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Errata article for "Three point boundary value problems for ordinary differential equations, uniqueness implies existence"

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Abstract. This paper serves as an errata for the article "P. W. Eloe, J. Henderson, J. Neugebauer, *Electron. J. Qual. Theory Differ. Equ.* **2020**, No. 74, 1–15." In particular, the proof the authors give in that paper of Theorem 3.6 is incorrect, and so, that alleged theorem remains a conjecture. In this erratum, the authors state and prove a correct theorem.

Keywords: uniqueness implies existence, nonlinear interpolation, ordinary differential equations, three point boundary value problems.

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1 Introduction

Let $n \geq 2$ denote an integer and let $a < T_1 < T_2 < T_3 < b$. Let $a_i \in \mathbb{R}$, $i = 1, \dots, n$. We shall consider the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t), \dots, y^{(n-1)}(t)), \quad t \in [T_1, T_3], \tag{1.1}$$

where $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$, or the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t)), \quad t \in [T_1, T_3], \tag{1.2}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$. We shall consider three point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-2, \quad y(T_2) = a_{n-1}, \quad y^{(j-1)}(T_3) = a_n, \tag{1.3}$$

and we shall have need to consider two point boundary value problems for either (1.1) or (1.2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-1, \quad y^{(j-1)}(T_2) = a_n. \tag{1.4}$$

With respect to (1.1), common assumptions for the types of results that we consider are:

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(A) $f(t, y_1, \ldots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;

(B) Solutions of initial value problems for (1.1) are unique and extend to (a, b);

(C) For $j \in \{1, 2\}$, solutions of the two-point boundary value problems (1.1), (1.3) are unique if they exist;

(D) For $j \in \{1, 2\}$, solutions of the two-point boundary value problems (1.1), (1.4) are unique if they exist.

With respect to (1.2), the assumptions (A, (B), (C) and (D) are replaced, respectively, by

(A') $f(t, y) : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B') Solutions of initial value problems for (1.2) are unique and extend to (a, b).

(C') For $j \in \{1, 2\}$, solutions of the two-point boundary value problems (1.2), (1.3) are unique if they exist.

(D') For $j \in \{1, 2\}$, solutions of the two-point boundary value problems (1.2), (1.4) are unique if they exist.

In [3, Theorem 3.6], the authors claimed to have proved the following theorem.

Theorem 1.1. *Assume that with respect to* (1.1)*, Conditions (A), (B), (C) and (D) are satisfied. Then for each* $a < T_1 < T_2 < T_3 < b$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, *and for* $j = 1$, *the three point boundary value problem* (1.1), (1.3) *has a solution.*

The proof that is offered in [3] is incorrect and so, the alleged theorem remains a conjecture. In this erratum, we state and prove a correct theorem. With the statement and proof of this correct theorem, the remainder of the results produced in [3] are correct.

Theorem 1.2. *Assume that with respect to* (1.2)*, Conditions* (A'), (B'), (C') *and* (D') *are satisfied. Then for each* $a < T_1 < T_2 < T_3 < b$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, *and for* $j = 1$, *the three point boundary value problem* (1.2), (1.3) *has a solution.*

Before proving Theorem 1.2, we state several results to which we refer in the proof. The first two are results about the continuous dependence of solutions of (1.1), (1.4) or (1.2), (1.4) on boundary conditions. The third is a known generalized mean value theorem.

Theorem 1.3. *Assume that with respect to* (1.1)*, Conditions (A), (B), and (D) are satisfied. Let* $j \in \{1, 2\}$.

(i) Given any $a < T_1 < T_2 < T_3 < b$, *and any solution* y *of* (1.1)*, there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$, $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$, $i = 1, \ldots, n-2$, $|T_{21} - T_2| < \epsilon$, *and* $|T_{31} - T_3| < \epsilon$, $|y(T_2) - y_{(n-1)1}| < \epsilon$, $|y(T_3) - y_{n1}| < \epsilon$, *then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$, $i = 1, \ldots, n-2$, $z(T_{21}) = y_{(n-1)1}$, *and* $z^{(j-1)}(T_{31}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$, $T_{2k} \to T_2$, $T_{3k} \to T_3$, $y_{ik} \to y_i$, $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$, $i = 1, \ldots, n-2$, $z_k(T_{2k}) = y_{(n-1)k}$, $z_k^{(j-1)}(T_{3k}) = y_{nk}$, *then for each* $i \in \{1, \ldots, n\}$, $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Theorem 1.3 was proved in [3] with a standard application of the Brouwer invariance of domain theorem; technically we shall apply the following theorem for which the proof is completely analogous to the proof of Theorem 1.3.

