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Existence and exact multiplicity of positive periodic solutions to forced non-autonomous Duffing type differential equations

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Abstract. The paper studies the existence, exact multiplicity, and a structure of the set of positive solutions to the periodic problem

$$u'' = p(t)u + q(t,u)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega),$$

where $p, f \in L([0,\omega])$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function. Obtained general results are applied to the forced non-autonomous Duffing equation

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u + f(t),$$

with $\lambda > 1$ and a non-negative $h \in L([0,\omega])$. We allow the coefficient p and the forcing term f to change their signs.

Keywords: positive periodic solution, second-order differential equation, Duffing equation, existence, uniqueness, multiplicity.

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1 Introduction

On an interval $[0,\omega]$, we consider the periodic problem

$$u'' = p(t)u + q(t,u)u + f(t), \tag{1.1}$$

$$u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.2}$$

where $p, f \in L([0,\omega])$ and $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. By a solution to problem (1.1), (1.2), as usual, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies (1.1) almost everywhere, and meets periodic conditions (1.2). A periodic boundary value problem for differential equations of different types has been extensively studied in the literature. To make the list of references shorter,

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the reader is referred to the well-known monographs [2,3] for a historical background and an extensive list of relevant references.

In this paper, we study the existence and multiplicity of **positive** solutions to problem (1.1), (1.2). Since we are interested in a Duffing type equation, which is originally characterized by a super-linear non-linearity, we write a non-linear term in the form $q(t,u)u$. We continue our previous studies presented in [8], where problem (1.1), (1.2) with $f(t) \equiv 0$ is considered. We have shown, among other things, that, if the function q is non-negative, then for the existence of a positive solution to (1.1), (1.2) with $f(t) \equiv 0$, it is necessary that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ (see Definitions 2.2 and 2.3). Therefore, we restrict ourselves to the case of (1.1), in which the "linear part" satisfies $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.

A particular case of (1.1) is the non-autonomous Duffing equation

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + f(t), \tag{1.3}$$

with $p, h, f \in L([0,\omega])$ and $\lambda > 1$, that is frequently studied in the literature (not only for ODEs), because arises in mathematical modelling in mechanics (mainly with $\lambda = 3$). Such an equation (with constant coefficients p, h) is the central topic of the monograph [1] by Duffing published in 1918 and still bears his name (see also [5]). Let us show, as a motivation, what happens in the autonomous case. If $p(t) := -a$, then $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ if and only if $a > 0$ (see Remark 2.4). Therefore, consider the equation

$$x'' = -ax + b|x|^\lambda \operatorname{sgn} x + c, \tag{1.4}$$

where $a > 0$ and $b, c \in \mathbb{R}$. In this paper, we are interested in the equation (1.3) with a non-negative h and, thus, we assume that $b > 0$ in (1.4). By direct calculation, the phase portraits of (1.4) can be elaborated depending on the choice of c, which leads to the following proposition.

Proposition 1.1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

(1) If $c \leq 0$, then equation (1.4) has a unique positive equilibrium (saddle) and no other positive periodic solutions occur.

(2) If $0 < c < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) possesses exactly two positive equilibria $x_1 > x_2$ (x_1 is a saddle and x_2 is a center), a unique negative equilibrium x_3 (saddle), and non-constant (both positive and sign-changing) periodic solutions with different periods. Moreover, all non-constant periodic solutions are smaller then x_1 and oscillate around x_2.

(3) If $c = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) has a unique positive equilibrium (cusp), a unique negative equilibrium (saddle), and no other periodic solutions occur.

(4) If $c > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation (1.4) has a unique negative equilibrium (saddle) and no other periodic solutions occur.

In [4], the authors study the stability and exact multiplicity of solutions to the periodic problem

$$x'' + cx' + ax - x^3 = d(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.5}$$

where $c > 0$, $0 < a < \frac{\pi^2}{\omega^2} + \frac{c^2}{4}$, and $d\colon [0,\omega] \to \mathbb{R}$ is a positive continuous function. It follows from the proof of Theorem 1.1 in [4] that all the conclusions of Theorem 1.1 remain true, except of the asymptotic stability, even in the case of $c = 0$. Therefore, [4, Theorem 1.1] yields

Proposition 1.2. *Let* $0 < a < \frac{\pi^2}{\omega^2}$ *and* $d_0 := \frac{2a}{3}\sqrt{\frac{a}{3}}$. *Then, the following conclusions hold:*

(1) Problem (1.5)*, with* $c = 0$*, has a unique solution that is negative if* $d(t) > d_0$ *for* $t \in [0, \omega]$.

(2) Problem (1.5)*, with* $c = 0$*, has exactly three ordered solutions if* $0 < d(t) < d_0$ *for* $t \in [0, \omega]$. *Moreover, the minimal solution is negative and the other two solutions are positive.*

In Section 3, we generalize some conclusions of Propositions 1.1 and 1.2. We use a technique developed in [8] and determine a **well-ordered** pair of positive lower and upper functions, which allows us to establish general results guaranteeing the existence and exact multiplicity of positive solutions to (1.1), (1.2) as well as to provide some properties of the set of all positive solutions to (1.1), (1.2). The obtained results and their consequences for (1.3), (1.2) will be compared with the conclusions of Propositions 1.1 and 1.2 (see Remarks 3.18, 3.20, 3.21, 3.23, 3.28, and 3.35).

It is worth mentioning that, in contrast to [4], our results cover also the case of a **sign-changing coefficient** p and a **sign-changing forcing term** f.

2 Notation and definitions

The following notation is used throughout the paper:

- $\mathbb{R}$ is the set of real numbers. For $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.
- $C(I)$ denotes the set of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For $u \in C([a, b])$, we put $\|u\|_C = \max\{|u(t)| : t \in [a, b]\}$.
- $AC^1([a, b])$ is the set of functions $u\colon [a, b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives.
- $AC_\ell([a, b])$ (resp. $AC_u([a, b])$) is the set of absolutely continuous functions $u\colon [a, b] \to \mathbb{R}$ such that u' admits the representation $u'(t) = \gamma(t) + \sigma(t)$ for a. e. $t \in [a, b]$, where $\gamma\colon [a, b] \to \mathbb{R}$ is absolutely continuous and $\sigma\colon [a, b] \to \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a, b]$.
- $L([a, b])$ is the Banach space of Lebesgue integrable functions $p\colon [a, b] \to \mathbb{R}$ equipped with the norm $\|p\|_L = \int_a^b |p(s)| \mathrm{d}s$. The symbol $\operatorname{Int} A$ stands for the interior of the set $A \subset L([a, b])$.

Definition 2.1. Let $I \subseteq \mathbb{R}$. A function $f\colon [a, b] \times I \to \mathbb{R}$ is said to be Carathéodory function if

(a) the function $f(\cdot, x)\colon [a, b] \to \mathbb{R}$ is measurable for every $x \in I$,

(b) the function $f(t, \cdot)\colon I \to \mathbb{R}$ is continuous for almost every $t \in [0, \omega]$,

(c) for any $r > 0$, there exists $q_r \in L([a, b])$ such that $|f(t, x)| \le q_r(t)$ for a. e. $t \in [a, b]$ and all $x \in I$, $|x| \le r$.

Definition 2.2 ([6, Definitions 0.1 and 15.1, Propositions 15.2 and 15.4])**.** We say that a function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if, for any function $u \in AC^1([0, \omega])$ satisfying

$$u''(t) \ge p(t)u(t) \quad \text{for a. e. } t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) \ge u'(\omega),$$

the inequality

$$u(t) \geq 0 \quad \text{for } t \in [0,\omega] \quad \big(\text{resp. } u(t) \leq 0 \quad \text{for } t \in [0,\omega]\big)$$

holds.

Definition 2.3 ([6, Definition 0.2]). We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.1}$$

has a positive solution.

Remark 2.4. Let $\omega > 0$. If $p(t) := p_0$ for $t \in [0,\omega]$, then one can show by direct calculation that:

- ▷ $p \in \mathcal{V}^-(\omega)$ if and only if $p_0 > 0$,
- ▷ $p \in \mathcal{V}_0(\omega)$ if and only if $p_0 = 0$,
- ▷ $p \in \mathcal{V}^+(\omega)$ if and only if $p_0 \in \big[-\frac{\pi^2}{\omega^2}, 0\big[$,
- ▷ $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ if and only if $p_0 \in \big]-\frac{\pi^2}{\omega^2}, 0\big[$.

If the function $p \in L([0,\omega])$ is not constant, efficient conditions for p to belong to each of the sets $\mathcal{V}^+(\omega)$ and $\mathcal{V}^-(\omega)$ are provided in [6].

Remark 2.5. It is well known that, if the homogeneous problem (2.1) has only the trivial solution, then, for any $f \in L([0,\omega])$, the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.2}$$

possesses a unique solution u and this solution satisfies

$$|u(t)| \leq \Delta(p) \int_0^\omega |f(s)|\mathrm{d}s \quad \text{for } t \in [0,\omega],$$

where $\Delta(p)$, depending only on p, denotes a norm of the Green's operator of problem (2.1). Clearly, $\Delta(p) > 0$.

Assume that $p \in \operatorname{Int}\mathcal{V}^+(\omega)$. Extend the function p periodically to the whole real axis denoting it by the same symbol. It is proved in [6, Section 6] that, for any $a \in \mathbb{R}$, the problem

$$u'' = p(t)u; \quad u(a) = 1,\ u(a+\omega) = 1$$

has a unique solution u_a and $u_a(t) > 0$ for $t \in [0,\omega]$. We put

$$\Gamma(p) := \sup\big\{\|u_a\|_C : a \in [0,\omega]\big\}\, \mathrm{e}^{\int_0^\omega [p(s)]_+\mathrm{d}s}. \tag{2.3}$$

It is clear that $\Gamma(p) \geq 1$.

Remark 2.6. If $p \in \mathcal{V}^+(\omega)$, then the number $\Delta(p)$ defined in Remark 2.5 can be estimated, for example, by using a maximal value of the Green's function of problem (2.1) (see, e. g., [9]). On the other hand, assuming $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, some estimates of the number $\Gamma(p)$ given by (2.3) are provided in [6, Section 6].

For instance, if $p(t) := p_0$ for $t \in [0,\omega]$ and $p_0 \in \big[-\frac{\pi^2}{\omega^2}, 0\big[$, resp. $p_0 \in \big]-\frac{\pi^2}{\omega^2}, 0\big[$, then

$$\Delta(p) \leq \left(2\sqrt{|p_0|}\sin\frac{\omega\sqrt{|p_0|}}{2}\right)^{-1}, \quad \text{resp.} \quad \Gamma(p) = \left(\cos\frac{\omega\sqrt{|p_0|}}{2}\right)^{-1}.$$

3 Main results

This section contains formulations of all the main results of the paper. Their proofs are presented in detail in Section 5.

3.1 Existence theorems

Let us introduce the hypothesis

$$\left.\begin{array}{l} q(t,x) \geq q_0(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \geq x_0, \\ x_0 \geq 0,\ q_0\colon [0,\omega] \times [x_0,+\infty[\to \mathbb{R} \text{ is a Carathéodory function,} \\ q_0(t,\cdot)\colon [x_0,+\infty[\to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega]. \end{array}\right\} \tag{H_1}$$

Theorem 3.1. *Let hypothesis (H_1) be fulfilled, and there exist $R > x_0$ such that $p + q_0(\cdot,R) \in \mathcal{V}^-(\omega)$. Let, moreover, there exist a positive function $\alpha \in AC_\ell([0,\omega])$ satisfying*

$$\alpha(0) = \alpha(\omega), \quad \alpha'(0) \geq \alpha'(\omega), \tag{3.1}$$

$$\alpha''(t) \geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) \quad \textit{for a. e. } t \in [0,\omega]. \tag{3.2}$$

Then, problem (1.1), (1.2) has a positive solution u satisfying

$$u(t) \geq \alpha(t) \quad \textit{for } t \in [0,\omega]. \tag{3.3}$$

We now provide an effective condition guaranteeing the existence of the function α in Theorem 3.1.

Corollary 3.2. *Let $p + [q(\cdot,0)]_+ \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, hypothesis ($H_1$) be fulfilled, and*

$$\lim_{x \to +\infty} \int_E q_0(s,x)\mathrm{d}s = +\infty \quad \textit{for every } E \subseteq [0,\omega],\ \operatorname{meas} E > 0. \tag{3.4}$$

Let, moreover,

$$\int_0^\omega [f(s)]_+\mathrm{d}s < \sup\left\{ \frac{r}{\Delta\big(p + q^*(\cdot,r)\big)} : r > 0,\ p + q^*(\cdot,r) \in \mathcal{V}^+(\omega) \right\}, \tag{3.5}$$

where Δ is defined in Remark 2.5 and

$$q^*(t,\varrho) := \max\big\{[q(t,x)]_+ : x \in [0,\varrho]\big\} \quad \textit{for a. e. } t \in [0,\omega] \textit{ and all } \varrho \geq 0. \tag{3.6}$$

Then, problem (1.1), (1.2) has at least one positive solution.

Remark 3.3. In Corollary 3.2, q^* is obviously a Carathéodory function satisfying $q^*(t,0) \equiv [q(t,0)]_+$. By Lemma 4.15, it follows from hypothesis (3.4) that there exists $R > x_0$ such that $p + q_0(\cdot,R) \in \mathcal{V}^-(\omega)$. Moreover, $q^*(t,R) \geq q_0(t,R)$ for a. e. $t \in [0,\omega]$ and, therefore, Lemma 4.12 yields $p + q^*(\cdot,R) \in \mathcal{V}^-(\omega)$. Since $p + q^*(\cdot,0) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, by virtue of Lemma 4.11 (with $\ell(t,x) := p(t) + q^*(t,x)$), there exists $r \in]0,R[$ such that $p + q^*(\cdot,r) \in \mathcal{V}^+(\omega)$ and, thus, hypothesis (3.5) of Corollary 3.2 is consistent.

Remark 3.4. If the supremum on the right-hand side of (3.5) is achieved at some $r_0 > 0$, then the strict inequality (3.5) in Corollary 3.2 (as well as Corollary 3.7) can be weakened to the non-strict one (see the end of the proof of Corollary 3.2).

