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Multiple positive solutions for singular anisotropic Dirichlet problems

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Abstract. We consider a nonlinear Dirichlet problem driven by the variable exponent (anisotropic) p-Laplacian and a reaction that has the competing effects of a singular term and of a superlinear perturbation. There is no parameter in the equation (nonparametric problem). Using variational tools together with truncation and comparison techniques, we show that the problem has at least two positive smooth solutions.

Keywords: variable exponent, anisotropic regularity, anisotropic maximum principle, positive solutions, critical point theory.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following anisotropic singular Dirichlet problem

$$-\Delta_{p(z)}u(z) = u(z)^{-\eta(z)} + f(z,u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0. \tag{1.1}$$

In this problem the exponent $p : \overline{\Omega} \to \mathbb{R}$ in the differential operator, is Lipschitz continuous (that is $p \in C^{0,1}(\overline{\Omega})$) and $1 < p_- = \min_{\overline{\Omega}} p$. By $\Delta_{p(z)}$ we denote the anisotropic p-Laplace operator defined by

$$\Delta_{p(z)}u = \operatorname{div}(|Du|^{p(z)-2}Du) \qquad \forall u \in W_0^{1,p(z)}(\Omega).$$

In problem (1.1) we have the competing effects of a singular term $x^{-\eta(z)}$ with $\eta \in C(\overline{\Omega}), 0 < \eta(z) < 1$ for all $z \in \overline{\Omega}$ and a Carathéodory perturbation $f(z,x)$ (that is, for all $x \in \mathbb{R}, z \to f(z,x)$ is measurable and for a.a. $z \in \Omega, x \to f(z,x)$ is continuous), which is $(p_+ - 1)$-superlinear as $x \to +\infty$ (here $p_+ = \max_{\overline{\Omega}} p$), but need not satisfy the usual for superlinear

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problems Ambrosetti–Rabinowitz condition (the AR-condition for short). We are looking for positive solutions. Using a combination of variational tools based on the critical point theory, together with truncation and comparison techniques, we show that the problem has at least two positive smooth solutions.

While anisotropic boundary value problems have been studied extensively in the last few years (see the books of Diening–Harjulehto–Hästö–Růžička [2] and of Rădulescu–Repovš [12] and the references therein), the study of singular anisotropic problems is lagging behind. Only a very limited number of works exist on this subject and they all concern parametric problems (see the works of Byun-Ko [1] and Saoudi–Ghanmi [13]). The presence of parameter in the equation is very helpful, since by varying the parameter, we achieve certain desirable geometric configurations which in turn permit the use of the minimax theorems of critical point theory. In problem (1.1) there is no parameter to facilitate the analysis.

2 Mathematical background – hypotheses

The study of problem (1.1) requires the use of Lebesgue and Sobolev spaces with variable exponents. A comprehensive presentation of these spaces can be found in the book of Diening–Harjulehto–Hästö–Růžička [2].

For every $r \in C(\overline{\Omega})$ we set

$$r_- = \min_{\overline{\Omega}} r \text{ and } r_+ = \max_{\overline{\Omega}} r.$$

Let $E_1 = \{r \in C(\overline{\Omega}) : 1 < r_-\}$ and $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. For $r \in E_1$, the variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined by

$$L^{r(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{r(z)} dz < \infty \right\}.$$

We equip this space with the so-called "Luxemburg norm" defined by

$$\|u\|_{r(z)} = \inf \left[\lambda > 0 : \int_\Omega \left(\frac{|u(z)|}{\lambda} \right)^{r(z)} dz \leq 1 \right], \qquad u \in L^{r(z)}(\Omega).$$

With this norm the space $L^{r(z)}(\Omega)$ is a Banach space which is separable and reflexive (in fact uniformly convex). Let $r' \in E_1$ be defined by $r'(z) = \frac{r(z)}{r(z)-1}$ for all $z \in \overline{\Omega}$ (that is, $\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$ for all $z \in \overline{\Omega}$). Then we have

$$L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$$

and the following version of Hölder's inequality is true

$$\int_\Omega |uv| dz \leq \left[\frac{1}{r_-} + \frac{1}{r'_-} \right] \|u\|_{r(z)} \|v\|_{r'(z)}, \qquad \forall u \in L^{r(z)}(\Omega), \forall v \in L^{r'(z)}(\Omega).$$