Theorem 1.4. *Assume that with respect to* (1.2)*, Conditions* (A')*,* (B')*, and* (D') *are satisfied. Let* $j \in \{1,2\}$.

(i) Given any $a < T_1 < T_2 < T_3 < b$*, and any solution* y *of* (1.1)*, there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$, $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$, $i = 1, \dots, n-2$, $|T_{21} - T_2| < \epsilon$*, and* $|T_{31} - T_3| < \epsilon$, $|y(T_2) - y_{(n-1)1}| < \epsilon$, $|y(T_3) - y_{n1}| < \epsilon$*, then there exists a solution* z *of* (1.1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$, $i = 1, \dots, n-2$, $z(T_{21}) = y_{(n-1)1}$*, and* $z^{(j-1)}(T_{31}) = y_{n1}$.

(ii) If $T_{1k} \to T_1$, $T_{2k} \to T_2$, $T_{3k} \to T_3$, $y_{ik} \to y_i$, $i = 1, \dots, n$ *and* z_k *is a sequence of solutions of* (1.1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$, $i = 1, \dots, n-2$, $z_k(T_{2k}) = y_{(n-1)k}$, $z_k^{(j-1)}(T_{3k}) = y_{nk}$*, then for each* $i \in \{1, \dots, n\}$, $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a,b).

For a proof of a generalized mean value theorem, we refer the reader to the text by Conte and de Boor [1, Theorem 2.2]. Let $t_0, \dots, t_i$ denote $i+1$ distinct real numbers and let $z\colon \mathbb{R} \to \mathbb{R}$. Define $z[t_l] = z(t_l)$, $l = 0, \dots, i$ and if $t_l, \dots, t_{k+1}$ denote $k - l + 2$ distinct points, define

$$z[t_l, \dots, t_{k+1}] = \frac{z[t_{l+1}, \dots, t_{k+1}] - z[t_l, \dots, t_k]}{t_{k+1} - t_l}.$$

Theorem 1.5. *Assume* $z(t)$ *is a real-valued function, defined on* $[a,b]$ *and* i *times differentiable in* (a,b)*. If* $t_0, \dots, t_i$ *are* $i+1$ *distinct points in* $[a,b]$*, then there exists*

$$c \in (\min\{t_0, \dots, t_i\}, \max\{t_0, \dots, t_i\})$$

such that

$$z[t_0, \dots, t_i] = \frac{z^{(i)}(c)}{i!}.$$

For our purposes, we shall set $h > 0$ and choose $t_0 = T_1, t_1 = T_1 + h, \dots, t_i = T_1 + ih$ to be equally spaced. In this setting

$$z[T_1, T_1 + h, \dots, T_1 + ih] = \frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T_1 + lh)}{i! h^i},$$

and, in general there exists $c \in (T_1, T_1 + ih)$ such that

$$\frac{\sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} z(T + ih)}{h^i} = z^{(i)}(c). \tag{1.5}$$

We now proceed to the proof of Theorem 1.2.

Proof. Let $a < T_1 < T_2 < T_3 < b$, and $a_i \in \mathbb{R}$, $i = 1, \dots, n$. Let $m \in \mathbb{R}$ and denote by $y(t;m)$ the solution of the initial value problem (1.2), with initial conditions

$$y^{(i-1)}(T_1;m) = a_i, \quad i = 1, \dots, n-1, \quad y^{(n-2)}(T_1;m) = m, \quad y(T_2) = a_{n-1}.$$

Let

$$\Omega = \{p \in \mathbb{R} : \text{ there exists } m \in \mathbb{R} \text{ with } y(T_3;m) = p\}.$$

The theorem is proved by showing $\Omega = \mathbb{R}$. It follows by Conditions (A'), (B') and (D') (see [2]), $\Omega \neq \emptyset$; thus, the theorem is proved by showing Ω is open and closed. That Ω is open follows from Theorem 1.4.

To show Ω is closed, let p_0 denote a limit point of Ω and without loss of generality let p_k denote a strictly increasing sequence of reals in Ω converging to p_0. Assume $y(T_3; m_k) = p_k$ for each $k \in \mathbb{N}_1$. It follows by the uniqueness of solutions, Condition (C'), that

$$y^{(j-1)}(t; m_{k_1}) \neq y^{(j-1)}(t; m_{k_2}), \quad t \in (T_2, b), \tag{1.6}$$

for each $j \in \{1, 2\}$, if $k_1 < k_2$, and in particular,

$$y(t; m_1) < y(t; m_k) \quad t \in (T_2, b), \tag{1.7}$$

for each k.