Remark 3.5. By Lemma 4.1, the hypothesis $p + [q(\cdot,0)]_+ \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ of Corollary 3.2 is satisfied provided that

$$\int_0^\omega \Big(p(s) + [q(s,0)]_+\Big)\mathrm{d}s \le 0, \qquad p(t) + [q(t,0)]_+ \not\equiv 0.$$

Remark 3.6. If

$$f(t) \le 0 \quad \text{for a. e. } t \in [0,\omega], \tag{3.7}$$

then condition (3.5) is obviously satisfied.

Assuming $p \in \mathcal{V}^+(\omega)$, hypothesis (3.4) of Corollary 3.2 can be weakened to

$$\lim_{x\to+\infty} \int_0^\omega q_0(s,x)\mathrm{d}s = +\infty. \tag{3.8}$$

Moreover, in such a case, another type of condition on $[f]_+$ can be provided instead of (3.5).

Corollary 3.7. *Let $p \in \mathcal{V}^+(\omega)$, $q(t,0) \equiv 0$, hypothesis (H_1) be fulfilled, (3.8) hold, and there exist $x_1 > x_0$ such that*

$$q_0(t,x_1) \ge 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{3.9}$$

Let, moreover, either (3.5) hold or $[f(t)]_+ \not\equiv 0$ and

$$\Delta(p) < \sup\left\{ \frac{r}{\int_0^\omega [f(s)]_+\mathrm{d}s + r\int_0^\omega q^*(s,r)\mathrm{d}s} : r > 0 \right\}, \tag{3.10}$$

where Δ is defined in Remark 2.5 and q^* is given by (3.6). Then, problem (1.1), (1.2) has at least one positive solution.

Remark 3.8. If the supremum on the right-hand side of (3.10) is achieved at some $r_0 > 0$, then the strict inequality (3.10) in Corollary 3.7 can be weakened to the non-strict one (see the end of the proof of Corollary 3.7).

It follows from Remark 2.6 that, in some particular cases, the number Δ defined in Remark 2.5 can be estimated from above and, thus, the effective conditions guaranteeing the validity of (3.5) and (3.10) can be found. In Section 3.3, we will provide such conditions for the Duffing equation (1.3).

3.2 Uniqueness and multiplicity theorems

Proposition 1.1 (1) implies that, if $a, b > 0$ and $c \le 0$, then, for any $\omega > 0$, equation (1.4) possesses a unique positive ω-periodic solution. Now we show that, under a certain monotonicity condition on q, a positive solution in Theorem 3.1 is unique provided that the function f is non-positive. Moreover, we generalize the ideas used in the proof of [4, Theorem 1.1] and, thus, we obtain some conditions on the forcing term f leading to the exact multiplicity of positive solutions to problem (1.1), (1.2).

Theorem 3.9. *Assume that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(t,0) \equiv 0$, (3.7) holds, and*

$$\left.\begin{array}{l} \text{for every } d > c > 0 \text{ and } e > 0, \text{ there exists } h_{cde} \in L([0,\omega]) \text{ such that} \\ h_{cde}(t) > 0 \quad \text{for a. e. } t \in [0,\omega], \\ q(t,x+e) - q(t,x) \ge h_{cde}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d]. \end{array}\right\} \tag{H_2}$$

Then, problem (1.1), (1.2) has at most one positive solution.

Combining Corollary 3.2 and Theorem 3.9, we get

Corollary 3.10. *Let $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $q(t,0) \equiv 0$, hypotheses (H_1) and (H_2) be fulfilled, and conditions (3.4) and (3.7) hold. Then, problem (1.1), (1.2) has a unique positive solution.*

In the next theorem, we assume that the non-linearity $q(t,u)u$ in (1.1) is "locally uniformly strictly concave/convex" in the sense of hypothesis (H_3^ℓ).

Proposition 3.11. *Assume that $p, f \in L([0,\omega])$, $\ell \in \{1,2\}$, and*

$$\left.\begin{aligned} &\text{for every } d_1 > c_1 > 0,\ d_2 > c_2 > 0,\ d_3 > c_3 > 0 \text{ there exists} \\ &h^* \in L([0,\omega]),\ h^*(t) \geq 0 \text{ for a. e. } t \in [0,\omega],\ h^*(t) \not\equiv 0, \\ &(-1)^\ell \left[\frac{q(t,x_3)x_3 - q(t,x_2)x_2}{x_3 - x_2} - \frac{q(t,x_2)x_2 - q(t,x_1)x_1}{x_2 - x_1}\right] \geq h^*(t) \\ &\quad \text{for a. e. } t \in [0,\omega] \text{ and all } c_1 \leq x_1 \leq d_1,\ x_1 + c_2 \leq x_2 \leq x_1 + d_2, \\ &\qquad\qquad x_2 + c_3 \leq x_3 \leq x_2 + d_3, \end{aligned}\right\} \tag{H_3^ℓ}$$

Then, there are no three solutions u_1, u_2, u_3 to problem (1.1), (1.2) satisfying

$$u_3(t) > u_2(t) > u_1(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{3.11}$$

Remark 3.12. Let $q(t,x) := h(t)\varphi(x)$, where $h \in L([0,\omega])$ and $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, q satisfies hypothesis (H_3^1) (resp. (H_3^2) provided that $h(t) \geq 0$ for a. e. $t \in [0,\omega]$, $h(t) \not\equiv 0$, and the function $x \mapsto \varphi(x)x$ is strictly concave (resp. convex) on $]0,+\infty[$.

If $p \in \mathcal{V}^+(\omega)$, then hypothesis (H_2) of Theorem 3.9 can be weakened to (H_2'). Moreover, one can show some other properties of solutions to problem (1.1), (1.2) in such a case. Introduce the hypothesis:

$$\left.\begin{aligned} &\text{For every } d > c > 0 \text{ and } e > 0, \text{ there exists } h_{cde} \in L([0,\omega]) \text{ such that} \\ &h_{cde}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega],\ h_{cde}(t) \not\equiv 0, \\ &q(t,x+e) - q(t,x) \geq h_{cde}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [c,d]. \end{aligned}\right\} \tag{H_2'}$$

Theorem 3.13. *Let $p \in \mathcal{V}^+(\omega)$. Then, the following conclusions hold:*

(1) If q satisfies hypothesis (H_2'),

$$q(t,0) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \tag{3.12}$$

and u, v are distinct positive solutions to problem (1.1), (1.2), then

$$u(t) \neq v(t) \quad \text{for } t \in [0,\omega]. \tag{3.13}$$

(2) If (3.7) and (3.12) hold and q satisfies hypothesis (H_2'), then problem (1.1), (1.2) has at most one positive solution.

(3) If $\ell \in \{1,2\}$, (3.12) holds and q satisfies hypotheses (H_2') and (H_3^ℓ), then problem (1.1), (1.2) has at most two positive solutions.

(4) If

$$q(t,x) \geq 0 \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}, \tag{3.14}$$

$$f(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad f(t) \not\equiv 0, \tag{3.15}$$

then every solution to (1.1), (1.2) is either positive or negative.

Combining Corollary 3.7 and Theorem 3.13 (2), we get

Corollary 3.14. *Let $p \in \mathcal{V}^+(\omega)$, $q(t,0) \equiv 0$, hypotheses (H_1) and (H_2') be fulfilled, there exist $x_1 > x_0$ such that (3.9) holds, and conditions (3.7) and (3.8) be satisfied. Then, problem (1.1), (1.2) has a unique positive solution.*

3.3 Consequences for the non-autonomous Duffing equation (1.3)

We now apply the above general results for the non-autonomous Duffing equation (1.3) and compare the obtained results with those stated in Propositions 1.1 and 1.2. In this section, we assume that the function h in (1.3) is non-negative. However, the properties of the given periodic problem differ in the following two cases: $h(t) > 0$ a. e. on $[0,\omega]$ and $h(t) \geq 0$ a. e. on $[0,\omega]$, $h(t) \not\equiv 0$. Such phenomenon does not occur in the autonomous case of (1.3) (i. e., in (1.4)).

Theorem 3.15. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, and*

$$h(t) > 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{3.16}$$

Then, the following conclusions hold:

(1) There are no three solutions u_1, u_2, u_3 to problem (1.3), (1.2) satisfying (3.11).

(2) Assume that there exists a positive function $\alpha \in AC_\ell([0,\omega])$ such that (3.1) holds and

$$\alpha''(t) \geq p(t)\alpha(t) + h(t)\alpha^\lambda(t) + f(t) \quad \text{for a. e. } t \in [0,\omega]. \tag{3.17}$$

Then, problem (1.3), (1.2) has a positive solution u^* satisfying

$$u^*(t) \geq \alpha(t) \quad \text{for } t \in [0,\omega] \tag{3.18}$$

such that every solution u to problem (1.3), (1.2) satisfies

$$\text{either} \quad u(t) < u^*(t) \quad \text{for } t \in [0,\omega], \quad \text{or} \quad u(t) \equiv u^*(t). \tag{3.19}$$

Moreover, for any couple of distinct positive solutions u_1, u_2 to (1.3), (1.2) satisfying

$$u_1(t) \not\equiv u^*(t), \qquad u_2(t) \not\equiv u^*(t), \tag{3.20}$$

the conditions

$$\begin{aligned} \min\{u_1(t) - u_2(t) : t \in [0,\omega]\} < 0, \\ \max\{u_1(t) - u_2(t) : t \in [0,\omega]\} > 0 \end{aligned} \tag{3.21}$$

hold.

(3) If (3.7) holds, then problem (1.3), (1.2) has a unique positive solution.

Now we provide a sufficient condition guaranteeing the existence of the function α in Theorem 3.15 (2)

Corollary 3.16. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, h satisfy (3.16), and*

$$\int_0^\omega [f(s)]_+ \mathrm{d}s < \sup\left\{ \frac{r}{\Delta(p + r^{\lambda-1}h)} : r > 0,\ p + r^{\lambda-1}h \in \mathcal{V}^+(\omega) \right\}, \tag{3.22}$$

where Δ is defined in Remark 2.5. Then, there exists a positive function $\alpha \in AC^1([0,\omega])$ satisfying (3.17) and

$$\alpha(0) = \alpha(\omega), \quad \alpha'(0) = \alpha'(\omega), \tag{3.23}$$

and, thus, the conclusions of Theorem 3.15 (2) hold.

Remark 3.17. It follows from the proof of Corollary 3.16 and Remark 3.4 that, if the supremum on the right-hand side of (3.22) is achieved at some $r_0 > 0$, then the strict inequality (3.22) can be weakened to the non-strict one.

Remark 3.18. Observe that Theorem 3.15 (and Corollary 3.16) extends the conclusions of Proposition 1.1 for the non-autonomous Duffing equation (1.3). Indeed, let $\omega > 0$ and

$$p(t) := -a, \quad h(t) := b \quad \text{for } t \in [0,\omega], \tag{3.24}$$

where $a, b > 0$. Then, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ (see Remark 2.4) and the function h satisfies (3.16). We emphasize, in particular, the conclusion of Corollary 3.16, which claims: If the forcing term f satisfies the integral-type condition (3.22), then problem (1.3), (1.2) has a maximal solution u^* that is positive. Moreover, every two positive solutions to problem (1.3), (1.2) (different from u^*) must intersect each other; compare it with Proposition 1.1 (2).

As we have mentioned in Remark 2.6, in the case of constant functions, the number Δ defined in Remark 2.5 can be estimated from above. Therefore, for the problem

$$u'' = -au + b|u|^\lambda \operatorname{sgn} u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3.25}$$

with $a, b > 0$, $\lambda > 1$, and $f \in L([0,\omega])$, Corollary 3.16 yields the following corollary.

Corollary 3.19. *Let $\lambda > 1$, $a, b > 0$, and*

$$\int_0^\omega [f(s)]_+ \mathrm{d}s \le \begin{cases} \frac{2\omega}{\pi} \frac{(\lambda-1)a}{\lambda} \left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} & \text{if } a < \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2, \\ \frac{2\pi}{\omega} \left[\frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right)\right]^{\frac{1}{\lambda-1}} & \text{if } a \ge \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2. \end{cases} \tag{3.26}$$

Then, problem (3.25) has at least one positive solution.

Remark 3.20. Observe that, if $f(t) \equiv c$ and $0 < a \le \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2$, then (3.26) reads as

$$c \le \frac{2}{\pi} \frac{(\lambda-1)a}{\lambda} \left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}. \tag{3.27}$$

The right-hand side of (3.27) is, up to the factor $\frac{2}{\pi}$, the number appearing in Proposition 1.1. Since condition (3.27) was derived from the integral-type condition (3.26) concerning non-constant forcing terms, it is not surprising that it can be improved in the autonomous case.

Remark 3.21. Let $f \in L([0,\omega])$ be such that

$$[f(t)]_+ \le f_0 \quad \text{for a. e. } t \in [0,\omega],$$

where

$$f_0 := \begin{cases} \frac{2}{\pi} \frac{2a}{3} \sqrt{\frac{a}{3}} & \text{if } a < \frac{3}{2}\left(\frac{\pi}{\omega}\right)^2, \\ \frac{2\pi}{\omega^2} \sqrt{a - \frac{\pi^2}{\omega^2}} & \text{if } a \ge \frac{3}{2}\left(\frac{\pi}{\omega}\right)^2. \end{cases}$$

Then, condition (3.26), with $b = 1$ and $\lambda = 3$, holds and, thus, Corollary 3.19 guarantees the existence of a positive solution to problem (1.5), with $c = 0$ and $d(t) \equiv f(t)$. Therefore, Corollary 3.19 complements the conclusions of Proposition 1.2 for the case of $a \ge \frac{\pi^2}{\omega^2}$ and a sign-changing forcing term d.

From Theorem 3.15 (3), we get the following generalization of Proposition 1.1 (1) for the Duffing equation with the constant coefficients and a non-constant forcing term.