Note that if $r_1, r_2 \in E_1$ and $r_1(z) \leq r_2(z)$ for all $z \in \overline{\Omega}$, then we have

$$L^{r_2(z)}(\Omega) \hookrightarrow L^{r_1(z)}(\Omega) \quad \text{continuously.}$$

Using the variable exponent Lebesgue spaces, we can introduce variable exponent Sobolev spaces. Given $r \in E_1$, the anisotropic Sobolev space $W^{1,r(z)}(\Omega)$ is defined by

$$W^{1,r(z)}(\Omega) = \{u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega)\},$$

where Du denotes the gradient of u in the weak sense. This space is equipped with the norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)}, \qquad u \in W^{1,r(z)}(\Omega) \quad (\text{here } \|Du\|_{r(z)} = \||Du|\|_{r(z)}).$$

If $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, then we define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

The spaces $W^{1,r(z)}(\Omega)$ and $W_0^{1,r(z)}(\Omega)$ are separable, reflexive (in fact uniformly convex). For the space $W_0^{1,r(z)}(\Omega)$ the Poincaré inequality holds, that is, there exists $\hat{c} > 0$ such that

$$\|u\|_{r(z)} \leq \hat{c}\|Du\|_{r(z)} \quad \text{for all } u \in W_0^{1,r(z)}(\Omega).$$

This implies that on $W_0^{1,r(z)}(\Omega)$ we can use the equivalent norm

$$|u|_{1,r(z)} = \|Du\|_{r(z)}, \qquad u \in W_0^{1,r(z)}(\Omega).$$

For $r \in E_1$, we set

$$r^*(z) = \begin{cases} \dfrac{Nr(z)}{N - r(z)}, & \text{if } r(z) < N \\ +\infty, & \text{if } N \leq r(z) \end{cases} \qquad \forall z \in \overline{\Omega}.$$

Let $r, q \in E_1 \cap C^{0,1}(\overline{\Omega})$ and suppose that $q(z) \leq r^*(z)$ (resp. $q(z) < r^*(z)$) for all $z \in \overline{\Omega}$. Then we have the anisotropic Sobolev embedding theorem

$$W_0^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text{continuously}$$

$$(\text{resp. } W_0^{1,r(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \quad \text{compactly}).$$

In the study of these spaces, central role plays the following modular function

$$\rho_r(u) = \int_\Omega |u|^{r(z)} dz \quad \text{for all } u \in L^{r(z)}(\Omega).$$

If $u \in W_0^{1,r(z)}(\Omega)$ or $u \in W^{1,r(z)}(\Omega)$, then $\rho_r(Du) = \rho_r(|Du|)$.
This modular function is closely related to the Luxemburg norm.

Proposition 2.1. *If $r \in E_1$ and $\{u_n, u\}_{n\in\mathbb{N}} \subseteq L^{r(z)}(\Omega)$, then we have*

(a) *For all $\lambda > 0$,*
$\|u\|_{r(z)} = \lambda \Leftrightarrow \rho_r(\frac{u}{\lambda}) = 1$;

(b) $\|u\|_{r(z)} < 1 \Leftrightarrow \|u\|_{r(z)}^{r_+} \leq \rho_r(u) \leq \|u\|_{r(z)}^{r_-}$,
$\|u\|_{r(z)} > 1 \Leftrightarrow \|u\|_{r(z)}^{r_-} \leq \rho_r(u) \leq \|u\|_{r(z)}^{r_+}$;

(c) $\|u_n\|_{r(z)} \to 0 \Leftrightarrow \rho_r(u_n) \to 0$;

(d) $\|u_n\|_{r(z)} \to \infty \Leftrightarrow \rho_r(u_n) \to +\infty$.

Also for $r \in E_1 \cap C^{0,1}(\overline{\Omega})$, we have

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

Consider the operator $A_r : W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ defined by

$$\langle A_{r(z)}(u), h\rangle = \int_\Omega |Du|^{r(z)-2}(Du, Dh)_{\mathbb{R}^N} dz, \quad \text{for all } u, h \in W_0^{1,r(z)}(\Omega).$$

This operator has the following properties (see Gasiński–Papageorgiou [5], Proposition 2.5 and Rădulescu–Repovš [12], p.40).

