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# Oscillatory solutions of Emden–Fowler type differential equation

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**Abstract.** The paper deals with the coexistence between the oscillatory dynamics and the nonoscillatory one for a generalized super-linear Emden–Fowler differential equation. In particular, the coexistence of infinitely many oscillatory solutions with unbounded positive solutions are proved. The asymptotics of the unbounded positive solutions are described as well.

**Keywords:** second order nonlinear differential equation, oscillatory solution, globally positive solution, intermediate solutions.

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## 1 Introduction

In the paper we investigate the second order nonlinear differential equation

$$x'' + b(t)t^{-\gamma}|x|^{\beta} \operatorname{sgn} x = 0, \qquad t \in [1, \infty), \tag{1.1}$$

where the function $b \in AC[1, \infty)$ is positive on $[1, \infty)$ and bounded away from zero, i.e.,

$$\inf_{t\in[1,\infty)} b(t) = b_0 > 0,$$

and the constants $\beta$ and $\gamma$ are positive and satisfy

$$\beta > 1, \qquad \gamma = \frac{\beta + 3}{2}.$$

Equation (1.1) is the so-called generalized super-linear Emden–Fowler differential equation; it is widely studied in the literature, see, e.g., [16,20,26] and references therein. Equation

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(1.1) arises also in the study for searching spherically symmetric solutions of the nonlinear elliptic equation

$$\operatorname{div}\,(r(\mathbf{x})\,\nabla u) + q(\mathbf{x})\,F(u) = 0,$$

where $r$ and $q$ are smooth functions defined on $\mathbb{R}^d$, $d \geq 2$, $r$ is positive, $F \in C(\mathbb{R})$. The search for radially symmetric solutions outside of a ball of radius $R$ leads to the equation

$$\left(t^{d-1}r(t)u'\right)' + t^{d-1}q(t)F(u) = 0, \qquad t \geq R, \tag{1.2}$$

where $t = |\mathbf{x}|$. In the special case $r(t) = t^{1-d}$, $q(t) = b(t)t^{1-\gamma-d}$ for $t \geq 1$ and $F(u) = |u|^\beta \operatorname{sgn} u$, we get (1.1).

By a solution of (1.1) we mean a function $x$, defined on some interval of positive measure contained on $[1, \infty)$, satisfying (1.1). Further, $x$ is said to be *proper* if it is defined on some interval $[t_x, \infty)$, $t_x \geq 1$, and $\sup_{t\in[\tau,\infty)}|x(t)| > 0$ for any $\tau \geq t_x$. In other words, a proper solution of (1.1) is a solution that is continuable to infinity and different from the trivial solution in any neighborhood of infinity. Since $\beta > 1$, the initial value problem associated to (1.1) has a unique local solution, that is a solution $x$ such that $x(\bar{t}) = x_0, x'(\bar{t}) = x_1$, defined in a suitable neighborhood of $\bar{t} \in [t_x, \infty)$ for arbitrary numbers $x_0, x_1$. Moreover, in view of the assumptions on the function $b$, any nontrivial local solution of (1.1) is a proper solution, see, e.g., [16, Theorem 17.1] or [26, Section 3]. Observe that, if $b(t) > 0$ but $b \notin AC[1, \infty)$, then equation (1.1) with uncontinuable to infinity solutions may exist, see, e.g., [10, 15].

As usual, a proper solution $x$ of (1.1) is said to be *nonoscillatory* if $x$ is different from zero for any large $t$ and *oscillatory* otherwise. Clearly, in view of the positiveness of $b$, any eventually positive solution $x$ of (1.1) is increasing for any large $t$. Thus, nonoscillatory solutions $x$ of (1.1) can be *a-priori* divided into three classes. More precisely, $x$ is called a *subdominant solution* if

$$\lim_{t\to\infty} x(t) = \ell_x, \qquad 0 < \ell_x < \infty,$$

or *intermediate solution* if

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} x'(t) = 0,$$

or *dominant solution* if

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} x'(t) = \ell_x, \qquad 0 < \ell_x < \infty,$$

see, e.g., [11, 18, 24, 25].