Corollary 3.22. *Let $\lambda > 1$, $a, b > 0$, and (3.7) hold. Then, problem (3.25) has a unique positive solution.*

Remark 3.23. Corollary 3.22 complements the conclusions of Proposition 1.2 by the existence and uniqueness of a negative solution to problem (1.5), with $c = 0$, provided that $a > 0$ and the forcing term d is non-negative.

We have shown in [8, Example 2.8] that, if $f(t) \equiv 0$, then hypothesis (3.16) in the above statements is optimal and cannot be weakened to

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad h(t) \not\equiv 0. \tag{3.28}$$

However, this weaker assumption on h can be considered instead of (3.16) under a stronger assumption on p, namely, $p \in \mathcal{V}^+(\omega)$. Moreover, one can show the exact multiplicity of solutions to problem (1.3), (1.2) in such a case. We first introduce the following definition.

Definition 3.24 ([6, Definition 16.1])**.** Let $p, f \in L([0, \omega])$. We say that the pair (p, f) belongs to the set $\mathcal{U}(\omega)$ if problem (2.2) has a unique solution which is positive.

Theorem 3.25. *Let $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, and (3.28) be fulfilled. Then, the following conclusions hold:*

(1) Problem (1.3), (1.2) has at most two positive solutions.

(2) Assume that (3.22) holds, where Δ is defined in Remark 2.5. Then, problem (1.3), (1.2) has either one or two positive solutions.

(3) Assume that there exists a positive function $\alpha \in AC_\ell([0, \omega])$ satisfying (3.1) and (3.17). Then, problem (1.3), (1.2) has a positive solution u^* satisfying (3.18) such that, for every solution u to problem (1.3), (1.2), condition (3.19) holds.

(4) Assume that $(p, f) \in \mathcal{U}(\omega)$ and there exist functions $\alpha_1 \in AC_\ell([0, \omega])$ and $\alpha_2 \in AC^1([0, \omega])$ such that

$$0 < \alpha_2(t) < \alpha_1(t) \quad \text{for } t \in [0, \omega], \tag{3.29}$$

$$\alpha_k(0) = \alpha_k(\omega), \quad \alpha_k'(0) \geq \alpha_k'(\omega) \quad \text{for } k = 1, 2, \tag{3.30}$$

$$\alpha_k''(t) \geq p(t)\alpha_k(t) + h(t)\alpha_k^\lambda(t) + f(t) \quad \text{for a. e. } t \in [0, \omega],\ k = 1, 2. \tag{3.31}$$

Then, problem (1.3), (1.2) possesses exactly two positive solutions u_1, u_2 and these solutions satisfy

$$u_1(t) > u_2(t) > 0 \quad \text{for } t \in [0, \omega]. \tag{3.32}$$

Moreover, for every solution u to problem (1.3), (1.2) different from u_1, the condition

$$u(t) < u_1(t) \quad \text{for } t \in [0, \omega] \tag{3.33}$$

holds.

(5) If (3.7) holds, then problem (1.3), (1.2) has a unique positive solution.

Remark 3.26. It follows from Lemma 4.3 that, if $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, then the inclusion $(p, f) \in \mathcal{U}(\omega)$ holds for every function $f \in L([0,\omega])$ satisfying $f(t) \not\equiv 0$ and

$$\int_0^\omega [f(s)]_+ \mathrm{d}s \geq \Gamma(p) \int_0^\omega [f(s)]_- \mathrm{d}s,$$

where Γ is given by (2.3).

On the other hand, if $p \in \mathcal{V}^+(\omega)$ and f satisfies (3.15), then $(p, f) \in \mathcal{U}(\omega)$ as well (see Lemma 4.2).

Remark 3.27. It follows from the proof of Theorem 3.25 that the solution u_1 in the conclusion of Theorem 3.25 (4) satisfies $u_1(t) \geq \alpha_1(t)$ for $t \in [0,\omega]$ and the solution u_2 is such that $u_2(t_0) \leq \alpha_2(t_0)$ for some $t_0 \in [0,\omega]$.

Remark 3.28. Let $\omega > 0$ and the functions p, h be defined by (3.24), where $0 < a \leq \frac{\pi^2}{\omega^2}$ and $b > 0$. Then, $p \in \mathcal{V}^+(\omega)$ (see Remark 2.4) and the function h satisfies (3.28). Therefore, it follows from Theorem 3.25 (1) that, for any $c \in \mathbb{R}$, equation (1.4) has at most two positive ω-periodic solutions. Consequently, if $0 < c \leq \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$ and u_0 be a non-constant positive periodic solution appearing in conclusion (2) of Proposition 1.1, then the minimal period T of the solution u_0 satisfies

$$T > \frac{\pi}{\sqrt{a}}.$$

Now we provide sufficient conditions guaranteeing the existence of the functions α and α_1, α_2 in Theorem 3.25(3,4).

Corollary 3.29. *Let $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, and h satisfy (3.28). Then, the following conclusions hold:*

(1) If

$$0 < \int_0^\omega [f(s)]_+ \mathrm{d}s \leq \frac{\lambda - 1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}, \tag{3.34}$$

where Δ is defined in Remark 2.5, then there exists a positive function $\alpha \in AC^1([0,\omega])$ satisfying (3.17) and (3.23) and, thus, the conclusion of Theorem 3.25 (3) holds.

(2) If $(p, f) \in \mathcal{U}(\omega)$ and

$$0 < \int_0^\omega [f(s)]_+ \mathrm{d}s < \frac{\lambda - 1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}, \tag{3.35}$$

where Δ is defined in Remark 2.5, then there exists functions $\alpha_1, \alpha_2 \in AC^1([0,\omega])$ satisfying (3.29), (3.31), and

$$\alpha_k(0) = \alpha_k(\omega), \quad \alpha_k'(0) = \alpha_k'(\omega) \quad \text{for } k = 1, 2 \tag{3.36}$$

and, thus, the conclusions of Theorem 3.25 (4) hold.

For the constant coefficient p in (1.3), we derive the following corollary.

Corollary 3.30. *Let $\lambda > 1$, $a \in \left]0, \frac{\pi^2}{\omega^2}\right]$, (3.28) hold, and*

$$0 < \int_0^\omega [f(s)]_+ \mathrm{d}s \leq \frac{\lambda - 1}{\lambda} \frac{\left[2\sqrt{a}\sin\frac{\omega\sqrt{a}}{2}\right]^{\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}. \tag{3.37}$$

Then, the problem

$$u'' = -au + h(t)|u|^{\lambda} \operatorname{sgn} u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{3.38}$$

has either one or two solutions.

Corollary 3.31. *Let $\lambda > 1$, $a \in \left]0, \frac{\pi^2}{\omega^2}\right]$, and conditions (3.7) and (3.28) hold. Then, problem (3.38) has a unique positive solution.*

Theorem 3.25 (2) and Corollary 3.29 say, among other things, that, if the forcing term f is such that $\int_0^{\omega}[f(s)]_+\mathrm{d}s$ is "small enough", then problem (1.3), (1.2) has at least one positive solution. The next theorem confirms that hypotheses of such a kind cannot be omitted. More precisely, Theorem 3.32 below claims that, if f is such that $\int_0^{\omega}[f(s)]_+\mathrm{d}s$ is "large enough", then problem (1.3), (1.2) has no positive solution.

Theorem 3.32. *Let $\lambda > 1$, $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, condition (3.28) hold, $f(t) \not\equiv 0$, and*

$$\int_0^{\omega}[f(s)]_+\mathrm{d}s - \Gamma(p)\int_0^{\omega}[f(s)]_-\mathrm{d}s \geq \frac{\lambda-1}{\lambda}\,\frac{\left|\Gamma(p)\int_0^{\omega}[p(s)]_-\mathrm{d}s - \int_0^{\omega}[p(s)]_+\mathrm{d}s\right|^{\frac{\lambda}{\lambda-1}}}{\left[\lambda\int_0^{\omega}h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}. \tag{3.39}$$

where Γ is given by (2.3). Then, problem (1.3), (1.2) has no non-negative solution.

If the forcing term f is non-negative, then the conclusions of Corollary 3.29 (2) and Theorem 3.32 can be extended as follows.

Theorem 3.33. *Let $\lambda > 1$ and conditions (3.15) and (3.28) be fulfilled. Then, the following conclusions hold:*

(1) Assume that $p \in \mathcal{V}^+(\omega)$ and

$$\int_0^{\omega} f(s)\mathrm{d}s < \frac{\lambda-1}{\lambda}\,\frac{[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\left[\lambda\int_0^{\omega}h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}, \tag{3.40}$$

where Δ is defined in Remark 2.5. Then, problem (1.3), (1.2) possesses exactly three solutions u_1, u_2, u_3 and these solutions satisfy

$$u_1(t) > u_2(t) > 0, \quad u_3(t) < 0 \quad \textit{for } t \in [0,\omega]. \tag{3.41}$$

(2) Assume that $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ and

$$\int_0^{\omega} f(s)\mathrm{d}s \geq \frac{\lambda-1}{\lambda}\,\frac{\left[\Gamma(p)\int_0^{\omega}[p(s)]_-\mathrm{d}s - \int_0^{\omega}[p(s)]_+\mathrm{d}s\right]^{\frac{\lambda}{\lambda-1}}}{\left[\lambda\int_0^{\omega}h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}, \tag{3.42}$$

where Γ is given by (2.3). Then, problem (1.3), (1.2) has a unique solution u_0 and this solution is negative.

Remark 3.34. If $\omega > 0$ and $p(t) := -a$ for $t \in [0,\omega]$, with $a \in \left]0, \frac{\pi^2}{\omega^2}\right[$, then $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ (see Remark 2.4) and, for any $h, f \in L([0,\omega])$ satisfying (3.15) and (3.28), conditions (3.40) and (3.42) are satisfied provided that

$$\int_0^{\omega} f(s)\mathrm{d}s < \frac{\lambda-1}{\lambda}\left[2\sqrt{a}\sin\frac{\omega\sqrt{a}}{2}\right]^{\frac{\lambda}{\lambda-1}}\frac{1}{\left[\lambda\int_0^{\omega}h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}$$

and

$$\int_0^\omega f(s)\mathrm{d}s \geq \frac{\lambda-1}{\lambda}\left[\frac{\omega a}{\cos\frac{\omega\sqrt{a}}{2}}\right]^{\frac{\lambda}{\lambda-1}} \frac{1}{\left[\lambda\int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}} \tag{3.43}$$

(see Remark 2.6). If, moreover, $h(t) := b$ for $t \in [0,\omega]$, with $b > 0$, then (3.43) reads as

$$\frac{1}{\omega}\int_0^\omega f(s)\mathrm{d}s \geq \left[\frac{1}{\cos\frac{\omega\sqrt{a}}{2}}\right]^{\frac{\lambda}{\lambda-1}} \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}};$$

compare this condition with those in Proposition 1.1 (4).

Remark 3.35. Theorem 3.33 extends the conclusions of Proposition 1.2 for the non-autonomous Duffing equation (1.3). Indeed, let $\omega > 0$ and the functions p, h be defined by (3.24), where $0 < a \leq \frac{\pi^2}{\omega^2}$ and $b = 1$. Then, $p \in \mathcal{V}^+(\omega)$ (see Remark 2.4) and the function h satisfies (3.28). As opposed to Proposition 1.2, where point conditions on the forcing term d are obtained, Theorem 3.33 (1) provides the integral-type conditions. This confirms conjecture (1) formulated by authors of [4] on p. 3930 – the graph of a forcing term may cross the line $y = \frac{2a}{3}\sqrt{\frac{a}{3}}$ mentioned therein.

4 Auxiliary statements

We first recall some results stated in [6,8].

Lemma 4.1 ([6, Proposition 10.8, Remark 0.7])**.** *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, then either $\int_0^\omega p(s)\mathrm{d}s > 0$ or $p(t) \equiv 0$.*

Lemma 4.2. *Let $g \in \mathcal{V}^+(\omega)$. Then, for any non-negative function $\ell \in L([0,\omega])$, the problem*

$$u'' = g(t)u + \ell(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{4.1}$$

has a unique solution u and this solution satisfies

$$0 \leq u(t) \leq \Delta(g)\int_0^\omega \ell(s)\mathrm{d}s \quad \text{for } t \in [0,\omega],$$

where Δ is defined in Remark 2.5. Moreover, if $\ell(t) \not\equiv 0$, then the solution u is positive.

Proof. The conclusions of the lemma follow from Definition 2.2, Remark 2.5, and [6, Remark 9.2]. ☐

Lemma 4.3 ([6, Theorem 16.4])**.** *Let $g \in \operatorname{Int}\mathcal{V}^+(\omega)$ and $\ell \in L([0,\omega])$ be such that $\ell(t) \not\equiv 0$ and*

$$\int_0^\omega [\ell(s)]_+\mathrm{d}s \geq \Gamma(g)\int_0^\omega [\ell(s)]_-\mathrm{d}s,$$

where Γ is given by (2.3). Then,

$$\Gamma(g)\int_0^\omega [g(s)]_-\mathrm{d}s > \int_0^\omega [g(s)]_+\mathrm{d}s$$

and problem (4.1) has a unique solution u, which satisfies

$$u(t) > \nu\left(\int_0^\omega [\ell(s)]_+\mathrm{d}s - \Gamma(g)\int_0^\omega [\ell(s)]_-\mathrm{d}s\right) \quad \text{for } t \in [0,\omega],$$

where

$$\nu := \left(\Gamma(g)\int_0^\omega [g(s)]_-\mathrm{d}s - \int_0^\omega [g(s)]_+\mathrm{d}s\right)^{-1}.$$

Lemma 4.4 ([6, Theorem 16.2])**.** *Let $g \in \mathcal{V}^-(\omega)$. Then, there exists $\nu_0 > 0$ such that, for any non-positive function $\ell \in L([0,\omega])$, problem* (4.1) *has a unique solution u and this solution satisfies*

$$u(t) \geq \nu_0 \int_0^\omega |\ell(s)| \mathrm{d}s \quad \text{for } t \in [0,\omega].$$

Lemma 4.5 ([6, Proposition 10.2])**.** *The set $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ is closed in $L([0,\omega])$.*

Definition 4.6 ([6, Definition 0.4])**.** We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}(\omega)$ if the problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \ u(b) = 0$$

has no non-trivial solution for any $a, b \in \mathbb{R}$ satisfying $0 < b - a < \omega$, where $\widetilde{p}$ is the ω-periodic extension of p to the whole real axis.