Proposition 2.2. *If $r \in E_1 \cap C^{0,1}(\overline{\Omega})$ and $A_r : W_0^{1,r(z)}(\Omega) \to W^{-1,r'(z)}(\Omega)$ is defined as above, then $A_r(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and is of type $(S)_+$, that is, it has the following property:*

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W_0^{1,r(z)}(\Omega) \text{ and } \limsup_{n\to\infty} \langle A_{r(z)}(u_n), u_n - u\rangle \le 0, \text{ then } u_n \to u \text{ in } W_0^{1,r(z)}(\Omega).\text{”}$$

For every $u \in W_0^{1,r(z)}(\Omega)$, we define $u^\pm = \max\{\pm u, 0\}$. Then

$$u^\pm \in W_0^{1,r(z)}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Suppose $u, v : \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \le v(z)$ for a.a $z \in \Omega$. We define

$$[u, v] = \Big\{h \in W_0^{1,r(z)}(\Omega) : u(z) \le h(z) \le v(z) \quad \text{for a.a. } z \in \Omega\Big\},$$
$$[u) = \Big\{h \in W_0^{1,r(z)}(\Omega) : u(z) \le h(z) \quad \text{for a.a. } z \in \Omega\Big\}.$$

Another space that we will need is $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. This is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \ge 0 \quad \text{for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\right\}.$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Let X be a Banach space and $\varphi \in C^1(X)$. We introduce the set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi).$$

We say that $\varphi(\cdot)$ satisfies the “C-condition”, if it has the following property:

“Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.”

Now we are ready to introduce our hypotheses on the data of problem (1.1).

H_0: $p \in C^{0,1}(\overline{\Omega})$, $1 < p_- = \min_{\overline{\Omega}} p, \eta \in C(\overline{\Omega}), 0 < \eta(z) < 1$ for all $z \in \overline{\Omega}$.

H_1: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)[1 + x^{r(z)-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $r \in C(\overline{\Omega})$ and $p(z) < r(z) < p^*(z)$ for all $z \in \overline{\Omega}$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{x^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exists $\tau \in C(\overline{\Omega})$ such that

$$\tau(z) \in \left((r_+ - p_-) \max\left\{\frac{N}{p_-}, 1\right\}, p^*(z)\right) \quad \text{for all } z \in \overline{\Omega},$$

$$0 < \widehat{\eta}_0 \leq \liminf_{x\to+\infty} \frac{f(z, x)x - p_+ F(z, x)}{x^{\tau(z)}} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\theta > 0$ such that

$$\theta^{-\eta(z)} + f(z, \theta) \leq -\widehat{c} < 0 \text{ for a.a. } z \in \Omega;$$

(iv) there exist $\delta > 0$ and $q \in E_1$ such that $q_+ < p_-$ such that

$$c_1 x^{q(x)-1} \leq f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta, \text{ with } c_1 > 0;$$

(v) there exists $\widehat{\xi}_\theta > 0$ such that for a.a.$z \in \Omega$, the function

$$x \to f(z, x) + \widehat{\xi}_\theta x^{p(z)-1} \quad \text{is nondecreasing on } [0, \theta].$$

Remark 2.3. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, \infty)$, we can always assume without any loss of generality that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Hypotheses $H_1(ii)$ implies that for a.a. $z \in \Omega$ $f(z, \cdot)$ is $(p_+ - 1)$-superlinear. However, it need not satisfy the AR-condition which is common in the literature when dealing with superlinear problems (see, for example, Saoudi–Ghanmi [13], hypothesis ($H4$) and Byun–Ko [1, p. 76]). Condition $H_1(ii)$ is less restrictive and incorporates in our framework also superlinear nonlinearities with "slower" growth as $x \to +\infty$, which fail to satisfy the AR-condition. For example, the following function $f(z, x)$ satisfies hypotheses H_1 but fails to satisfy the AR-condition:

$$f(z, x) = \begin{cases} (x^+)^{q(z)-1} - 2(x^+)^{k(z)-1} & \text{if } x \leq 1 \\ x^{p_+-1} \ln x - x^{p(z)-1} & \text{if } 1 < x, \end{cases}$$

with $q \in E_1$ as in hypothesis $H_1(iv)$, $k \in C(\overline{\Omega})$, $\tau(z) < k(z)$ for a $z \in \overline{\Omega}$. Evidently for this $f(z, x)$ we can choose $\theta = 1$. Hypotheses $H_1(iii), (iv)$ dictate an oscillatory behavior for $f(z, \cdot)$ near 0^+ since it starts positive near zero (see hypothesis $H_1(v)$) and drops to negative values as we approach $\theta > 0$ (see hypothesis $H_1(iii)$). Also, hypothesis $H_1(v)$ implies the presence of a concave term near zero.