Either $y'(T_3; m_k) \leq 0$ infinitely often or $y'(T_3; m_k) \geq 0$ infinitely often. Relabel if necessary and assume $y'(T_3; m_k) \leq 0$ or $y'(T_3; m_k) \geq 0$ for each k.

We first assume the case $y'(T_3; m_k) \leq 0$ for each k. We now consider two subcases. For the first subcase, assume $y'(T_3; m_k) < y'(T_3; m_1) \leq 0$ infinitely often. Relabeling if necessary, assume $y'(T_3; m_k) < y'(T_3; m_1) < 0$ for each k. Find $T_3 < T_4 < b$ such that $y'(t; m_1) \leq 0$, for $t \in [T_3, T_4]$. Then $y(t; m_1)$ is decreasing on $[T_3, T_4]$. Set $L = y(T_4; m_1)$; then, for $t \in [T_3, T_4]$,

$$L = y(T_4; m_1) \leq y(t; m_1) \leq y(T_3; m_1) \leq p_0.$$

Since $y'(T_2; m_k) < y'(T_2; m_1)$, then analogous to (1.7), it follows that

$$y'(t; m_k) < y'(t; m_1), \quad t \in (T_2, b),$$

and $y(t; m_k)$ is decreasing on $[T_3, T_4]$. Then for $t \in [T_3, T_4]$,

$$L = y(T_4; m_1) \leq y(t; m_1) \leq y(t; m_k) \leq y(T_3; m_k) \leq p_0. \tag{1.8}$$

In particular,

$$\{(t, y(t; m_k) : t \in [T_3, T_4], k \in \mathbb{N}_1\} \subset [T_3, T_4] \times [L, p_0]. \tag{1.9}$$

Since $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous, there exists $M > 0$ such that

$$\max_{t \in [T_3, T_4], k \in \mathbb{N}_1} |y^{(n)}(t; m_k)| \leq M. \tag{1.10}$$

We now proceed to adapt an observation made by Lasota and Opial [4] and apply the adapted observation to higher order derivatives. Lasota and Opial essentially observed that

$$0 > \frac{y(T_4; m_k) - y(T_3; m_k)}{T_4 - T_3} \geq \frac{L - p_0}{T_4 - T_3} = -K_1, \tag{1.11}$$

which implies

$$\{t \in [T_3, T_4] : -K_1 \leq y'(t; m_k) < 0\} \neq \emptyset.$$

For our purposes, define

$$S_{k1} = \{t \in [T_3, T_4] : |y'(t; m_k)| \leq K_1\},$$

and $S_{k1} \neq \emptyset$.

To proceed to higher order derivatives, employ Theorem 1.5. For example, set

$$h = \frac{T_4 - T_3}{2}$$

and consider
$$\frac{y(T_3;m_k) - 2y(T_3+h;m_k) + y(T_3+2h;m_k)}{h^2}.$$
Employing (1.8), it follows that
$$\begin{aligned}\left|\frac{y(T_3;m_k) - 2y(T_3+h;m_k) + y(T_3+2h;m_k)}{h^2}\right| &\leq \frac{2(p_0-L)}{h^2} \\ &= \frac{2^3(p_0-L)}{(T_4-T_3)^2} = K_2.\end{aligned}$$
Thus,
$$S_{k2} = \{t \in [T_3,T_4] : |y''(t;m_k)| \leq K_2\} \neq \emptyset.$$
So, in general, let $i \in \{1,\dots n-1\}$. Set $h = \frac{T_4-T_3}{i}$. Then,
$$\left|\frac{\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}y(T_3+lh;m_k)}{h^i}\right| \leq \frac{(i)^i 2^{i-1}(p_0-L)}{(T_4-T_3)^i} = K_i.$$
Apply (1.5) and the set,
$$S_{ki} = \{t \in [T_3,T_4] : |y^{(i)}(t;m_k)| \leq K_i\} \neq \emptyset.$$
Let $c_{n-1} \in S_{k(n-1)}$. Then for $t \in [T_3,T_4]$,
$$y^{(n-1)}(t;m_k) = y^{(n-1)}(c_{n-1};m_k) + \int_{c_{n-1}}^{t} y^{(n)}(s;m_k)ds$$
which implies
$$\max_{t\in[T_3,T_4]} |y^{(n-1)}(t;m_k)| \leq K_{n-1} + M(T_4-T_3) = M_{n-1}.$$
Since $S_{k(n-2)} \neq \emptyset$, the same argument implies that
$$\max_{t\in[T_3,T_4]} |y^{(n-2)}(t;m_k)| \leq K_{n-2} + M_{n-1}(T_4-T_3) = M_{n-2}.$$
Continuing with the same argument, define for $i \in \{n-2,\dots,1\}$,
$$M_i = K_i + M_{i+1}(T_4-T_3).$$
Then
$$\max_{t\in[T_3,T_4]} |y^{(i)}(t;m_k)| \leq M_i, \quad i = 1,\dots,n-1.$$
For each k, choose $t_k \in [T_3,T_4]$. Then
$$(t_k, y(t_k;m_k), y'(t_k;m_k), \dots, y^{(n-1)}(t_k;m_k)) \in [T_3,T_4] \times [L,p_0] \times \Pi_{i=1}^{n-1}[-M_i,M_i]. \tag{1.12}$$
The set on the righthand side of (1.12) is a compact subset of $\mathbb{R}^{n+1}$ and independent of k. Thus, there exists a convergent subsequence (relabeling if necessary)
$$\{(t_k, y(t_k;m_k), y'(t_k;m_k), \dots, y^{(n-1)}(t_k;m_k))\} \to (t_0, c_1, \dots, c_n)$$
where $t_0 \in [T_3,T_4]$. Since $t_0 \in (a,b)$, by the continuous dependence of solutions of initial value problems, $y(t;m_k)$ converges in $C^{n-1}[T_1,T_3]$ to a solution, say $z(t)$, of the initial value problem