Lemma 4.7. *$\mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)$ and $\operatorname{Int} \mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \operatorname{Int} \mathcal{V}^+(\omega)$.*

Proof. It follows from Propositions 2.1, 10.5, and 10.6 stated in [6]. □

Lemma 4.8 ([6, Proposition 2.5])**.** *Let $g \colon \mathbb{R} \to \mathbb{R}$ be an ω-periodic function such that $g \in \mathcal{D}(\omega)$. Then, for any $a, b \in \mathbb{R}$ and $w \in AC^1([a,b])$ satisfying $0 < b - a < \omega$ and*

$$w''(t) \geq g(t)w(t) \quad \text{for a. e. } t \in [a,b], \quad w(a) \leq 0, \quad w(b) \leq 0,$$

the inequality $w(t) \leq 0$ holds for $t \in [a,b]$.

Lemma 4.9 ([8, Lemma 3.10])**.** *Let $p \in \mathcal{D}(\omega)$ and $\ell \in L([0,\omega])$ be such that*

$$\ell(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad \ell(t) \not\equiv 0. \tag{4.2}$$

Then, $p + \ell \in \operatorname{Int} \mathcal{D}(\omega)$.

Lemma 4.10 ([6, Lemma 2.7])**.** *Let $g \in \mathcal{D}(\omega)$, $\ell \in L([0,\omega])$ be a function satisfying* (4.2)*, and u be a solution to problem* (4.1)*. Then, the function u is either positive or negative.*

Lemma 4.11. *Let $\ell \colon [0,\omega] \times [\lambda_1, \lambda_2] \to \mathbb{R}$ be a Carathéodory function such that*

$$\ell(\cdot, \lambda_1) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega), \qquad \ell(\cdot, \lambda_2) \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega). \tag{4.3}$$

Then, there exists $r \in]\lambda_1, \lambda_2[$ such that $\ell(\cdot, r) \in \operatorname{Int} \mathcal{V}^+(\omega)$.

Proof. Let

$$A := \big\{\lambda \in [\lambda_1, \lambda_2] : \ell(\cdot, x) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \text{ for } x \in [\lambda_1, \lambda]\big\}. \tag{4.4}$$

In view of (4.3), it is clear that $A \neq \emptyset$. Put

$$\lambda^* := \sup A. \tag{4.5}$$

Since the set $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ is closed (see Lemma 4.5), it follows from (4.4) and (4.5) that $\lambda^* > \lambda_1$ and

$$\ell(\cdot, x) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \text{for } x \in [\lambda_1, \lambda^*[\,. \tag{4.6}$$

We first show that

$$\ell(\cdot, \lambda^*) \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega). \tag{4.7}$$

Indeed, suppose on the contrary that $\ell(\cdot,\lambda^*) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Then, hypothesis (4.3) yields $\lambda^* < \lambda_2$. Since the set $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ is closed (see Lemma 4.5), there exists $\varepsilon > 0$ such that $\ell(\cdot,x) \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ for $x \in [\lambda^* - \varepsilon, \lambda^* + \varepsilon]$. However, this condition, together with (4.4) and (4.6), implies that $\lambda^* + \varepsilon \in A$, which contradicts (4.5).

Now, in view of (4.7), it follows from Lemma 4.7 that $\ell(\cdot,\lambda^*) \in \operatorname{Int}\mathcal{D}(\omega)$. Therefore, there exists $\eta \in]0, \lambda^* - \lambda_1[$ such that $\ell(\cdot,\lambda^* - \eta) \in \operatorname{Int}\mathcal{D}(\omega)$. By Lemma 4.7, we get $\ell(\cdot,\lambda^* - \eta) \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \operatorname{Int}\mathcal{V}^+(\omega)$ and, thus, condition (4.6) yields $\ell(\cdot,\lambda^* - \eta) \in \operatorname{Int}\mathcal{V}^+(\omega)$. Consequently, the conclusion of the lemma holds with $r := \lambda^* - \eta$. □

Lemma 4.12 ([6, Remark 8.5]). *Let $p \in \mathcal{V}^-(\omega)$. Then, for any $g \in L([0,\omega])$ satisfying $g(t) \geq p(t)$ for a. e. $t \in [0,\omega]$, the inclusion $g \in \mathcal{V}^-(\omega)$ holds.*

Lemma 4.13 ([6, Remark 8.4]). *Let $p \in \mathcal{V}_0(\omega)$. Then, for any $g \in L([0,\omega])$ satisfying $g(t) \geq p(t)$ for a. e. $t \in [0,\omega]$ and $g(t) \not\equiv p(t)$, the inclusion $g \in \mathcal{V}^-(\omega)$ holds.*

Lemma 4.14 ([8, Proposition 3.16]). *Let $g \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ and $\ell\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying $\ell(t,0) \equiv 0$. Then, for any $c > 0$, there exists a function $\alpha \in AC^1([0,\omega])$ such that (3.23) holds and*

$$\alpha''(t) \geq g(t)\alpha(t) + \ell(t,\alpha(t))\alpha(t) \quad \text{for a. e. } t \in [0,\omega],$$
$$0 < \alpha(t) \leq c \quad \text{for } t \in [0,\omega].$$

Lemma 4.15. *Let $p \in L([0,\omega])$, $x_0 \geq 0$, and $q_0\colon [0,\omega] \times [x_0,+\infty[\to \mathbb{R}$ be a Carathéodory function such that*

$$\text{the function } q_0(t,\cdot)\colon [x_0,+\infty[\to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega] \tag{4.8}$$

and (3.4) holds. Then, there exists $K > x_0$ such that $p + q_0(\cdot,x) \in \mathcal{V}^-(\omega)$ for $x \geq K$.

Proof. It follows from [8, Proposition 3.13] with

$$f(t,x) := \begin{cases} q_0(t,x) & \text{if } x > x_0, \\ q_0(t,x_0) & \text{if } x_0 \geq x > 0. \end{cases} \tag{4.9}$$

□

Lemma 4.16. *Let $p \in \mathcal{V}^+(\omega)$, $x_0 \geq 0$, $q_0\colon [0,\omega] \times [x_0,+\infty[\to \mathbb{R}$ be a Carathéodory function satisfying (3.8) and (4.8), and there exist $x_1 > x_0$ such that (3.9) holds. Then, there exists $K > x_0$ such that $p + q_0(\cdot,x) \in \mathcal{V}^-(\omega)$ for $x \geq K$.*

Proof. It follows from [8, Proposition 3.14] with f given by (4.9). □

Now we recall a classical results concerning the solvability of the periodic problem

$$u'' = g(t,u); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{4.10}$$

where $g\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (see, e. g., [3]).

Lemma 4.17. *Let there exist functions $\alpha \in AC_\ell([a,b])$ and $\beta \in AC_u([a,b])$ satisfying*

$$\alpha(t) \leq \beta(t) \quad \text{for } t \in [a,b], \tag{4.11}$$
$$\alpha''(t) \geq g(t,\alpha(t)) \quad \text{for a. e. } t \in [a,b], \quad \alpha(0) = \alpha(\omega), \quad \alpha'(0) \geq \alpha'(\omega), \tag{4.12}$$
$$\beta''(t) \leq g(t,\beta(t)) \quad \text{for a. e. } t \in [a,b], \quad \beta(0) = \beta(\omega), \quad \beta'(0) \leq \beta'(\omega). \tag{4.13}$$

Then, problem (4.10) has at least one solution u such that

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \text{for } t \in [a,b]. \tag{4.14}$$

The next existence result is also known.

Lemma 4.18 ([7, Theorem 1.1 and Remark 1.2]). *Let there exist $p_0 \in \operatorname{Int}\mathcal{D}(\omega)$ and a Carathéodory function $z\colon [0,\omega] \times [0,+\infty[\to [0,+\infty[$ such that*

$$g(t,x)\operatorname{sgn} x \geq p_0(t)|x| - z(t,|x|) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}$$

and

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^\omega z(s,x)\mathrm{d}s = 0.$$

Let, moreover, there exist functions $\alpha \in AC_\ell([0,\omega])$ and $\beta \in AC_u([0,\omega])$ satisfying (4.12) and (4.13). Then, problem (4.10) has a solution u such that

$$\min\{\alpha(t_u),\beta(t_u)\} \leq u(t_u) \leq \max\{\alpha(t_u),\beta(t_u)\} \quad \text{for some } t_u \in [0,\omega].$$

The following three propositions concern the existence of the functions α, β appearing in Lemmas 4.17 and 4.18 (with $g(t,x) := p(t)x + q(t,x)x + f(t)$), which are usually referred to as lower and upper functions of problem (1.1), (1.2).

Proposition 4.19. *Let $p, f \in L([0,\omega])$, $q\colon [0,\omega]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function, and there exist $r_0 > 0$ such that $p + q^*(\cdot,r_0) \in \mathcal{V}^+(\omega)$ and*

$$0 < \int_0^\omega [f(s)]_+\mathrm{d}s \leq \frac{r_0}{\Delta\big(p+q^*(\cdot,r_0)\big)}, \tag{4.15}$$

where Δ is defined in Remark 2.5 and q^* is given by (3.6). Then, there exists a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and

$$0 < \alpha(t) \leq r_0 \quad \text{for } t \in [0,\omega]. \tag{4.16}$$

Moreover, if both inequalities in (4.15) are strict, then there exists $A > 0$ such that

$$\alpha''(t) \geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) + A \quad \text{for a. e. } t \in [0,\omega]. \tag{4.17}$$

Proof. Hypothesis (4.15) implies that there exists $\varepsilon \geq 0$ such that

$$0 < \int_0^\omega [f(s)]_+\mathrm{d}s \leq \frac{r_0-\varepsilon}{\Delta\big(p+q^*(\cdot,r_0)\big)} \tag{4.18}$$

and $\varepsilon > 0$ if both inequalities in (4.15) are strict.

Since we assume that $p + q^*(\cdot,r_0) \in \mathcal{V}^+(\omega)$ and $[f(t)]_+ \not\equiv 0$, it follows from Lemma 4.2 (with $g(t) := p(t)+q^*(t,r_0)$ and $\ell(t) := [f(t)]_+ + \frac{\varepsilon}{\omega\Delta(p+q^*(\cdot,r_0))}$) that the problem

$$\alpha'' = \big(p(t)+q^*(t,r_0)\big)\alpha + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta\big(p+q^*(\cdot,r_0)\big)}; \qquad \alpha(0)=\alpha(\omega), \quad \alpha'(0)=\alpha'(\omega)$$

has a unique solution α and this solution satisfies

$$0 < \alpha(t) \leq \varepsilon + \Delta\big(p+q^*(\cdot,r_0)\big)\int_0^\omega [f(s)]_+\mathrm{d}s \quad \text{for } t \in [0,\omega].$$

Therefore, in view of (4.18), conditions (3.23) and (4.16) hold. Moreover, (3.6) implies that

$$\text{the function } q^*(t,\cdot)\colon [0,+\infty[\to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega] \tag{4.19}$$

and, thus, the function α satisfies

$$\begin{aligned}\alpha''(t) &\geq p(t)\alpha(t) + q^*(t,\alpha(t))\alpha(t) + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta\big(p+q^*(\cdot,r_0)\big)}\\ &\geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) + \frac{\varepsilon}{\omega\Delta\big(p+q^*(\cdot,r_0)\big)}\\ &\geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) \quad \text{for a. e. } t\in[0,\omega],\end{aligned}$$

i. e., (3.2) holds. Furthermore, if both inequalities in (4.15) are strict, then $\varepsilon > 0$ and, therefore, condition (4.17) is fulfilled with $A := \frac{\varepsilon}{\omega\Delta\left(p+q^*(\cdot,r_0)\right)}$. □

Proposition 4.20. *Let $p \in \mathcal{V}^+(\omega)$, $f \in L([0,\omega])$, and $q\colon [0,\omega]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Let, moreover, $[f(t)]_+ \not\equiv 0$ and there exist $r_0 > 0$ such that*

$$\Delta(p) \leq \frac{r_0}{\int_0^\omega [f(s)]_+ \mathrm{d}s + r_0\int_0^\omega q^*(s,r_0)\mathrm{d}s}, \tag{4.20}$$

where Δ is defined in Remark 2.5 and q^* is given by (3.6). Then, there exists a function $\alpha \in AC^1([0,\omega])$ satisfying(3.2), (3.23), (4.16), and

$$\alpha''(t) \geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + [f(t)]_+ \quad \textit{for a. e. } t\in[0,\omega]. \tag{4.21}$$

Moreover, if inequality (4.20) is strict, then there exists $A > 0$ such that α satisfies (4.17).

Proof. Hypothesis (4.20) implies that there exists $\varepsilon \geq 0$ such that

$$\Delta(p) \leq \frac{r_0-\varepsilon}{\int_0^\omega [f(s)]_+ \mathrm{d}s + r_0\int_0^\omega q^*(s,r_0)\mathrm{d}s} \tag{4.22}$$

and $\varepsilon > 0$ if inequality (4.20) is strict.