3 An auxiliary problem

When dealing with singular problems, a major difficulty that we encounter, is that the presence of the singularity leads to an energy functional which is not C^1. This fact prevents us from using the results of critical point theory. So, we need to find a way to bypass the singularity and deal with C^1-functions in order to use the minimax theorems of critical point theory. This is done by using the solution of an auxiliary problem which we introduce and solve in this section. The auxiliary problem is suggested by a unilateral growth condition satisfied by $f(z,\cdot)$. More precisely note that on account of hypotheses $H_1(i),(iv)$, we can find $c_2>0$ such that

$$f(z,x)\geq c_1x^{q(z)-1}-c_2x^{r(z)-1}\quad \text{for a.a } z\in\Omega,\ \text{all } x\geq 0. \tag{3.1}$$

Motivated by this unilateral growth condition on $f(z,\cdot)$ and using hypothesis $H_1(iii)$, we introduce the Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$g(z,x)=\begin{cases} c_1(x^+)^{q(z)-1}-c_2(x^+)^{r(z)-1} & \text{if } x\leq\theta\\ c_1\theta^{q(z)-1}-c_2\theta^{r(z)-1} & \text{if } \theta<x. \end{cases} \tag{3.2}$$

Then we consider the following Dirichlet problem

$$-\Delta_{p(z)}u(z)=g(z,u(z))\quad \text{in } \Omega,\ u|_{\partial\Omega}=0, u>0. \tag{3.3}$$

Proposition 3.1. *If hypotheses H_0 hold, then problem (3.3) has a unique positive solution $\overline{u}\in \operatorname{int} C_+$ and $0\leq\overline{u}(z)\leq\theta$ for all $z\in\overline{\Omega}$.*

Proof. First we show the existence of a positive solution for problem (3.3). To this end, let $\psi_0:W_0^{1,p(z)}(\Omega)\to\mathbb{R}$ be the C^1-functional defined by $\psi_0(u)=\int_\Omega\frac{1}{p(z)}|Du|^{p(z)}dz-\int_\Omega G(z,u)dz$ for all $u\in W_0^{1,p(z)}(\Omega)$, where $G(z,x)=\int_0^x g(z,s)ds$. From (3.2), we see that

$$\begin{aligned} &\psi_0(u)\geq\frac{1}{p}\rho_p(Du)-c_3\quad \text{for some } c_3>0,\\ \Rightarrow\ &\psi_0(\cdot)\quad \text{is coercive (see Proposition 2.1).}\end{aligned}$$

Also, from the anisotropic Sobolev embedding theorem, we see that $\psi_0(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, we can find $\overline{u}\in W_0^{1,p(z)}(\Omega)$ such that

$$\psi_0(\overline{u})=\min[\psi_0(u):u\in W_0^{1,p(z)}(\Omega)]. \tag{3.4}$$

Let $u\in\operatorname{int}C_+$ and choose $t\in(0,1)$ small so that $0\leq tu(z)\leq\theta$ for all $z\in\overline{\Omega}$. Then using (3.2), we have

$$\begin{aligned}\psi_0(tu)&\leq\frac{t^{p_-}}{p_-}\rho_p(Du)+\frac{t^{r_-}}{r_-}\rho_\tau(u)-\frac{t^{q_+}}{q_+}\rho_q(u)\\ &\leq c_4t^{p_-}-c_5t^{q_+}\quad \text{for some } c_4,c_5>0.\\ &\quad(\text{since } 1<q_+<p_-<r_- \text{ and } t\in(0,1)).\end{aligned} \tag{3.5}$$