In the literature great attention has been devoted to the existence of unbounded solutions which are dominant solutions, sometimes called asymptotically linear solutions. However, unbounded nonoscillatory solutions, which are not asymptotically linear solutions, are very difficult to treat. Indeed, as far we know, until now no general necessary and sufficient conditions for existence of intermediate solutions of (1.1) are known; this fact mainly is due to the lack of sharp upper and lower bounds for intermediate solutions, see, e.g., [1, page 241], [13, page 3], [18, page 2].

For the special case of (1.1) with $b(t) = 1/4$, that is for the equation

$$x'' + \frac{1}{4}t^{-\gamma}|x|^\beta \operatorname{sgn} x = 0, \qquad t \in [1, \infty), \tag{1.3}$$

the above three types of nonoscillatory solutions cannot simultaneously coexist, as Moore and Nehari proved in [21]. The problem of this triple coexistence has been solved in a negative way for the more general equation

$$(a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0,$$

where $a$ is a positive continuous function on $[1, \infty)$ and $\alpha$ is a positive constant, $\alpha \neq \beta$, in [14, 22] and [5, 7], according to $\alpha > \beta$ or $\alpha < \beta$, respectively.

A much more subtle question concerns the possible coexistence between oscillatory solutions and nonoscillatory solutions. The particular equation (1.3), as it is shown in [21], has both oscillatory solutions and nonoscillatory solutions. These nonoscillatory solutions are either subdominant solutions or intermediate solutions and both types exist. Moreover, intermediate solutions of (1.3) intersect the intermediate solution $\sqrt{t}$ infinitely many times.

Many efforts have been made to obtain the existence of at least one oscillatory solution for more general equations than (1.3). A classical approach is due to Jasný [12] and Kurzweil [17], see also [16, Theorem 18.4.], and is based on certain properties of an auxiliary energy-type function. In particular, in [12, 17] it is proved that, if the function $b$ is nondecreasing for large $t$, then any proper solution $x$ of (1.1), with $x(t_1) = 0$ and $|x'(t_1)|$ sufficiently large, $t_1 \geq 1$, is oscillatory. The sharpness of this monotonicity condition follows from a Skhalyakho–Kiguradze result, see e.g., [20, Theorem 14.3.], where it is shown that if the function $t^\varepsilon b(t)$ is nonincreasing for any large $t$ and some $\varepsilon > 0$, then every proper solution of (1.1) is nonoscillatory.

Roughly speaking, in view of the above quoted results by Jasný, Kurzweil and Kiguradze, equation (1.3) can be considered as the border equation between oscillation and nonoscillation.

Our aim here is to study how the quoted results in [21] for (1.3) can be extended to the perturbed equation (1.1).

Since $b \in AC[1, \infty)$, there exists the derivative of $b$ almost everywhere on $[1, \infty)$. Thus, under the additional assumption

$$\int_1^\infty |b'(t)|\, dt < \infty, \tag{1.4}$$

we will study the existence of at least one oscillatory solution to (1.1) and its coexistence with intermediate solutions. Observe that in view of (1.4), the function $b$ is of bounded variation on $[1, \infty)$, but $b$ could not be monotone for large $t$.

Our main results are the following.

**Theorem 1.1.** *Assume (1.4) holds. Then (1.1) has infinitely many oscillatory solutions.*

**Theorem 1.2.** *Assume (1.4) holds. Equation (1.1) has infinitely many intermediate solutions $x$ defined on $[1, \infty)$ such that*

$$C_0 t^{1/2} \leq x(t) \leq C_1 t^{1/2} \quad \text{for large } t \tag{1.5}$$

*where $C_0$ is a suitable positive constant which does not depends on the choice of $x$, and*

$$C_1 = \left( \frac{\beta + 1}{8 b_0} \right)^{1/(\beta - 1)}.$$

*Moreover, intermediate solutions intersect the function*

$$\left( \frac{1}{4b(t)} \right)^{1/(\beta-1)} \sqrt{t}$$

*infinitely many times.*

**Corollary 1.3.** *Assume (1.4) holds. Equation (1.1) admits simultaneously infinitely many oscillatory solutions, subdominant solutions, and intermediate solutions.*