It follows from Lemma 4.2 (with $g(t) := p(t)$ and $\ell(t) := r_0 q^*(t,r_0) + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta(p)}$) that the problem

$$\alpha'' = p(t)\alpha + r_0 q^*(t,r_0) + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta(p)}; \qquad \alpha(0)=\alpha(\omega), \quad \alpha'(0)=\alpha'(\omega)$$

has a unique solution α and this solution satisfies

$$0 < \alpha(t) \leq \varepsilon + \Delta(p)\left(r_0\int_0^\omega q^*(s,r_0)\mathrm{d}s + \int_0^\omega [f(s)]_+\mathrm{d}s\right) \quad \text{for } t\in[0,\omega].$$

Therefore, in view of (4.22), conditions (3.23) and (4.16) hold. Moreover, (3.6) yields (4.19) and, thus, the function α satisfies

$$\begin{aligned}\alpha''(t) &\geq p(t)\alpha(t) + q^*(t,\alpha(t))\alpha(t) + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta(p)}\\ &\geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + [f(t)]_+ + \frac{\varepsilon}{\omega\Delta(p)}\\ &\geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) + \frac{\varepsilon}{\omega\Delta(p)}\\ &\geq p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) + f(t) \quad \text{for a. e. } t\in[0,\omega],\end{aligned}$$

i. e., (3.2) and (4.21) hold. Furthermore, if inequality (4.20) is strict, then $\varepsilon > 0$ and, therefore, condition (4.17) is fulfilled with $A := \frac{\varepsilon}{\omega\Delta(p)}$. □

Proposition 4.21. *Let $p, f \in L(0,\omega)$, $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying hypothesis* (H_1)*, and there exist $R > x_0$ such that $p + q_0(\cdot, R) \in \mathcal{V}^-(\omega)$. Then, for any $c > 0$, there exist $B > 0$ and a function $\beta \in AC^1([0,\omega])$ such that*

$$\beta''(t) \leq p(t)\beta(t) + q(t,\beta(t))\beta(t) + f(t) - B \quad \textit{for a. e. } t \in [0,\omega], \tag{4.23}$$

$$\beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega), \tag{4.24}$$

$$\beta(t) \geq c \quad \textit{for } t \in [0,\omega]. \tag{4.25}$$

Proof. Let $\nu_0 > 0$ be the number appearing in the conclusion of Lemma 4.4 (with $g(t) := p(t) + q_0(t,R)$) and let $c > 0$ be arbitrary. Then, it follows from Lemma 4.4 that the problem

$$\beta'' = \big(p(t) + q_0(t,R)\big)\beta - [f(t)]_- - \frac{\max\{c,R\}}{\nu_0\omega}; \qquad \beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega)$$

has a unique solution β and this solution satisfies

$$\beta(t) \geq \max\{c,R\} \quad \text{for } t \in [0,\omega].$$

Obviously, (4.24) and (4.25) hold. Since $\beta(t) \geq R > x_0$ for $t \in [0,\omega]$, by hypothesis (H_1), we get

$$\begin{aligned}\beta''(t) &\leq p(t)\beta(t) + q_0(t,\beta(t))\beta(t) - [f(t)]_- - \frac{\max\{c,R\}}{\nu_0\omega} \\ &\leq p(t)\beta(t) + q(t,\beta(t))\beta(t) + f(t) - \frac{\max\{c,R\}}{\nu_0\omega} \quad \text{for a. e. } t \in [0,\omega],\end{aligned}$$

i. e., (4.23) is fulfilled with $B := \frac{\max\{c,R\}}{\nu_0\omega}$. □

The following lemma concerning problem (1.3), (1.2) we use in the proof of Theorem 3.15.

Lemma 4.22. *Let u_1, u_2 be solutions to problem* (1.3), (1.2) *such that*

$$u_2(t) \geq u_1(t) \quad \textit{for } t \in [0,\omega], \qquad u_2(t) \not\equiv u_1(t). \tag{4.26}$$

Then,

$$u_2(t) > u_1(t) \quad \textit{for } t \in [0,\omega]. \tag{4.27}$$

Proof. Suppose on the contrary that (4.27) does not hold. Then, there exists $t_0 \in [0,\omega]$ such that

$$u_2(t_0) = u_1(t_0). \tag{4.28}$$

Extend the functions p, h, f, u_1, u_2 periodically to the whole real axis denoting them by the same symbols. Then, in view of (4.26) and (4.28), we get

$$u_2'(t_0) = u_1'(t_0). \tag{4.29}$$

Since the function $x \mapsto |x|^\lambda \operatorname{sgn} x$ is Lipschitz on every compact interval, for any $c_1, c_2 \in \mathbb{R}$, the Cauchy problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + f(t); \qquad u(t_0) = c_1, \quad u'(t_0) = c_2$$

is uniquely solvable. Therefore, (4.28) and (4.29) yield $u_2(t) \equiv u_1(t)$, which contradicts (4.26). □

We finally provide a technical lemma, which we use in the proof of Theorem 3.32.

Lemma 4.23. *Let $\varrho \geq 1$ and $f, g \in L([0,\omega])$ be such that*

$$g(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega]. \tag{4.30}$$

Then,

$$\begin{aligned} &\int_0^\omega [g(s)+f(s)]_+ \mathrm{d}s - \varrho \int_0^\omega [g(s)+f(s)]_- \mathrm{d}s \\ &\qquad \geq \int_0^\omega g(s)\mathrm{d}s + \int_0^\omega [f(s)]_+ \mathrm{d}s - \varrho \int_0^\omega [f(s)]_- \mathrm{d}s. \end{aligned} \tag{4.31}$$

Proof. Put

$$A^+ := \{t \in [0,\omega] : g(t)+f(t) \geq 0\}, \qquad A^- := \{t \in [0,\omega] : g(t)+f(t) < 0\}.$$

Then, by (4.30) and the hypothesis $\varrho \geq 1$, we get

$$\begin{aligned} \int_0^\omega [g(s)+f(s)]_+ \mathrm{d}s &= \int_{A^+} g(s)\mathrm{d}s + \int_{A^+} [f(s)]_+ \mathrm{d}s - \int_{A^+} [f(s)]_- \mathrm{d}s \\ &\geq \int_{A^+} g(s)\mathrm{d}s + \int_{A^+} [f(s)]_+ \mathrm{d}s - \varrho \int_{A^+} [f(s)]_- \mathrm{d}s \end{aligned}$$

and

$$\begin{aligned} \varrho \int_0^\omega [g(s)+f(s)]_- \mathrm{d}s &= \varrho \left(- \int_{A^-} g(s)\mathrm{d}s - \int_{A^-} [f(s)]_+ \mathrm{d}s + \int_{A^-} [f(s)]_- \mathrm{d}s \right) \\ &\leq - \int_{A^-} g(s)\mathrm{d}s - \int_{A^-} [f(s)]_+ \mathrm{d}s + \varrho \int_{A^-} [f(s)]_- \mathrm{d}s, \end{aligned}$$

which yields (4.31). □

5 Proofs of main results

Proof of Theorem 3.1. Let $\alpha \in AC_\ell([0,\omega])$ be a positive function such that (3.1) and (3.2) hold. It follows from Proposition 4.21 that there exists a function $\beta \in AC^1([0,\omega])$ satisfying (4.11), (4.24), and

$$\beta''(t) \leq p(t)\beta(t) + q(t,\beta(t))\beta(t) + f(t) \quad \text{for a. e. } t \in [0,\omega]. \tag{5.1}$$

Consequently, all the hypotheses of Lemma 4.17 (with $g(t,x) := p(t)x + q(t,x)x + f(t)$) are fulfilled and, thus, problem (1.1), (1.2) has a positive solution u such that (4.14) holds. □

Proof of Corollary 3.2. By Lemma 4.15 (with $p(t) + [q(t,0)]_+$ and $q_0(t,x) - [q(t,0)]_+$ instead of $p(t)$ and $q_0(t,x)$), there exists $R > x_0$ such that the inclusion $p + q_0(\cdot, R) \in \mathcal{V}^-(\omega)$ holds. It follows from (3.5) that there exists $r_0 > 0$ such that $p + q^*(\cdot, r_0) \in \mathcal{V}^+(\omega)$ and

$$\int_0^\omega [f(s)]_+ \mathrm{d}s < \frac{r_0}{\Delta\big(p + q^*(\cdot, r_0)\big)}.$$

If $[f(t)]_+ \not\equiv 0$, then, by Proposition 4.19, there exists a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and (4.16).

Assume that $[f(t)]_+ \equiv 0$. In view of (3.6) and the hypothesis $p + [q(\cdot,0)]_+ \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, it follows from Lemma 4.14 (with $g(t) := p(t) + [q(t,0)]_+$, $\ell(t,x) := q^*(t,|x|) -$

$[q(t,0)]_+$, and $c := r_0$) that there exists a function $\alpha \in AC^1([0,\omega])$ satisfying (3.23), (4.16), and

$$\alpha''(t) \geq \big(p(t) + [q(t,0)]_+\big)\alpha(t) + \big(q^*(t,|\alpha(t)|) - [q(t,0)]_+\big)\alpha(t) \quad \text{for a. e. } t \in [0,\omega].$$

Since α is positive and $f(t) \leq 0$ for a. e. $t \in [0,\omega]$, the latter inequality yields (3.2).

Consequently, the conclusion of the corollary follows from Theorem 3.1.

We finally prove the assertion stated in Remark 3.4. Assume that a supremum on the right-hand side of (3.5) is achieved at some $r_0 > 0$ and

$$\int_0^\omega [f(s)]_+ \mathrm{d}s = \frac{r_0}{\Delta\big(p + q^*(\cdot, r_0)\big)}. \tag{5.2}$$

Then, by Proposition 4.19, there exist a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution. □

Proof of Corollary 3.7. By Lemma 4.16, there exists $R > x_0$ such that the inclusion $p + q_0(\cdot, R) \in \mathcal{V}^-(\omega)$ holds.

First assume that (3.5) is fulfilled, where Δ is defined in Remark 2.5 and q^* is given by (3.6). In much the same way as in the proof of Corollary 3.2, we show that there exists a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and (4.16).

Now assume that $[f(t)]_+ \not\equiv 0$ and (3.10) holds, where Δ is defined in Remark 2.5 and q^* is given by (3.6). It follows from (3.10) that there exists $r_0 > 0$ such that

$$\Delta(p) < \frac{r_0}{\int_0^\omega [f(s)]_+ \mathrm{d}s + r_0 \int_0^\omega q^*(s, r_0)\mathrm{d}s}$$

and, thus, Proposition 4.20 guarantees the existence of a function $\alpha \in AC^1([0,\omega])$ such that (3.2), (3.23), and (4.16) hold.

Consequently, in both cases (3.5) and (3.10), the conclusion of the corollary follows from Theorem 3.1.

We finally prove the assertions stated in Remarks 3.4 and 3.8. First assume that a supremum on the right-hand side of (3.5) is achieved at some $r_0 > 0$ and (5.2) holds. Then, by Proposition 4.19, there exist a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution.

Now assume that $[f(t)]_+ \not\equiv 0$, a supremum on the right-hand side of (3.10) is achieved at some $r_0 > 0$, and

$$\Delta(p) = \frac{r_0}{\int_0^\omega [f(s)]_+ \mathrm{d}s + r_0 \int_0^\omega q^*(s, r_0)\mathrm{d}s}.$$

Then, by Proposition 4.20, there exist a function $\alpha \in AC^1([0,\omega])$ satisfying (3.2), (3.23), and (4.16). Therefore, it follows from Theorem 3.1 that problem (1.1), (1.2) has at least one positive solution. □

Proof of Theorem 3.9. It follows from hypothesis (H_2) that

$$\text{the function } q(t,\cdot)\colon [0,+\infty[\, \to \mathbb{R} \text{ is non-decreasing for a. e. } t \in [0,\omega]. \tag{5.3}$$

Suppose on the contrary that u, w are positive solutions to problem (1.1), (1.2) satisfying

$$\max\big\{u(t) - w(t) : t \in [0,\omega]\big\} > 0.$$

Put

$$\beta_0(t) := \min\{u(t), w(t)\} \quad \text{for } t \in [0, \omega].$$

It is not difficult to verify that $\beta_0 \in AC_u([0,\omega])$,

$$\beta_0''(t) = p(t)\beta_0(t) + q(t,\beta_0(t))\beta_0(t) + f(t) \quad \text{for a.e. } t \in [0,\omega], \tag{5.4}$$

$$\beta_0(0) = \beta_0(\omega), \quad \beta_0'(0) \le \beta_0'(\omega), \tag{5.5}$$

$$\beta_0(t) \le u(t) \quad \text{for } t \in [0,\omega], \quad \beta_0(t) \not\equiv u(t). \tag{5.6}$$

By Lemma 4.14 (with $g(t) := p(t)$ and $\ell(t,x) := q(t,x)$), there exists a function $\alpha \in AC^1([0,\omega])$ such that (3.23) holds,

$$\begin{gathered}\alpha''(t) \ge p(t)\alpha(t) + q(t,\alpha(t))\alpha(t) \quad \text{for a.e. } t \in [0,\omega],\\ \alpha(t) \le \beta_0(t) \quad \text{for } t \in [0,\omega].\end{gathered} \tag{5.7}$$

In view of hypothesis (3.7), it is clear that the function α satisfies also (3.2). Therefore, by virtue of (3.2), (3.23), (5.4), (5.5), and (5.7), all the hypotheses of Lemma 4.17 (with $g(t,x) := p(t)x + q(t,x)x + f(t)$ and $\beta(t) := \beta_0(t)$) that there exists a solution v to problem (1.1), (1.2) such that

$$\alpha(t) \le v(t) \le \beta_0(t) \quad \text{for } t \in [0,\omega].$$

However, the latter condition and (5.6) imply that there exist $t_1, t_2 \in [0,\omega]$ such that $t_1 < t_2$ and

$$u(t) \ge v(t) > 0 \quad \text{for } t \in [0,\omega], \quad u(t) > v(t) \quad \text{for } t \in [t_1,t_2]. \tag{5.8}$$