From (3.5) we see that by taking $t\in(0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\psi_0(tu)<0,\\ \Rightarrow\ &\psi_0(\overline{u})<0=\psi_0(0)\quad(\text{see (3.4)}),\\ \Rightarrow\ &\overline{u}\neq 0.\end{aligned}$$

From (3.4) we have that

$$\begin{aligned}
&\psi_0'(\overline{u}) = 0,\\
\Rightarrow\ &\langle A_{p(z)}(\overline{u}), h\rangle = \int_\Omega g(z,\overline{u})h dz, \quad \text{for all } h \in W_0^{1,p(z)}(\Omega).
\end{aligned} \tag{3.6}$$

In (3.6) first we choose $h = -\overline{u}^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\begin{aligned}
&\rho_p(D\overline{u}^-) = 0 \quad (\text{see } (3.2)),\\
\Rightarrow\ &\overline{u} \geq 0,\ \overline{u} \neq 0.
\end{aligned}$$

Next in (3.6) first we choose $h = [\overline{u} - \theta]^+ \in W_0^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned}
\langle A_{p(z)}(\overline{u}), (\overline{u}-\theta)^+\rangle &= \int_\Omega [c_1\theta^{q(z)-1} - c_2\theta^{r(z)-1}](\overline{u}-\theta)^+ dz \quad (\text{see } (3.2))\\
&\leq \int_\Omega f(z,\theta)(\overline{u}-\theta)^+ dz \quad (\text{see } (3.1))\\
&\leq 0 = \langle A_{p(z)}(\theta), (\overline{u}-\theta)^+\rangle \quad (\text{see } H_1(iii)),\\
&\Rightarrow \overline{u} \leq \theta.
\end{aligned}$$

So, we have proved that

$$\overline{u} \in [0,\theta], \qquad \overline{u} \neq 0. \tag{3.7}$$

From (3.7),(3.2) and (3.6), we infer that $\overline{u} \neq 0$ is a positive solution of problem (3.3). From Fan [3] (Theorem 1.3), we have that $\overline{u} \in C_+ \setminus \{0\}$. Moreover, we have

$$\Delta_{p(z)}(\overline{u}) \leq c_2\theta^{r(z)-p(z)}\overline{u}(z)^{p(z)-1} \leq c_6\overline{u}(z)^{p(z)-1} \quad \text{in } \Omega \text{ for some } c_6 > 0.$$

Then the anisotropic maximum principle of Zhang [15, Theorem 1.2] implies that

$$\overline{u} \in \operatorname{int} C_+. \tag{3.8}$$

Next we show that this positive solution of (3.3) is in fact unique. Let $\overline{v} \in W_0^{1,p(z)}(\Omega)$ be another positive solution of (3.3). Again we have

$$\overline{v} \in \operatorname{int} C_+. \tag{3.9}$$

From (3.8) and (3.9) and using Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu-Repovš [9], we have that

$$\frac{\overline{u}}{\overline{v}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\overline{v}}{\overline{u}} \in L^\infty(\Omega). \tag{3.10}$$

Let $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)}|Du^{1/p_-}|^{p(z)}dz & \text{if } u \geq 0, u^{1/p_-} \in W_0^{1,p(z)}(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

Let $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$). From Theorem 2.2 of Takač–Giacomoni [14], we know that $j(\cdot)$ is convex. Let $h = \overline{u}^{p_-} - \overline{v}^{p_-} \in W_0^{1,p(z)}(\Omega)$. On account of (3.10), for $|t| < 1$ small, we have

$$\overline{u}^{p_-} + th \in \operatorname{dom} j \quad \text{and} \quad \overline{v}^{p_-} + th \in \operatorname{dom} j.$$

Then the convexity of $j(\cdot)$ implies the Gateaux differentiability of $j(\cdot)$ at $\overline{u}^{p_-}$ and at $\overline{v}^{p_-}$ in the direction h. Moreover, using Green's theorem, we obtain

$$j'(\overline{u}^{p_-})(h) = \frac{1}{p_-}\int_\Omega \frac{-\Delta_{p(z)}\overline{u}}{\overline{u}^{p_--1}} h dz = \frac{1}{p_-}\int_\Omega \left[\frac{c_1}{\overline{u}^{p_--q(z)}} - c_2\overline{u}^{r(z)-p_-}\right] hdz,$$
$$j'(\overline{v}^{p_-})(h) = \frac{1}{p_-}\int_\Omega \frac{-\Delta_{p(z)}\overline{v}}{\overline{v}^{p_--1}} h dz = \frac{1}{p_-}\int_\Omega \left[\frac{c_1}{\overline{v}^{p_--q(z)}} - c_2\overline{v}^{r(z)-p_-}\right] hdz.$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. So, we have