For equation (1.1), Theorem 1.2 extends analogues results in [6, Theorem 2.1] and [3, Theorem 3.1], where $b$ is required to be nonincreasing for $t \geq 1$. Recently, the existence of intermediate solutions of (1.1) has been considered in [23,24]. More precisely, in these papers, the existence problem is reduced, by means of an ingenious change of variables, to the solvability of a system of two integral equations on the half-line $[1, \infty)$. Moreover, an asymptotic formula for these solutions is presented, too. Observe that asymptotic forms of intermediate solutions of (1.1) are given also in [13], where the existence problem is not studied. Hence, Theorem 1.2 extends also these quoted results in [13,23,24].

## 2 Preliminaries

We start by recalling the following asymptotic property of nonoscillatory solutions of (1.1).

**Lemma 2.1.** *Any nonoscillatory solution $x$ of (1.1) satisfies $\lim_{t\to\infty} x'(t) = 0$. Consequently, $x$ is either subdominant solution or intermediate solution.*

*Proof.* Since

$$\beta - \gamma = \frac{\beta - 3}{2} > -1,$$

and $b$ is bounded away from zero, we obtain

$$\int_1^\infty t^{\beta-\gamma} b(t)\, dt = \infty.$$

Hence, in view of [8, Theorem 1], equation (1.1) does not have nonoscillatory solutions $x$ such that $\lim_{t\to\infty} x'(t) \neq 0$. □

The approach for proving our main results is based on the following lemma.

**Lemma 2.2.** *The change of variable*

$$x(t) = t^{1/2} u(s), \qquad s = \log t, \qquad t \in [1, \infty), \tag{2.1}$$

*transforms equation (1.1) into equation*

$$\ddot{u} - \frac{u}{4} + b(e^s)|u(s)|^\beta \operatorname{sgn} u(s) = 0, \qquad s \in [0, \infty), \tag{2.2}$$

*where " · " denotes the derivative with respect to the variable $s$.*

*Proof.* We have

$$\begin{aligned} x'(t) &= \frac{1}{2t^{1/2}} u(s) + t^{1/2} \dot{u}(s) \frac{1}{t} = \frac{1}{t^{1/2}} \left( \frac{u(s)}{2} + \dot{u}(s) \right) \\ x''(t) &= -\frac{1}{2t^{3/2}} \left( \frac{u(s)}{2} + \dot{u}(s) \right) + \frac{1}{t^{1/2}} \left( \frac{\dot{u}(s)}{2} + \ddot{u}(s) \right) \frac{1}{t} \\ &= \frac{1}{t^{3/2}} \left( -\frac{u(s)}{4} + \ddot{u}(s) \right). \end{aligned}$$

Substituting into (1.1) we get (2.2). □

**Lemma 2.3.** *All the solutions of (2.2) are defined on $[0, \infty)$. Moreover, any solution $u$ of (2.2) such that $u(S) = 0$, $\dot{u}(S) = 0$ at some $S \geq 0$, satisfies $u(s) \equiv 0$ for $s \geq 0$.*

*Proof.* The continuability at infinity follows from the same property for (1.1), see, e.g., [16, Theorem 17.1]. Another approach employs an idea of Conti [4] and uses two Lyapunov functions, see [9, Theorem 3.1.] and [27, Appendix A]. The second statement follows, e.g., from [19, Lemma 1.1.] and Lemma 2.2. □

Set, for $u \geq 0$,

$$Q(u) = -u^2 + \frac{8b_0}{\beta+1}\, u^{\beta+1} \tag{2.3}$$

and

$$A_0 = \left(\frac{1}{4b_0}\right)^{\frac{1}{\beta-1}}, \qquad A = \left(\frac{\beta+1}{8b_0}\right)^{\frac{1}{\beta-1}}. \tag{2.4}$$

Since $\beta > 1$, we have $A_0 < A$. The following holds.