Consequently, there exist $v_*, v^*, e_0 > 0$ such that

$$u(t) \ge v(t) + e_0, \quad v^* \ge v(t) \ge v_* \quad \text{for } t \in [t_1,t_2] \tag{5.9}$$

and, thus, in view of (5.3), (5.8), (5.9), and (H_2), we get

$$q(t,u(t)) \ge q(t,v(t)) \quad \text{for a.e. } t \in [0,\omega] \tag{5.10}$$

and

$$q(t,u(t)) - q(t,v(t)) \ge q(t,v(t)+e_0) - q(t,v(t)) \ge h_{v_*v^*e_0}(t) \tag{5.11}$$

for a.e. $t \in [t_1,t_2]$. It follows immediately from (1.1) that u and v are solutions to the equations

$$\begin{aligned} z'' &= \left(p(t) + q(t,v(t)) + \frac{f(t)}{v(t)}\right) z + [q(t,u(t)) - q(t,v(t))]u(t) - \frac{f(t)}{v(t)}[u(t)-v(t)],\\ z'' &= \left(p(t) + q(t,v(t)) + \frac{f(t)}{v(t)}\right) z, \end{aligned} \tag{5.12}$$

respectively. Therefore, by virtue of (3.7), (5.8), (5.10), and (5.11), third Fredholm's theorem yields the contradiction

$$\begin{aligned} 0 &= \int_0^\omega \left([q(t,u(t)) - q(t,v(t))]u(t) - \frac{f(t)}{v(t)}[u(t)-v(t)] \right) v(t)\mathrm{d}t \\ &\ge \int_{t_1}^{t_2} [q(t,u(t)) - q(t,v(t))]u(t)v(t)\mathrm{d}t \ge v_*^2 \int_{t_1}^{t_2} h_{v_*v^*e_0}(t)\mathrm{d}t > 0. \end{aligned}$$

□

Proof of Proposition 3.11. Suppose on the contrary that u_1, u_2, u_3 are solutions to problem (1.1), (1.2) satisfying (3.11). It is clear that there exist $d_1 > c_1 > 0$, $d_2 > c_2 > 0$, and $d_3 > c_3 > 0$ such that

$$\begin{aligned} c_1 \leq u_1(t) \leq d_1 \quad & \text{for } t \in [0, \omega], \\ c_k \leq u_k(t) - u_{k-1}(t) \leq d_k \quad & \text{for } t \in [0, \omega],\ k = 2, 3. \end{aligned}$$

Put

$$\varphi_k(t) := \frac{q(t, u_{k+1}(t))u_{k+1}(t) - q(t, u_k(t))u_k(t)}{u_{k+1}(t) - u_k(t)} \quad \text{for a. e. } t \in [0, \omega],\ k = 1, 2.$$

It follows from hypothesis (H_3^ℓ) with ($x_1 := u_1(t)$, $x_2 := u_2(t)$, and $x_3 := u_3(t)$) that

$$(-1)^\ell \big[\varphi_2(t) - \varphi_1(t)\big] \geq h^*(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \quad h^*(t) \not\equiv 0. \tag{5.13}$$

Now let $z_k(t) := u_{k+1}(t) - u_k(t)$ for $t \in [0, \omega]$, $k = 1, 2$. Then, (1.1) yields

$$z_k''(t) = p(t)z_k(t) + q(t, u_{k+1}(t))u_{k+1}(t) - q(t, u_k(t))u_k(t) = \big(p(t) + \varphi_k(t)\big)z_k(t)$$

for a. e. $t \in [0, \omega]$, $k = 1, 2$, and, in view of (3.11), we get

$$z_1(t) > 0, \quad z_2(t) > 0 \quad \text{for } t \in [0, \omega].$$

Therefore, by Definition 2.3, we get

$$p + \varphi_1 \in \mathcal{V}_0(\omega), \qquad p + \varphi_2 \in \mathcal{V}_0(\omega). \tag{5.14}$$

On the other hand, (5.13) yields

$$p(t) + \varphi_\ell(t) \geq p(t) + \varphi_{3-\ell}(t) \quad \text{for a. e. } t \in [0, \omega]$$

and

$$p(t) + \varphi_\ell(t) \not\equiv p(t) + \varphi_{3-\ell}(t),$$

which, by virtue of Lemma 4.13, contradicts (5.14). □

Proof of Theorem 3.13. Conclusion (1): Assume that (H_2') and (3.12) hold and u, v are positive solutions to problem (1.1), (1.2) such that

$$\max\big\{u(t) - v(t) : t \in [0, \omega]\big\} > 0. \tag{5.15}$$

It follows from hypothesis (H_2') that (5.3) is fulfilled which, together with (3.12), yields

$$q(t, x) \geq 0 \quad \text{for a. e. } t \in [0, \omega] \text{ and all } x \geq 0. \tag{5.16}$$

Suppose on the contrary that (3.13) does not hold. Then, either

$$u(t) \geq v(t) \quad \text{for } t \in [0, \omega], \quad u(t) \not\equiv v(t), \tag{5.17}$$

$$\text{there exists } t_0 \in [0, \omega] \text{ such that } u(t_0) = v(t_0), \tag{5.18}$$

or

$$\min\big\{u(t) - v(t) : t \in [0, \omega]\big\} < 0. \tag{5.19}$$

First assume that (5.17) and (5.18) are satisfied. Then, in view of (5.16), condition (5.3) yields

$$q(t,u(t))u(t) \geq q(t,v(t))u(t) \geq q(t,v(t))v(t) \quad \text{for a. e. } t \in [0,\omega].$$

Put $z(t) := u(t) - v(t)$ for $t \in [0,\omega]$. The function z is a solution to the linear periodic problem

$$z'' = p(t)z + q(t,u(t))u(t) - q(t,v(t))v(t); \quad z(0) = z(\omega),\ z'(0) = z'(\omega).$$

If $q(t,u(t))u(t) \not\equiv q(t,v(t))v(t)$, then, in view of Lemma 4.2 (with $g(t) := p(t)$ and $\ell(t) := q(t,u(t))u(t) - q(t,v(t))v(t)$), we get $z(t) > 0$ for $t \in [0,\omega]$, which contradicts (5.18). On the other hand, if $q(t,u(t))u(t) \equiv q(t,v(t))v(t)$, then Lemma 4.2 (with $g(t) := p(t)$ and $\ell(t) := 0$) yields $z(t) \equiv 0$, which is in contradiction with (5.17).

Now assume that (5.19) holds. Extend the functions u, v, p, f, $q(\cdot,x)$ periodically to the whole real axis denoting them by the same symbols. Then, in view of (5.15) and (5.19), there exist $a,b \in \mathbb{R}$ such that $0 < b - a < \omega$ and

$$u(t) > v(t) \quad \text{for } t \in \,]a,b[\,, \quad u(a) = v(a), \quad u(b) = v(b). \tag{5.20}$$

Put $w(t) := u(t) - v(t)$ for $t \in [a,b]$. By virtue of (5.3), (5.16), and (5.20), it follows from (1.1) that

$$\begin{aligned} w''(t) &= p(t)w(t) + \big[q(t,u(t)) - q(t,v(t))\big]u(t) + q(t,v(t))\big[u(t) - v(t)\big] \\ &\geq p(t)w(t) \quad \text{for a. e. } t \in [a,b]. \end{aligned}$$

Since $w(a) = 0$ and $w(b) = 0$, by Lemma 4.7 and Lemma 4.8 (with $g(t) := p(t)$), we get $w(t) \leq 0$ for $t \in [a,b]$, which is in contradiction with (5.20).

Conclusion (2)*:* Assume that (3.7), (3.12), and (H_2') are fulfilled. It follows from hypothesis (H_2') that (5.3) holds.

Suppose on the contrary that u, v are positive solutions to problem (1.1), (1.2) satisfying (5.15). Then, the above-proved conclusion (1) yields

$$u(t) > v(t) \quad \text{for } t \in [0,\omega] \tag{5.21}$$

and, thus, there exist $v_*, v^*, e_0 > 0$ such that

$$u(t) \geq v(t) + e_0, \quad v^* \geq v(t) \geq v_* \quad \text{for } t \in [0,\omega].$$

Therefore, by using (5.3) and (H_2'), we get

$$q(t,u(t)) - q(t,v(t)) \geq q(t,v(t)+e_0) - q(t,v(t)) \geq h_{v_* v^* e_0}(t) \tag{5.22}$$

for a. e. $t \in [0,\omega]$. It follows immediately from (1.1) that u and v are solutions to equations (5.12) and, thus, by virtue of (3.7), (5.21), and (5.22), third Fredholm's theorem yields the contradiction

$$\begin{aligned} 0 &= \int_0^\omega \left(\big[q(t,u(t)) - q(t,v(t))\big]u(t) - \frac{f(t)}{v(t)}\big[u(t) - v(t)\big] \right) v(t)\mathrm{d}t \\ &\geq \int_0^\omega \big[q(t,u(t)) - q(t,v(t))\big]u(t)v(t)\mathrm{d}t \geq v_*^2 \int_0^\omega h_{v_* v^* e_0}(t)\mathrm{d}t > 0. \end{aligned}$$

Conclusion (3)*:* Assume that $\ell \in \{1,2\}$, condition (3.12) holds, and hypotheses (H_2') and (H_3^ℓ) are fulfilled.

Suppose on the contrary that u_1, u_2, u_3 are mutually distinct positive solutions to problem (1.1), (1.2). Then, the above-proved conclusion (1) implies that we can assume without loss of generality that u_1, u_2, u_3 satisfy (3.11), which is in contradiction with the conclusion of Proposition 3.11.

Conclusion (4)*:* Assume that (3.14) and (3.15) hold and let u be a solution to problem (1.1), (1.2). Then, u is a solution to the linear periodic problem

$$z'' = \big(p(t) + q(t, u(t))\big)z + f(t); \quad z(0) = z(\omega),\ z'(0) = z'(\omega).$$

By virtue of (3.14), Lemmas 4.7 and 4.9 yield $p + q(\cdot, u(\cdot)) \in \mathcal{D}(\omega)$. Since f satisfies (3.15), by Lemma 4.10 (with $g(t) := p(t) + q(t, u(t))$ and $\ell(t) := f(t)$), we conclude that the function u is either positive or negative. □

Proof of Theorem 3.15. Put

$$q(t, x) := h(t)|x|^{\lambda-1} \quad \text{for a.e. } t \in [0, \omega] \text{ and all } x \in \mathbb{R}. \tag{5.23}$$

In view of (3.16), it is clear that q is a Carathéodory function satisfying hypothesis (H_1) with $q_0(t, x) := h(t)x^{\lambda-1}$ and $x_0 := 0$. Moreover, $q(t, 0) \equiv 0$, condition (3.4) holds and hypothesis (H_2) is fulfilled. Furthermore, since the function $x \mapsto x^\lambda$ is strictly convex on $]0, +\infty[$, one can show that q satisfies also hypothesis (H_0^2).

Conclusion (1)*:* It follows immediately from Proposition 3.11.

Conclusion (2)*:* Let $\alpha \in AC_\ell([0, \omega])$ be a positive function satisfying (3.1) and (3.17). By Lemma 4.15, there exists $R > 0$ such that

$$p + q_0(\cdot, R) \in \mathcal{V}^-(\omega). \tag{5.24}$$

Consequently, all the hypotheses of Theorem 3.1 are fulfilled and, thus, problem (1.3), (1.2) has a positive solution u_0 such that

$$u_0(t) \geq \alpha(t) \quad \text{for } t \in [0, \omega]. \tag{5.25}$$

We now determine a solution u^* to problem (1.3), (1.2) satisfying (3.18) such that, for any solution u to problem (1.3), (1.2), condition (3.19) is fulfilled.

First assume that problem (1.3), (1.2) has a unique positive solution. Put $u^* := u_0$. In view of (5.25), it is clear that (3.18) holds. We show that every solution u to problem (1.3), (1.2) satisfies (3.19). Suppose on the contrary that u is a solution to (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^*(t)$ for $t \in [0, \omega]$ and $u(t) \not\equiv u^*(t)$, then $u(t) < u^*(t)$ for $t \in [0, \omega]$. Therefore, u satisfies

$$\max\big\{u(t) - u^*(t) : t \in [0, \omega]\big\} > 0. \tag{5.26}$$

Put

$$\alpha_0(t) := \max\big\{u(t), u^*(t)\big\} \quad \text{for } t \in [0, \omega].$$

It is not difficult to verify that $\alpha_0 \in AC_\ell([0, \omega])$,

$$\alpha_0''(t) = p(t)\alpha_0(t) + q(t, \alpha_0(t))\alpha_0(t) + f(t) \quad \text{for a.e. } t \in [0, \omega], \tag{5.27}$$

$$\alpha_0(0) = \alpha_0(\omega), \quad \alpha_0'(0) \geq \alpha_0'(\omega), \tag{5.28}$$

$$\alpha_0(t) \geq u^*(t) \quad \text{for } t \in [0, \omega], \quad \alpha_0(t) \not\equiv u^*(t). \tag{5.29}$$

In view of (5.24), Proposition 4.21 implies that there exists $\beta \in AC^1([0,\omega])$ satisfying (4.24), (5.1), and

$$\beta(t) \geq \alpha_0(t) \quad \text{for } t \in [0,\omega].$$

Therefore, by virtue of (4.24), (5.1), (5.27), and (5.28), it follows from Lemma 4.17 (with $g(t,x) := p(t)x + q(t,x)x + f(t)$ and $\alpha(t) := \alpha_0(t)$) that there exists a solution $\tilde{u}$ to problem (1.3), (1.2) such that

$$\alpha_0(t) \leq \tilde{u}(t) \leq \beta(t) \quad \text{for } t \in [0,\omega].$$

However, in view of (5.29), the latter condition yields

$$\tilde{u}(t) \geq u^*(t) \quad \text{for } t \in [0,\omega], \quad \tilde{u}(t) \not\equiv u^*(t).$$

Consequently, by Lemma 4.22, we get

$$\tilde{u}(t) > u^*(t) > 0 \quad \text{for } t \in [0,\omega], \tag{5.30}$$

which contradicts our assumption that problem (1.3), (1.2) has a unique positive solution.

Now assume that problem (1.3), (1.2) has at least two positive solutions. Then, there exists a positive solution v to problem (1.3), (1.2) different from u_0. We can assume without loss of generality that

$$\min\left\{v(t) - u_0(t) : t \in [0,\omega]\right\} < 0. \tag{5.31}$$

We first determine a positive solution u^* to problem (1.3), (1.2) satisfying (3.18) and

$$u^*(t) > v(t) \quad \text{for } t \in [0,\omega]. \tag{5.32}$$

It is clear that either

$$\max\left\{v(t) - u_0(t) : t \in [0,\omega]\right\} \leq 0 \tag{5.33}$$

or

$$\max\left\{v(t) - u_0(t) : t \in [0,\omega]\right\} > 0. \tag{5.34}$$

Let (5.33) hold. Then, $v(t) \leq u_0(t)$ for $t \in [0,\omega]$ and, in view of (5.31), Lemma 4.22 yields $v(t) < u_0(t)$ for $t \in [0,\omega]$. We put $u^* := u_0$ and, in view of (5.25), we conclude immediately that (3.18) and (5.32) are satisfied.