$$0 \le \int_\Omega \left[c_1\left(\frac{1}{\overline{u}^{p_--q(z)}} - \frac{1}{\overline{v}^{p_--q(z)}}\right) - c_2(\overline{u}^{r(z)-p_-} - \overline{v}^{r(z)-p_-})\right](\overline{u}^{p_-} - \overline{v}^{p_-})dz \le 0$$
$$(\text{since } q_+ < p_- < r_-)$$
$$\Rightarrow \overline{u} = \overline{v}.$$

This proves the uniqueness of the positive solution $\overline{u} \in \operatorname{int} C_+$. ☐

In what follows, let $\widehat{d}(\cdot) = d(\cdot, \partial\Omega)$ and $\widehat{u}_1$ is the positive, L^{p_+}-normalized (that is, $\|\widehat{u}_1\|_{p_+} = 1$) eigenfunction corresponding to the principal eigenvalue of $(-\Delta_{p_+}, W^{1,p_+}(\Omega))$. We know that $\widehat{u}_1 \in \operatorname{int} C_+$ (see, for example, Gasiński–Papageorgiou [4, p. 739]).

Proposition 3.2. *If Hypotheses H_0 hold and $\overline{u} \in \operatorname{int} C_+$ is the unique solution of problem (3.3), then $\overline{u}(\cdot)^{-\eta(\cdot)} \in L^1(\Omega)$ and for every $h \in W_0^{1,p(z)}(\Omega), \overline{u}(\cdot)^{-\eta(\cdot)}h(\cdot) \in L^1(\Omega)$.*

Proof. From Lemma 14.16, p. 355 of Gilbarg–Trudinger [6], we can find $\delta_0 > 0$ such that, if $\Omega_{\delta_0} = \{z \in \overline{\Omega} : \widehat{d}(z) < \delta_0\}$, then $\widehat{d} \in C^2(\Omega_{\delta_0})$. If follows that $\widehat{d} \in \operatorname{int} C_+$ and so by Proposition 4.1.22, p. 274, of Papageorgiou–Rădulescu–Repovš [9], we can find $c_7 > 0$ such that

$$c_7\widehat{u}_1 \le \widehat{d} \quad \text{and} \quad c_7\widehat{d} \le \overline{u} \quad (\text{recall } \overline{u} \in \operatorname{int} C_+). \tag{3.11}$$

From (3.11) we infer that

$$\overline{u}^{-\eta(\cdot)} \le c_8\widehat{u}_1^{-\eta(\cdot)} \quad \text{for some } c_8 > 0.$$

Then the Lemma (in fact its proof to be precise) of Lazer–McKenna [8], implies that $\widehat{u}_1^{-\eta(\cdot)} \in L^1(\Omega)$. Therefore we have

$$\overline{u}^{-\eta(\cdot)} \in L^1(\Omega).$$

On the other hand, for every $h \in W_0^{1,p(z)}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega |\overline{u}^{-\eta(z)}h|dz &= \int_\Omega \overline{u}^{1-\eta(z)}\frac{|h|}{\overline{u}}dz \\
&\le c_9\int_\Omega \frac{|h|}{\overline{u}}dz \quad \text{for some } c_9 > 0 \\
&\quad (\text{recall that } \overline{u} \in \operatorname{int} C_+ \text{ and see hypotheses } H_0) \\
&\le c_{10}\int_\Omega \frac{|h|}{\widehat{d}}dz \quad \text{for some } c_{10} > 0 \quad (\text{see (3.11)}) \\
&\le c_{11}\|\frac{h}{\widehat{d}}\|_{p(z)} \quad \text{for some } c_{11} > 0 \\
&\le c_{12}\|Dh\|_{p(z)} \quad \text{for some } c_{12} > 0.
\end{aligned}$$

This last inequality is a consequence of the anisotropic Hardy inequality due to Harjulehto–Hästö–Koskenoja [7]. So, finally we have

$$\overline{u}(\cdot)^{-\eta(\cdot)}h(\cdot) \in L^1(\Omega) \quad \text{for all } h \in W_0^{1,p(z)}(\Omega). \qquad \square$$

4 Multiple positive solutions

In this section using $\overline{u} \in \operatorname{int} C_+$, the unique positive solution of (3.3), we are able to bypass the singularity and have C^1-functionals. Working with them, we show that problem (1.1) has at least two positive smooth solutions.