**Lemma 2.4.** *The function $Q$ satisfies*

$$Q(0) = Q(A) = 0, \qquad Q(A_0) = -A_0^2\,\frac{\beta-1}{\beta+1}.$$

*Moreover, $Q$ is decreasing on $[0, A_0)$ and increasing on $(A_0, A]$.*

*Proof.* Since $8b_0 A^{\beta+1}/(\beta+1) = A^2$, we obtain

$$Q(A) = -A^2 + \frac{8b_0}{\beta+1} A^{\beta+1} = 0.$$

From $dQ/du = 2u(-1 + 4b_0 u^{\beta-1})$ we get $dQ/du = 0$ for $u = A_0$, $dQ/du < 0$ for $u \in (0, A_0)$, and $dQ/du > 0$ for $u \in (A_0, A)$. This gives the assertion. □

**Lemma 2.5.** *Let $u$ be a solution of (2.2). For fixed $\bar{s} \in [0,\infty)$, the solution $u$ satisfies for $s \in [0,\infty)$*

$$\begin{aligned} 4\dot{u}^2(s) + Q(|u(s)|) &= 4\dot{u}^2(\bar{s}) + Q(|u(\bar{s})|) + \frac{8}{\beta+1}\Big(b_0 - b(e^s)\Big)|u(s)|^{\beta+1} \\ &\quad - \frac{8}{\beta+1}\Big(b_0 - b(e^{\bar{s}})\Big)|u(\bar{s})|^{\beta+1} + \frac{8}{\beta+1}\int_{\bar{s}}^{s} b'(e^\sigma)e^\sigma |u(\sigma)|^{\beta+1}\, d\sigma. \end{aligned} \tag{2.5}$$

*Proof.* Multiplying equation (2.2) by $8\dot{u}$, we get

$$8\ddot{u}\dot{u} - 2\dot{u}u + 8b_0|u|^\beta \dot{u}\,\mathrm{sgn}\, u = 8\Big(b_0 - b(e^s)\Big)|u|^\beta \dot{u}\,\mathrm{sgn}\, u\,.$$

Integrating this equality on $[\bar{s}, s]$ we obtain

$$4\dot{u}^2(s) + Q(|u(s)|) = 4\dot{u}^2(\bar{s}) + Q(|u(\bar{s})|) + 8\int_{\bar{s}}^{s}\Big(b_0 - b(e^\sigma)\Big)|u(\sigma)|^\beta\, \dot{u}(\sigma)\,\mathrm{sgn}\, u(\sigma)\, d\sigma.$$

Hence (2.5) follows by integrating by parts. □

**Lemma 2.6.** *Let $0 < b_1 \leq b_0$ and $T \geq 1$ be such that $b(t) \geq b_1$ on $[T,\infty)$. Let $x$ be a nonoscillatory solution of (1.1) such that $x(t) \neq 0$ on $[T,\infty)$ and $u$ be given by (2.1) with $s_0 = \log T$. Then we have for $t \geq T$*

$$|x(t)| \leq K t^{1/2} \quad \text{with } K = \left(\frac{\beta+1}{4b_1}\right)^{\frac{1}{\beta-1}} \tag{2.6}$$

*and*

$$|u(s)| \leq K \quad \text{for } s \geq s_0. \tag{2.7}$$

*Moreover, set* $b_2 = \sup_{t \geq T} b(t)$. *Then we have for* $t \geq T$

$$|x'(t)| \leq K_1 t^{-1/2}, \quad \text{with } K_1 = 2K^\beta b_2 \tag{2.8}$$

*and*

$$|\dot{u}(s)| \leq K_1 + K/2 \quad \text{for } s \geq s_0. \tag{2.9}$$

*Proof.* Let $x$ be nonoscillatory solution of (1.1) such that

$$x(t) > 0, \qquad x'(t) > 0 \quad \text{for } t \geq T.$$

Using Lemma 2.1, we have $\lim_{t\to\infty} x'(t) = 0$. Integrating (1.1) on $[t, \infty)$, $t \geq T$, we get

$$x'(t) = \int_t^\infty b(\tau)\tau^{-\gamma} x^\beta(\tau) d\tau \geq b_1 x^\beta(t) \int_t^\infty \sigma^{-\frac{\beta+3}{2}} d\sigma = C t^{-\frac{\beta+1}{2}} x^\beta(t)$$

with $C = \frac{2}{\beta+1} b_1$. Hence,

$$\frac{x'(t)}{x^\beta(t)} \geq C t^{-\frac{\beta+1}{2}}$$

or

$$\frac{x^{-\beta+1}(t)}{\beta - 1} \geq \frac{2C}{\beta - 1} t^{-\frac{\beta-1}{2}}.$$