Let (5.34) hold. Put

$$\alpha_0(t) := \max\left\{v(t), u_0(t)\right\} \quad \text{for } t \in [0,\omega].$$

In much the same way as above, we determine a solution u^* to problem (1.3), (1.2) such that

$$u^*(t) \geq \max\left\{v(t), u_0(t)\right\} \quad \text{for } t \in [0,\omega].$$

By virtue of (5.25) and (5.31), the solution u^* satisfies (3.18) and

$$u^*(t) \geq v(t) \quad \text{for } t \in [0,\omega], \quad u^*(t) \not\equiv v(t).$$

Therefore, in view Lemma 4.22, (5.32) holds.

Hence, in both cases (5.33) and (5.34), we have determined a solution u^* to problem (1.3), (1.2) satisfying (3.18) and (5.32). Now we show that every solution u to (1.3), (1.2) satisfies (3.19). Suppose on the contrary that u is a solution to problem (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^*(t)$ for $t \in [0,\omega]$ and $u(t) \not\equiv u^*(t)$, then

$u(t) < u^*(t)$ for $t \in [0, \omega]$. Therefore, u satisfies (5.26). In much the same way as above, we determine a solution $\tilde{u}$ to problem (1.3), (1.2) such that (5.30) holds. Hence, conditions (5.30) and (5.32) yields

$$\tilde{u}(t) > u^*(t) > v(t) > 0 \quad \text{for } t \in [0, \omega],$$

which contradicts the above-proved conclusion (1).

It remains to show that, for any couple of distinct positive solutions u_1, u_2 to problem (1.3), (1.2) satisfying (3.20), conditions (3.21) hold. Assume that u_1, u_2 are distinct positive solutions to (1.3), (1.2) satisfying (3.20). We have proved above that

$$u_1(t) < u^*(t), \quad u_2(t) < u^*(t) \quad \text{for } t \in [0, \omega]. \tag{5.35}$$

Suppose on the contrary that (3.21) does not hold, i. e., there exists $k \in \{1, 2\}$ such that

$$u_k(t) \leq u_{3-k}(t) \quad \text{for } t \in [0, \omega], \quad u_k(t) \not\equiv u_{3-k}(t).$$

Then, Lemma 4.22, together with (5.35), yields

$$0 < u_k(t) < u_{3-k}(t) < u^*(t) \quad \text{for } t \in [0, \omega],$$

which contradicts the above-proved conclusion (1).

Conclusion (3)*:* Assume that (3.7) holds. Then, the existence and uniqueness of a positive solution to problem (1.3), (1.2) follows from Corollary 3.10. □

Proof of Corollary 3.16. Let the function q be defined by formula (5.23). In view of (3.16), it is clear that q is a Carathéodory function satisfying hypothesis (H_1) with $q_0(t, x) := h(t)x^{\lambda-1}$ and $x_0 := 0$. Moreover, $q(t, 0) \equiv 0$ and condition (3.4) holds. According to (3.22), inequality (3.5) is obviously satisfied, because we have $q^*(t, \varrho) = h(t)\varrho^{\lambda-1}$. Therefore, by Corollary 3.2, problem (1.3), (1.2) has a positive solution u_0 and, thus, all the hypotheses of Theorem 3.15 (2) (with $\alpha(t) := u_0(t)$) are fulfilled. □

Proof of Corollary 3.19. Put

$$p(t) := -a, \quad h(t) := b \quad \text{for } t \in [0, \omega]. \tag{5.36}$$

It is clear that (3.16) holds and, by Remark 2.4, we get $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.

Let us show that condition (3.22) is satisfied, where Δ is defined in Remark 2.5. It follows from Remark 2.4 that

$$p + r^{\lambda-1}h \in \mathcal{V}^+(\omega) \quad \text{if and only if} \quad -\frac{\pi^2}{\omega^2} \leq -a + br^{\lambda-1} < 0.$$

Moreover, by Remark 2.6, we get

$$\Delta(p + r^{\lambda-1}h) \leq \left(2\sqrt{a - br^{\lambda-1}} \sin \frac{\omega\sqrt{a - br^{\lambda-1}}}{2}\right)^{-1}$$

for $r > 0$, $-\frac{\pi^2}{\omega^2} \leq -a + br^{\lambda-1} < 0$. It is easy to see that $\sin x > \frac{2}{\pi}\, x$ for $x \in]0, \frac{\pi}{2}[$ and, thus,

$$\frac{1}{\Delta(p + r^{\lambda-1}h)} > \frac{2\omega}{\pi}\left(a - br^{\lambda-1}\right) \quad \text{for } r > 0,\ -\frac{\pi^2}{\omega^2} < -a + br^{\lambda-1} < 0 \tag{5.37}$$

and

$$\frac{1}{\Delta(p + r^{\lambda-1}h)} \geq \frac{2\omega}{\pi}(a - br^{\lambda-1}) \quad \text{for } r > 0,\ -\frac{\pi^2}{\omega^2} = -a + br^{\lambda-1}. \tag{5.38}$$

Put

$$\varphi(r) := ar - br^{\lambda} \quad \text{for } 0 \leq r \leq \left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}}.$$

By direct calculation, we show that

$$\max\left\{\varphi(r) : 0 \leq r \leq \left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}}\right\} = \varphi(r^*), \quad \varphi'(r) < 0 \quad \text{for } r^* < r \leq \left(\frac{a}{b}\right)^{\frac{1}{\lambda-1}},$$

where $r^* := \left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$.

If $a < \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2$, then either $a \leq \frac{\pi^2}{\omega^2}$ or $a > \frac{\pi^2}{\omega^2}$, $\left[\frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right)\right]^{\frac{1}{\lambda-1}} < r^*$. Hence, we get

$$\max\left\{\varphi(r) : r > 0,\ \frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right) \leq r^{\lambda-1} < \frac{a}{b}\right\} = \varphi(r^*) \tag{5.39}$$

and, moreover, (5.37) yields

$$\frac{2\omega}{\pi}\varphi(r^*) < \frac{r^*}{\Delta(p + (r^*)^{\lambda-1}h)}. \tag{5.40}$$

If $a \geq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2$, then $r^* \leq \left[\frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right)\right]^{\frac{1}{\lambda-1}}$ and, thus,

$$\max\left\{\varphi(r) : r > 0,\ \frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right) \leq r^{\lambda-1} < \frac{a}{b}\right\} = \varphi(r_0), \tag{5.41}$$

where $r_0 := \left[\frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right)\right]^{\frac{1}{\lambda-1}}$. Moreover, (5.38) implies

$$\frac{2\omega}{\pi}\varphi(r_0) \leq \frac{r_0}{\Delta(p + r_0^{\lambda-1}h)}. \tag{5.42}$$

Therefore, from (3.26), (5.37), (5.38), (5.39), and (5.41), we conclude that the function f satisfies

$$\begin{aligned}\int_0^{\omega}[f(s)]_+\mathrm{d}s &\leq \frac{2\omega}{\pi}\max\left\{\varphi(r) : r > 0,\ \frac{1}{b}\left(a - \frac{\pi^2}{\omega^2}\right) \leq r^{\lambda-1} < \frac{a}{b}\right\} \\ &\leq \sup\left\{\frac{r}{\Delta(p + r^{\lambda-1}h)} : r > 0,\ p + r^{\lambda-1}h \in \mathcal{V}^+(\omega)\right\}.\end{aligned} \tag{5.43}$$

Furthermore, it follows from (5.40) and (5.42) that, if (5.43) holds in the form of equalities, then $a \geq \frac{\lambda}{\lambda-1}\left(\frac{\pi}{\omega}\right)^2$ and a supremum on the right-hand side of (5.43) is achieved at r_0. Consequently, taking into account Remark 3.17, all the hypotheses of Corollary 3.16 are fulfilled and, thus, problem (3.25) has at least one positive solution. □

Proof of Corollary 3.22. Let the functions p and h be defined by (5.36). Then, (3.16) holds and, by Remark 2.4, we get $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Consequently, the conclusion of the corollary follows from Theorem 3.15 (3). □

Proof of Theorem 3.25. Let the function q be defined by formula (5.23). In view of (3.28), it is clear that q is a Carathéodory function satisfying hypothesis (H_1) with $q_0(t,x) := h(t)x^{\lambda-1}$ and $x_0 := 0$. Moreover, hypothesis (H_2') holds, $q(t,0) \equiv 0$, and conditions (3.8), (3.9) with $x_1 := 1$, and (3.14) are fulfilled. Furthermore, since the function $x \mapsto x^\lambda$ is strictly convex on $]0,+\infty[$, one can show that q satisfies hypothesis (H_0^2).

Conclusion (1)*:* It follows from Theorem 3.13 (3) with $\ell := 2$.

Conclusion (2)*:* Assume that (3.22) holds, where Δ is defined in Remark 2.5. Then, inequality (3.5) is obviously satisfied, because we have $q^*(t,\varrho) = h(t)\varrho^{\lambda-1}$. Consequently, all the hypotheses of Corollary 3.7 are fulfilled and, thus, problem (1.3), (1.2) has at least one positive solution.

On the other hand, in view of the above-proved conclusion (1), problem (1.3), (1.2) has at most two positive solutions.

Conclusion (3)*:* Let $\alpha \in AC_\ell([0,\omega])$ be a positive function satisfying (3.1) and (3.17). According to Lemma 4.16, there exists $R > 0$ such that (5.24) holds. Consequently, all the hypotheses of Theorem 3.1 are fulfilled and, thus, problem (1.3), (1.2) has a positive solution u satisfying (3.3). By the above-proved conclusion (1), problem (1.3), (1.2) has either one or two positive solutions.

If (1.3), (1.2) has a unique positive solution u_0, then we put $u^* := u_0$. If (1.3), (1.2) has exactly two positive solutions u_1, u_2, then it follows from Theorem 3.13 (1) that $u_1(t) \neq u_2(t)$ for $t \in [0,\omega]$, and we put

$$u^*(t) := \max\{u_1(t), u_2(t)\} \quad \text{for } t \in [0,\omega].$$

It is clear that, in both these cases, u^* satisfies (3.18).

We now show that every solution u to problem (1.3), (1.2) satisfies (3.19). Suppose on the contrary that u is a solution to problem (1.3), (1.2) such that (3.19) does not hold. Lemma 4.22 implies that, if $u(t) \leq u^*(t)$ for $t \in [0,\omega]$ and $u(t) \not\equiv u^*(t)$, then $u(t) < u^*(t)$ for $t \in [0,\omega]$. Therefore, u satisfies (5.26). Put

$$\alpha_0(t) := \max\{u(t), u^*(t)\} \quad \text{for } t \in [0,\omega].$$

Since we have proved that (5.24) holds for some $R > 0$, in much the same way as in the proof of Theorem 3.15 (2), we determine a solution $\tilde{u}$ to problem (1.3), (1.2) satisfying (5.30), which is in contradiction with the definition of u^*.

Conclusion (4)*:* Let $\alpha_1 \in AC_\ell([0,\omega])$ and $\alpha_2 \in AC^1([0,\omega])$ be such that (3.29), (3.30), and (3.31) hold. According to Lemma 4.16, there exists $R > 0$ such that (5.24) holds. Consequently, it follows from Proposition 4.21 that there exists a function $\beta \in AC^1([0,\omega])$ satisfying (4.24), (5.1), and

$$\beta(t) \geq \alpha_1(t) \quad \text{for } t \in [0,\omega].$$

Therefore, by virtue of (3.30), (3.31), (4.24), and (5.1), all the hypotheses of Lemma 4.17 (with $g(t,x) := p(t)x + q(t,x)x + f(t)$) are fulfilled and, thus, problem (1.3), (1.2) has a solution u_1 such that

$$\alpha_1(t) \leq u_1(t) \leq \beta(t) \quad \text{for } t \in [a,b]. \tag{5.44}$$

We further determine a solution u_2 to problem (1.3), (1.2) satisfying (3.32). It follows from the hypothesis $(p,f) \in \mathcal{U}(\omega)$ (see Definition 3.24) that the problem

$$v'' = p(t)v + f(t); \quad v(0) = v(\omega),\ v'(0) = v'(\omega) \tag{5.45}$$

has a unique solution v, which is positive. Since h satisfies (3.28) and α_2 is positive, it follows from (3.30), (3.31), and (5.45) that

$$v''(t) \le p(t)v(t) + h(t)v^{\lambda}(t) + f(t) \quad \text{for a. e. } t \in [0,\omega], \tag{5.46}$$
$$\alpha_2(0) - v(0) = \alpha_2(\omega) - v(\omega), \quad \alpha_2'(0) - v'(0) \ge \alpha_2'(\omega) - v'(\omega) \quad \text{for } k = 1,2,$$

and

$$(\alpha_2(t) - v(t))'' \ge p(t)(\alpha_2(t) - v(t)) \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, by the hypothesis $p \in \mathcal{V}^+(\omega)$, the latter inequality yields

$$v(t) \le \alpha_2(t) \quad \text{for } t \in [0,\omega]. \tag{5.47}$$

Put

$$\delta := \min\left\{v(t) : t \in [0,\omega]\right\} \tag{5.48}$$

and consider the periodic problem

$$u'' = p(t)u + h(t)\left[\chi(u)\right]^{\lambda-1}u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{5.49}$$

where

$$\chi(x) = \begin{cases} x & \text{for } x \ge \delta, \\ \delta & \text{for } x < \delta. \end{cases} \tag{5.50}$$

In view of (3.28), it is clear that $\chi(x) \ge \delta$ for $x \in \mathbb{R}$ and

$$\begin{aligned}&\Big(p(t)x + h(t)\left[\chi(x)\right]^{\lambda-1}x + f(t)\Big)\operatorname{sgn} x \\ &\qquad \ge \Big(p(t) + \delta^{\lambda-1}h(t)\Big)|x| - |f(t)| \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in \mathbb{R}.\end{aligned}$$