Theorem 4.1. *If hypotheses H_0, H_1 hold, then problem (1.1) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+, u_0 \neq \widehat{u}, u_0(z) < \theta$ for all $z \in \overline{\Omega}$.*

Proof. Let $\overline{u} \in \operatorname{int} C^+$ be the unique positive solution of problem (3.3) produced in Proposition 3.1. We introduce the Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(z,x) = \begin{cases} \overline{u}^{-\eta(z)} + f(z,\overline{u}(z)) & \text{if } x \leq \overline{u}(z) \\ x^{-\eta(z)} + f(z,x) & \text{if } \overline{u}(z) < x. \end{cases} \tag{4.1}$$

From Proposition 3.1 we know that $0 \leq \overline{u}(z) \leq \theta$ for all $z \in \overline{\Omega}$. Hence we can consider the truncation of $g(z,\cdot)$ at θ, that is, the Carathéodory function $\widehat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{g}(z,x) = \begin{cases} g(z,x) & \text{if } x \leq \theta \\ g(z,\theta) & \text{if } \theta < x. \end{cases} \tag{4.2}$$

We set $G(z,x) = \int_0^x g(z,s)ds$ and $\widehat{G}(z,x) = \int_0^x \widehat{g}(z,s)ds$ and consider the functions $\psi, \widehat{\psi} : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \psi(u) &= \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz - \int_\Omega G(z,u)dz, \\ \widehat{\psi}(u) &= \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz - \int_\Omega \widehat{G}(z,u)dz, \quad \text{for all } u \in W_0^{1,p(z)}(\Omega). \end{aligned}$$

On account of Proposition 3.2, these functionals are well-defined and in fact Proposition 3.1 of Papageorgiou–Smyrlis [11] implies that $\psi, \widehat{\psi} \in C^1(W_0^{1,p(z)}(\Omega))$.

For every $u \in W_0^{1,p(z)}(\Omega)$, we have

$$\begin{aligned} \widehat{\psi}(u) &\geq \frac{1}{p_+}\rho_p(Du) - c_{13} \quad \text{for some } c_{13} > 0, \\ &\quad \text{(see (4.1),(4.2) and Proposition 3.2)} \\ \Rightarrow\ & \widehat{\psi}(\cdot) \text{ is coercive.} \\ &\quad \text{(see Proposition 2.1 and use Poincaré's inequality).} \end{aligned}$$

The anisotropic Sobolev embedding theorem implies that $\widehat{\psi}(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\widehat{\psi}(u_0) = \min[\widehat{\psi}(u) : u \in W_0^{1,p(z)}(\Omega)], \tag{4.3}$$

From (4.3) we have

$$\langle \widehat{\psi}'(u_0), h\rangle = 0 \quad \text{for all } h \in W_0^{1,p(z)}(\Omega),$$
$$\Rightarrow \langle A_{p(z)}(u_0), h\rangle = \int_\Omega \widehat{g}(z, u_0) h dz \quad \text{for all } h \in W_0^{1,p(z)}(\Omega). \tag{4.4}$$

In (4.4) first we choose $h = [\overline{u} - u_0]^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\langle A_{p(z)}(u_0), (\overline{u} - u_0)^+\rangle &= \int_\Omega [\overline{u}^{-\eta(z)} + f(z, \overline{u})](\overline{u} - u_0)^+ dz \quad \text{(see (4.1),(4.2))}\\ &\geq \int_\Omega f(z, \overline{u})(\overline{u} - u_0)^+ dz \quad (\text{since } \overline{u} \in \operatorname{int} C_+)\\ &= \langle A_{p(z)}(\overline{u}), (\overline{u} - u_0)^+\rangle \quad \text{(see Proposition 3.1)},\\ &\Rightarrow \overline{u} \leq u_0 \quad \text{(see Proposition 2.2)}.\end{aligned}$$