Thus, we have for $t \geq T$

$$x(t) \leq \left(\frac{1}{2C}\right)^{\frac{1}{\beta-1}} t^{1/2} = K t^{1/2}.$$

Since $b(t) \leq b_2 < \infty$ on $[T, \infty)$, integrating (1.1) and using (2.6) we obtain for $t \geq T$

$$x'(t) = \int_t^\infty b(\tau)\tau^{-\gamma} x^\beta(\tau) d\tau \leq b_2 K^\beta \int_t^\infty \tau^{-3/2} d\tau = 2K^\beta b_2 t^{-1/2} = K_1 t^{-1/2}.$$

Thus, (2.8) holds and using the transformation (2.1), the estimations for $u$ and $\dot{u}$ follow. □

**Lemma 2.7.** *Equation* (2.2) *has two types of nonoscillatory solutions. Namely:*
*Type (a): solution* $u$ *satisfies for large* $s$

$$0 < |u(s)| \leq D e^{-s/2} \tag{2.10}$$

*where* $|u|$ *is decreasing and* $D > 0$ *is a suitable constant.*
*Type (b): solution* $u$ *intersects the function*

$$Z(s) = \left(\frac{1}{4b(e^s)}\right)^{\frac{1}{\beta-1}}, \tag{2.11}$$

*infinitely many times, i.e., there exists a sequence* $\{s_n\}_{n=1}^\infty$, $\lim_n s_n = \infty$, *such that* $|u(s_n)| = Z(s_n)$.

*Proof.* First, observe that the function $Z$ in (2.11) satisfies

$$\lim_{s\to\infty} Z(s) = \left(\frac{1}{4b_0}\right)^{\frac{1}{\beta-1}} = A_0. \tag{2.12}$$

Let $u$ be a nonoscillatory solution of (2.2) and, for sake of simplicity, assume

$$u(s) > 0 \quad \text{for } s \geq S \geq 0\,, \tag{2.13}$$

where $S$ is chosen such that for any $s \geq S$

$$b(e^s) \geq b_0/2.$$

According to (2.7), we get for $s \geq S$

$$0 < u(s) \leq K, \tag{2.14}$$

where $K$ is given by (2.6) with $b_1 = b_0/2$.

Then, from (2.2), we get the following:

$$\begin{aligned} \ddot{u}(s) > 0 &\quad \text{if and only if} \quad u(s) < Z(s) \\ \ddot{u}(s) < 0 &\quad \text{if and only if} \quad u(s) > Z(s) \\ \ddot{u}(s) = 0 &\quad \text{if and only if} \quad u(s) = Z(s). \end{aligned} \tag{2.15}$$

Since $A_0 < K$, from (2.14) and (2.15), *a-priori*, only one of the following possibilities holds:

(i) $A_0 < \lim_{s\to\infty} u(s) \leq K$, $\ddot{u}(s) < 0$ for large $s$;

(ii) $0 \leq \lim_{s\to\infty} u(s) \leq A_0$, $\ddot{u}(s) > 0$ for large $s$;

(iii) $u$ intersects infinitely many times the function $Z$.

Observe that in case (iii), the solution $u$ is of Type (b) and the corresponding solution $x$ of (1.1) satisfies $\lim_{t\to\infty} x(t) = \infty$, $\lim_{t\to\infty} x'(t) = 0$. Thus, $x$ is an intermediate solution of (1.1).

To prove the lemma, it is sufficient to prove that in cases (i) and (ii), the solution $u$ is of Type (a).