By Lemmas 4.7 and 4.9, we get $p + \delta^{\lambda-1}h \in \operatorname{Int}\mathcal{D}(\omega)$, because $\delta > 0$ and h satisfies (3.28). Therefore, in view of (3.30), (3.31), (5.45), (5.46), (5.47), and (5.50), all the hypotheses of Lemma 4.18 (with $g(t,x) := p(t)x + [\chi(x)]^{\lambda-1}x + f(t)$, $p_0(t) := p(t) + \delta^{\lambda-1}h(t)$, $z(t,x) := |f(t)|$, $\alpha(t) := \alpha_2(t)$, and $\beta(t) := v(t)$) are fulfilled and, thus, problem (5.49) possesses a solution u_2 such that

$$v(t_0) \le u_2(t_0) \le \alpha_2(t_0) \quad \text{for some } t_0 \in [0,\omega]. \tag{5.51}$$

Let $z(t) := u_2(t) - v(t)$ for $t \in [0,\omega]$. It is clear that z is a solution to the linear problem

$$z'' = \big(p(t) + \delta^{\lambda-1}h(t)\big)z + h(t)\left(\left[\chi(u_2(t))\right]^{\lambda-1} - \delta^{\lambda-1}\right)u_2(t) + \delta^{\lambda-1}h(t)v(t),$$
$$z(0) = z(\omega), \quad z'(0) = z'(\omega)$$

and, by virtue of (3.28), (5.50), and the condition $\delta > 0$, we get

$$h(t)\left(\left[\chi(u_2(t))\right]^{\lambda-1} - \delta^{\lambda-1}\right)u_2(t) + \delta^{\lambda-1}h(t)v(t) \ge 0 \quad \text{for a. e. } t \in [0,\omega],$$
$$h(t)\left(\left[\chi(u_2(t))\right]^{\lambda-1} - \delta^{\lambda-1}\right)u_2(t) + \delta^{\lambda-1}h(t)v(t) \not\equiv 0.$$

Therefore, in view of the inclusion $p + \delta^{\lambda-1}h \in \operatorname{Int}\mathcal{D}(\omega)$ and condition (5.51), it follows from Lemma 4.10 (with $g(t) := p(t) + \delta^{\lambda-1}h(t)$ and $\ell(t) := h(t)([\chi(u_2(t))]^{\lambda-1} - \delta^{\lambda-1})u_2(t) + \delta^{\lambda-1}h(t)v(t)$) that $z(t) > 0$ for $t \in [0,\omega]$, i. e.,

$$u_2(t) > v(t) \quad \text{for } t \in [0,\omega].$$

Consequently, (5.48) yields $u_2(t) > \delta$ for $t \in [0,\omega]$, which, in view of (5.50), yields $\chi(u_2(t)) = u_2(t)$ for $t \in [0,\omega]$ and, thus, u_2 is a positive solution to problem (1.3), (1.2). Moreover, (3.29), (5.44), and (5.51) yield $u_1(t_0) > u_2(t_0)$ for some $t_0 \in [0,\omega]$. Therefore, by Theorem 3.13 (1), we conclude that the solutions u_1, u_2 satisfy (3.32). Furthermore, the above-proved conclusion (1) implies that problem (1.3), (1.2) has exactly two positive solutions.

Finally, let u be a solution to problem (1.3), (1.2) different from u_1. Then, it follows from the above-proved conclusion (3) that u satisfies (3.33).

Conclusion (5)*:* Assume that (3.7) holds. Then, the existence and uniqueness of a positive solution to problem (1.1), (1.2) follow from Corollary 3.14. □

Proof of Corollary 3.29. Let the function q be defined by formula (5.23).

Conclusion (1)*:* Assume that (3.34) holds, where Δ is defined in Remark 2.5. Observe that the function q^* given by (3.6) is of the form $q^*(t,\varrho) = h(t)\varrho^{\lambda-1}$. Put

$$H := \int_0^\omega h(s)\mathrm{d}s, \qquad F := \int_0^\omega [f(s)]_+\mathrm{d}s.$$

Since $[f(t)]_+ \not\equiv 0$, by direct calculation, we get

$$\begin{aligned}\sup\left\{\frac{r}{\int_0^\omega [f(s)]_+\mathrm{d}s + r\int_0^\omega q^*(s,r)\mathrm{d}s} : r>0\right\} &= \sup\left\{\frac{r}{F+Hr^\lambda} : r>0\right\}\\ &= \frac{(\lambda-1)^{\frac{\lambda-1}{\lambda}}}{\lambda}F^{-\frac{\lambda-1}{\lambda}}H^{-\frac{1}{\lambda}}\end{aligned}$$

and this supremum is achieved at $r_0 := \left[\frac{F}{(\lambda-1)H}\right]^{\frac{1}{\lambda}}$. Therefore, (3.34) yields

$$\Delta(p) \le \frac{r_0}{\int_0^\omega [f(s)]_+\mathrm{d}s + r_0\int_0^\omega q^*(s,r_0)\mathrm{d}s}$$

and, thus, Proposition 4.20 guarantees that there exists a positive function $\alpha \in AC^1([0,\omega])$ satisfying (3.17) and (3.23). Consequently, all the hypotheses of Theorem 3.25 (3) are fulfilled.

Conclusion (2)*:* Assume that (3.35) holds, where Δ is defined in Remark 2.5. Then, there exits $\varepsilon > 1$ such that

$$0 < \int_0^\omega [\varepsilon f(s)]_+\mathrm{d}s \le \frac{\lambda-1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda\int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}.$$

In much the same way as in the proof of conclusion (1), we show that there exists $r_0 > 0$ such that

$$\Delta(p) \le \frac{r_0}{\int_0^\omega [\varepsilon f(s)]_+\mathrm{d}s + r_0\int_0^\omega q^*(s,r_0)\mathrm{d}s}.$$

By Proposition 4.20 (with $[\varepsilon f]_+$ instead of $[f]_+$), there exists a positive function $\alpha_1 \in AC^1([0,\omega])$ such that

$$\alpha_1(0) = \alpha_1(\omega), \quad \alpha_1'(0) = \alpha_1'(\omega), \tag{5.52}$$

$$\alpha_1''(t) \ge p(t)\alpha_1(t) + q(t,\alpha_1(t))\alpha_1(t) + \varepsilon[f(t)]_+ \quad \text{for a. e. } t \in [0,\omega]. \tag{5.53}$$

Since $\varepsilon > 1$, the function α_1 satisfies

$$\alpha_1''(t) \ge p(t)\alpha_1(t) + h(t)\alpha_1^\lambda(t)) + f(t) \quad \text{for a. e. } t \in [0,\omega].$$

Put $\alpha_2(t) := \frac{1}{\varepsilon}\alpha_1(t)$ for $t \in [0,\omega]$. Then, (3.29) holds and from (5.52) and (5.53), we get

$$\alpha_2(0) = \alpha_2(\omega), \quad \alpha_2'(0) = \alpha_2'(\omega)$$

and

$$\begin{aligned}\alpha_2''(t) &\geq p(t)\alpha_2(t) + q(t, \varepsilon\alpha_2(t))\alpha_2(t) + [f(t)]_+ \\ &= p(t)\alpha_2(t) + \varepsilon^{\lambda-1}h(t)\alpha_2^{\lambda}(t) + [f(t)]_+ \\ &\geq p(t)\alpha_2(t) + h(t)\alpha_2^{\lambda}(t) + f(t) \quad \text{for a. e. } t \in [0,\omega],\end{aligned}$$

because $\varepsilon > 1$ and h satisfies (3.28). Consequently, α_1, α_2 satisfy (3.29), (3.31), and (3.36) and, thus, all the hypotheses of Theorem 3.25 (4) are fulfilled. □

Proof of Corollary 3.30. Put

$$p(t) := -a \quad \text{for } t \in [0,\omega]. \tag{5.54}$$

By Remarks 2.4 and 2.6, we get $p \in \mathcal{V}^+(\omega)$ and

$$\Delta(p) \leq \left(2\sqrt{a}\sin\frac{\omega\sqrt{a}}{2}\right)^{-1}.$$

Consequently, hypothesis (3.37) yields (3.34) and, thus, problem (3.38) has either one or two positive solutions as follows from Corollary 3.29 (1) and Theorem 3.25 (1,3). □

Proof of Corollary 3.31. Let the function p be defined by (5.54). By Remark 2.4, we get $p \in \mathcal{V}^+(\omega)$ and, thus, Theorem 3.25 (5) implies that problem (3.38) has a unique positive solution. □

Proof of Theorem 3.32. Suppose on the contrary that u is a non-negative solution to problem (1.3), (1.2). In view of (3.28) and (3.39), it follows from Lemma 4.23 (with $g(t) := h(t)u^{\lambda}(t)$ and $\varrho := \Gamma(p)$) that

$$\begin{aligned}&\int_0^{\omega}\big[h(s)u^{\lambda}(s) + f(s)\big]_+\mathrm{d}s - \Gamma(p)\int_0^{\omega}\big[h(s)u^{\lambda}(s) + f(s)\big]_-\mathrm{d}s \\ &\quad\geq \int_0^{\omega}h(s)u^{\lambda}(s)\mathrm{d}s + \int_0^{\omega}[f(s)]_+\mathrm{d}s - \Gamma(p)\int_0^{\omega}[f(s)]_-\mathrm{d}s \geq 0.\end{aligned} \tag{5.55}$$

Assuming $h(t)u^{\lambda}(t) + f(t) \equiv 0$, we conclude easily that u is a solution to problem (2.1) which, together with the hypothesis $p \in \operatorname{Int}\mathcal{V}^+(\omega)$, yields $u(t) \equiv 0$. However, this is in contradiction with the hypothesis $f(t) \not\equiv 0$. Therefore, $h(t)u^{\lambda}(t) + f(t) \not\equiv 0$ and, thus, from Lemma 4.3 (with $g(t) := p(t)$ and $\ell(t) := h(t)u^{\lambda}(t) + f(t)$), we get

$$\Gamma(p)\int_0^{\omega}[p(s)]_-\mathrm{d}s > \int_0^{\omega}[p(s)]_+\mathrm{d}s \tag{5.56}$$

and

$$\begin{aligned}u(t) > \nu\bigg(&\int_0^{\omega}\big[h(s)u^{\lambda}(s) + f(s)\big]_+\mathrm{d}s \\ &-\Gamma(p)\int_0^{\omega}\big[h(s)u^{\lambda}(s) + f(s)\big]_-\mathrm{d}s\bigg) \quad \text{for } t \in [0,\omega],\end{aligned} \tag{5.57}$$

where

$$\nu := \left(\Gamma(p)\int_0^{\omega}[p(s)]_-\mathrm{d}s - \int_0^{\omega}[p(s)(s)]_+\mathrm{d}s\right)^{-1}.$$

The latter condition, together with (5.55) and (5.56) yields

$$m > 0, \tag{5.58}$$

where $m := \min\{u(t) : t \in [0, \omega]\}$. Put

$$H := \int_0^\omega h(s)\mathrm{d}s, \qquad F := \int_0^\omega [f(s)]_+\mathrm{d}s - \Gamma(p)\int_0^\omega [f(s)]_-\mathrm{d}s.$$

Then, it follows (3.28), (3.39), and (5.56) that $H > 0$ and $F > 0$. Moreover, (5.55) and (5.57) lead to the inequality

$$m > \nu H m^\lambda + \nu F. \tag{5.59}$$

Put

$$\varphi(x) := -x + \nu H x^\lambda + \nu F \quad \text{for } x > 0.$$

One can show by direct calculation that

$$\inf\left\{\varphi(x) : x > 0\right\} = \nu F - \frac{\lambda - 1}{\lambda}\left(\frac{1}{\lambda\nu H}\right)^{\frac{1}{\lambda-1}}$$

and, thus, hypothesis (3.39) implies that $\inf\left\{\varphi(x) : x > 0\right\} \geq 0$. Hence,

$$-x + \nu H x^\lambda + \nu F \geq 0 \quad \text{for } x > 0,$$

which, in view of (5.58), contradicts (5.59). □

Proof of Theorem 3.33. Let the function q be defined by formula (5.23). In view of (3.28), it is clear that q is a Carathéodory function satisfying (3.14).

Conclusion (1): Assume that $p \in \mathcal{V}^+(\omega)$ and (3.40) holds, where Δ is defined in Remark 2.5. Since f satisfies (3.15), the inclusion $(p, f) \in \mathcal{U}(\omega)$ holds (see Remark 3.26) and condition (3.35) is fulfilled. Therefore, it follows from Corollary 3.29 (2) and Theorem 3.25 (4) that problem (1.3), (1.2) has exactly two positive solutions u_1, u_2 and these solutions satisfy (3.32).

Since u is a negative solution to problem (1.3), (1.2) if and only if the function $-u$ is a positive solution to the problem

$$z'' = p(t)z + h(t)|z|^\lambda \operatorname{sgn} z - f(t); \qquad z(0) = z(\omega), \quad z'(0) = z'(\omega), \tag{5.60}$$

it follows from Theorem 3.25 (5) that problem (1.3), (1.2) possesses a unique negative solution u_3.

Finally, by Theorem 3.13 (4), we conclude that problem (1.3), (1.2) has exactly three solutions u_1, u_2, u_3 and these solutions satisfy (3.41).

Conclusion (2): Assume that $p \in \operatorname{Int}\mathcal{V}^+(\omega)$ and (3.42) holds, where Γ is given by (2.3). Since u is a negative solution to problem (1.3), (1.2) if and only if the function $-u$ is a positive solution to problem (5.60), it follows from Theorem 3.25 (5) that problem (1.3), (1.2) possesses a unique negative solution u_0. Moreover, Theorem 3.32 implies that problem (1.3), (1.2) has no positive solution.

Therefore, by Theorem 3.13 (4), we conclude that problem (1.3), (1.2) has exactly one solutions u_0 and this solution is negative. □

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