(1.2), with initial conditions, $y^{(i-1)}(t_0) = c_i$, $i = 1, \ldots, n$. Thus, $p_0 = z(T_3)$ which implies $p_0 \in \Omega$ and Ω is closed. This completes the proof if, for each k,

$$y'(T_3; m_k) < y'(T_3; m_1) \le 0.$$

Moving to the second subcase, assume $y'(T_3; m_1) < y'(T_3; m_k) \le 0$ infinitely often. Relabeling if necessary, assume $y'(T_3; m_1) < y'(T_3; m_k) \le 0$ for each k. For this case, we work on an interval to the left of T_3. Find $T_2 < T_4 < T_3$ such that $y'(t; m_1) \le 0$ and $y(T_3; m_1) \le y(t; m_1) \le p_0$ for $t \in [T_4, T_3]$. The inequality (1.7) remains valid and

$$y'(t; m_1) < y'(t; m_k), \quad t \in (T_2, b).$$

So, for $t \in [T_4, T_3]$,

$$y(T_3; m_1) \le y(t; m_1) < y(t; m_k)$$

and there exists $c_k \in (t, T_3)$ such that

$$\begin{aligned} y(t; m_k) &= y(T_3; m_k) + y'(c_k; m_k)(t - T_3) \le y(T_3; m_k) + y'(c_k; m_1)(t - T_3) \\ &\le p_0 + \max_{t \in [T_4, T_3]} |y'(t; m_1)|(T_3 - T_4). \end{aligned}$$

Set $L = y(T_3; m_1)$ and $P_0 = p_0 + \max_{t \in [T_4, T_3]} |y'(t; m_1)|(T_3 - T_4)$ and analogous to (1.8) we have for $t \in [T_4, T_3]$, $k \in \mathbb{N}_1$,

$$L \le y(t; m_k) \le P_0.$$

The proof of the second subcase now proceeds precisely as the proof of the first case.

For these two subcases we have assumed $y'(T_3; m_k) \le 0$ for each k. If $y'(T_3; m_k) > 0$ for each k, one again considers two subcases, $y'(T_3; m_k) > y'(T_3; m_1) > 0$ for each k, or $y'(T_3; m_1) > y'(T_3; m_k) \ge 0$ for each k. If $y'(T_3; m_k) > y'(T_3; m_1) > 0$ for each k, produce an analogue to the preceding first subcase on an interval $[T_4, T_3]$ where $T_2 < T_4 < T_3$ and define $L = y(T_4; m_1)$. If $y'(T_3; m_1) > y'(T_4; m_k) \ge 0$ for each k, produce an analogue to the preceding second subcase on an interval $[T_3, T_4]$ where $T_3 < T_4 < b$. The proof is complete. □

Remark 1.6. In [3], the authors claim to have constructed a sequence of solutions of (1.1), (1.3) for $j = 1$ and a compact set analogous to (1.12). The calculations to obtain an interval analogous to $[T_3, T_4]$ of positive length are incorrect which in turn implies the calculations to obtain a priori bounds on higher order derivatives are incorrect. Thus, the conjecture, stated as Theorem 3.6 in [3] is unproven.

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