*Case (i).* Since $\lim_{s\to\infty} u(s) = B > A_0$, we get from (2.2)

$$\begin{aligned} \lim_{s\to\infty} \ddot{u}(s) &= \lim_{s\to\infty} \left[\frac{u(s)}{4} - b(e^s)u^\beta(s)\right] = \frac{B}{4} - B^\beta b_0 \\ &= \frac{B}{4}(1 - 4b_0 B^{\beta-1}) < \frac{B}{4}(1 - 4b_0 A_0^{\beta-1}) = 0. \end{aligned}$$

Hence, $\lim_{s\to\infty} \dot{u}(s) = \lim_{s\to\infty} u(s) = -\infty$, which is a contradiction with the positiveness of the constant $B$. Thus, the case (i) cannot occur.

*Case (ii).* If $0 < B = \lim_{s\to\infty} u(s) < 1$, reasoning in a similar way as in case (i), we get a contradiction. Now suppose $\lim_{s\to\infty} u(s) = 0$. According to (2.12), there exists $S_1 \geq S$ such that for $s \geq S_1$,

$$u(s) < Z(s), \qquad 0 < u(s) \leq \left(\frac{\beta+1}{24b_0}\right)^{1/(\beta-1)}, \qquad b(e^s) \leq \frac{3}{2}b_0. \tag{2.16}$$

From this and (2.15) we obtain $\ddot{u}(s) > 0$. Thus, we have for $s \in [S_1, \infty)$

$$\dot{u}(s) < 0 \quad \text{and} \quad \lim_{s\to\infty} \dot{u}(s) = 0. \tag{2.17}$$

Let $S_1 \le s < \bar{s}$. Multiplying (2.2) by $8\dot{u}$ and integrating on $[s, \bar{s}]$ we get

$$4\dot{u}^2(\bar{s}) - u^2(\bar{s}) = 4\dot{u}^2(s) - u^2(s) - 8\int_s^{\bar{s}} b(e^\sigma)u^\beta(\sigma)\,\dot{u}(\sigma)\,d\sigma.$$

From this, (2.16) and (2.17), as $\bar{s}$ tends to infinity, we have

$$4\dot{u}^2(s) - u^2(s) - 8\int_s^{\infty} b(e^\sigma)u^\beta(\sigma)\,\dot{u}(\sigma)\,d\sigma = 0,$$

and

$$\begin{aligned}\frac{4\dot{u}^2(s)}{u^2(s)} &= 1 + \frac{8}{u^2(s)}\int_s^{\infty} b(e^\sigma)u^\beta(\sigma)\,\dot{u}(\sigma)\,d\sigma \ge 1 + \frac{12b_0}{u^2(s)}\int_s^{\infty} u^\beta(\sigma)\,\dot{u}(\sigma)\,d\sigma \\ &= 1 - \frac{12b_0}{\beta+1}u^{\beta-1}(s) > 0.\end{aligned}$$

Since $\dot{u}(s) < 0$, we obtain

$$\frac{\dot{u}(s)}{u(s)} \le -\frac{1}{2}\sqrt{1 - \frac{12b_0}{\beta+1}u^{\beta-1}(s)} \le -\frac{1}{2}\left(1 - \frac{12b_0}{\beta+1}u^{\beta-1}(s)\right). \tag{2.18}$$

Using the estimation for $u$ in (2.16), we get for $s \ge S_1$

$$\frac{\dot{u}(s)}{u(s)} \le -\frac{1}{4},$$

or

$$u(s) \le u(S_1)e^{(-s+S_1)/4}.$$

Applying this estimation to the inequality (2.18), we have for $s \ge S_1$

$$\frac{\dot{u}(s)}{u(s)} \le -\frac{1}{2} + \frac{6b_0}{\beta+1}u^{\beta-1}(S_1)e^{-(\beta-1)(s-S_1)/4}$$

or

$$\log\frac{u(s)}{u(S_1)} \le -\frac{1}{2}(s-S_1) + \frac{24b_0}{\beta^2-1}u^{\beta-1}(S_1)e^{(\beta-1)S_1/4}e^{-(\beta-1)s/4} \le -\frac{s}{2} + C,$$

where

$$C = \frac{1}{2}S_1 + \frac{24b_0u^{\beta-1}(S_1)e^{(\beta-1)S_1/4}}{\beta^2-1}.$$

Therefore, setting $K_2 = u(S_1)e^C$, we obtain

$$u(s) \le K_2e^{-s/2},$$

and in view of (2.17), $u$ is of Type (a). □

**Remark 2.8.** Solutions $u$ of Type (a) in Lemma 2.7 correspond, via the transformation (2.1), to subdominant solutions of equation (1.1) because

$$x(t) = t^{1/2}u(s) \le t^{1/2}K_2e^{-s/2} = K_2,$$

while solutions $u$ of Type (b) correspond to intermediate solutions of (1.1).