Next in (4.4) we choose $h = [u_0 - \theta]^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}\langle A_{p(z)}(u_0), (u_0 - \theta)^+\rangle &= \int_\Omega [\theta^{-\eta} + f(z, \theta)](u_0 - \theta)^+ dz \quad \text{(see (4.1),(4.2))}\\ &\leq 0 = \langle A_{p(z)}(\theta), (u_0 - \theta)^+\rangle \quad \text{(see hypothesis } H_1(iii)),\\ &\Rightarrow u_0 \leq \theta.\end{aligned}$$

So, we have proved that

$$u_0 \in [\overline{u}, \theta]. \tag{4.5}$$

From (4.5), (4.1), (4.2) and (4.4), we have that u_0 is a positive solution of (1.1). Invoking Theorem 13.1 of Saaudi–Ghanmi [13] (see also Theorem 3.2 of Byun–Ko [1]), we have that $u_0 \in \operatorname{int} C_+$ (recall $\overline{u} \in \operatorname{int} C_+$).

Now let $\widehat{\xi}_\theta > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned}&-\Delta_{p(z)} u_0 + \widehat{\xi}_\theta u_0^{p(z)-1} - u_0^{-\eta(z)}\\ &\quad = f(z, u_0) + \widehat{\xi}_\theta u_0^{p(z)-1}\\ &\quad \leq f(z, \theta) + \widehat{\xi}_\theta \theta^{p(z)-1} \quad \text{(see (4.5) and hypothesis } H_1(v))\\ &\quad \leq -\Delta_{p(z)}\theta + \widehat{\xi}_\theta \theta^{p(z)-1} - \theta^{-\eta(z)} \quad \text{(see hypothesis } H_1(iii)),\\ &\quad \Rightarrow u_0(z) < \theta \quad \text{for all } z \in \overline{\Omega} \qquad (4.6)\\ &\qquad \text{(from Proposition A4 of Papageorgiou–Rădulescu–Zhang [10])}.\end{aligned}$$

It is clear from (4.1) and (4.2) that

$$\psi|_{[0,\theta]} = \widehat{\psi}|_{[0,\theta]}.$$

Since $u_0 \in \operatorname{int} C_+$, we infer that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega}) \text{ minimizer of } \psi(\cdot) \quad \text{(see (4.6))},$$
$$\Rightarrow u_0 \text{ is a local } W_0^{1,p(z)}(\Omega) \text{ minimizer of } \psi(\cdot) \quad \text{(see [10,13])}. \tag{4.7}$$

Using (4.1) and the anisotropic regularity theory, we can see that $K_\psi \subseteq [\overline{u}) \cap \operatorname{int} C_+$. So, we may assume that K_ψ is finite or otherwise on account of (4.1) we see that we already have a whole sequence of distinct positive smooth solutions and so we are done. Then from (4.7) and Theorem 5.7.6, p. 449, of Papageorgiou–Rădulescu–Repovš [9], we know that we can find $\rho \in (0,1)$ small such that

$$\psi(u_0) < \inf[\psi(u) : \|u - u_0\| = \rho] = m_\rho. \tag{4.8}$$

Moreover, hypothesis $H_1(ii)$ implies that if $u \in \operatorname{int} C_+$, then

$$\psi(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{4.9}$$

Finally from Proposition 4.1 of Gasiński–Papageorgiou [5] (see hypothesis $H_1(ii)$), we have that

$$\psi(\cdot) \quad \text{satisfies the C-condition.} \tag{4.10}$$

Then (4.8), (4.9) and (4.10) permit the use of the mountain pass theorem. Therefore we can find $\hat{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &\hat{u} \in K_\psi \subseteq [\overline{u}) \cap \operatorname{int} C_+, m_\rho \le \psi(\hat{u}), && (4.11)\\ \Rightarrow\ &\hat{u} \in \operatorname{int} C_+ \text{ is a positive solution of (1.1) (see (4.1)),} \\ &\hat{u} \ne u_0 \quad (\text{ see (4.8) and (4.11)}), u_0(z) < \theta \text{ for all } z \in \overline{\Omega}. \end{aligned}$$

□

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