The above proofs are well documented. It is a repetition of the arguments. □

Next we prove Theorem 1.5.

*Proof.* The solution $u = u_\lambda$ satisfies

$$\begin{aligned} \|u\|^2_{H^2 \cap H_0^1} &\le \int_\Omega |\Delta u|^2 + \left(a + b \int_\Omega |\nabla u|^2\right)^\gamma \int_\Omega |\nabla u|^2 = \int_\Omega f(u)u \\ &= \int_\Omega \lambda u^{q+1} + u^{p+1} \le \lambda C_{q+1}^{q+1} \|u\|^{q+1}_{H^2 \cap H_0^1} + C_{p+1}^{p+1} \|u\|^{p+1}_{H^2 \cap H_0^1}. \end{aligned}$$

Then

$$\|u\|^{1-q}_{H^2 \cap H_0^1} \le \frac{\lambda C_{q+1}^{q+1}}{1 - C_{p+1}^{p+1} \|u\|^{p-1}_{H^2 \cap H_0^1}}.$$

By the choice of $R$ we get

$$1 - C_{p+1}^{p+1} \|u\|^{p-1}_{H^2 \cap H_0^1} \ge 1/2.$$

Hence

$$\|u\|_{H^2 \cap H_0^1} \le \left(2\lambda C_{q+1}^{q+1}\right)^{1/(1-q)} \to 0 \quad \text{as } \lambda \to 0.$$

The proof for (1.2) is similar. □

We conclude the paper proving Theorem 1.6.

*Proof.* We denote the existing solution of Theorem 1.3 by $u = u_\lambda$ and assume that $\|u\|_{H^2 \cap H_0^1} \le R$. Since $\|u\|^2_{H_0^1} \le \|u\|_{L^2} \|u\|_{H^2 \cap H_0^1}$, the term $a + b\|u\|^2_{H_0^1}$ is bounded. Multiply the equation (1.1) by $w_1$, integrate and use (2.1) and (2.2), hence

$$\int_\Omega f(u) w_1 = \lambda_1 \int_\Omega u w_1 (1 + (a + b\|u\|^2_{H_0^1})^\gamma) \le \lambda_1 M \int_\Omega u w_1, \tag{3.17}$$

for a constant $M > 0$ independent on $\lambda$. Notice that $f(t) = \lambda\left(\frac{1}{t^\theta} + t^q + t\right) + t^p \ge \lambda t^q + t^p$ for $t \ge 0$. Then $f(t) \ge \lambda^{(p-1)(p-q)} C_{p,q} t$ for $t \ge 0$, where $C_{p,q} > 0$ is a constant depending only on $p$ and $q$. Hence (3.17) gives

$$\lambda^{(p-1)(p-q)} C_{p,q} \int_\Omega u w_1 \le \lambda_1 M \int_\Omega u w_1,$$

which makes $\lambda$ bounded, a contradiction. Again the reasoning for (1.2) is similar. □

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