## 3 Proof of Theorem 1.1

*Proof of Theorem 1.1.* Consider equation (2.2) and the function $Q$ given by (2.3). In view of (1.4), there exists $s_0 \geq 0$ such that for $s \geq s_0$

$$\int_{s_0}^{\infty} |b'(e^\sigma)|e^\sigma d\sigma \leq \frac{b_0}{8}, \qquad |b_0 - b(e^s)| \leq \frac{b_0}{8}. \tag{3.1}$$

Let $u$ be a solution of (2.2) such that

$$u(s_0) = 0, \qquad \dot{u}(s_0) = d > 0, \tag{3.2}$$

where

$$d > \sqrt{3}K_3, \qquad K_3 = \left(\frac{9}{4}b_0K^\beta + \frac{K}{2}\right), \tag{3.3}$$

and $K$ is given by (2.6) with $b_1 = 7b_0/8$, i.e.,

$$K = \left(\frac{2(\beta+1)}{7b_0}\right)^{1/(\beta-1)}.$$

Let us prove that $u$ is oscillatory. By contradiction, suppose that there exists $s_2 \geq s_0$ such that

$$u(s_2) = 0, \qquad u(s) \neq 0 \quad \text{for } s > s_2. \tag{3.4}$$

Applying Lemma 2.6 with $b_1 = 7b_0/8$, $b_2 = 9b_0/8$, we have for $s \geq s_2$

$$|u(s)| \leq K.$$

Using (2.9), we obtain for $s \geq s_2$

$$|\dot{u}(s)| \leq 2K^\beta b_2 + \frac{K}{2} = \frac{9}{2}b_0K^\beta + \frac{K}{2} = K_3. \tag{3.5}$$

If $s_2 = s_0$, inequality (3.5) contradicts (3.2) and (3.3). Thus, suppose that $s_0 < s_2$. From (3.2) and (3.4), there exists $s_1$, $s_0 < s_1 < s_2$, such that

$$|u(s_1)| = \max_{s_0 \leq s \leq s_2} |u(s)|.$$

Obviously, $\dot{u}(s_1) = 0$. Put

$$B = (\beta+1)/(4b_0)$$

and consider two cases:

$$\text{(i)} \;\; |u(s_1)| < B^{1/(\beta-1)}, \qquad \text{(ii)} \;\; |u(s_1)| \geq B^{1/(\beta-1)}.$$

Assume case (i) holds. Applying Lemma 2.5 with $\bar{s} = s_0$, $s = s_2$, using (3.1), (3.2), and (3.4), we get

$$\begin{aligned} 4\dot{u}^2(s_2) &= 4d^2 + \frac{8}{\beta+1}\int_{s_0}^{s_2} b'(e^\sigma)e^\sigma |u(\sigma)|^{\beta+1} d\sigma \\ &\geq 4d^2 - \frac{2}{Bb_0}B^{\frac{\beta+1}{\beta-1}} \int_{s_0}^{\infty} |b'(e^\sigma)|e^\sigma d\sigma \geq 4d^2 - \frac{1}{4}B^{\frac{2}{\beta-1}} \\ &\geq 4d^2 - \left(\frac{K}{2}\right)^2 \geq 4d^2 - (K_3)^2 \geq 4d^2 - \frac{d^2}{3} = \frac{11}{3}d^2. \end{aligned}$$

Therefore,

$$|\dot{u}(s_2)| \geq \sqrt{\frac{11}{12}}d \geq \sqrt{\frac{33}{12}}K_3,$$

which contradicts (3.5).

Assume case (ii) holds. We have

$$\frac{|u(s_1)|^{\beta+1}}{B} \geq u^2(s_1).$$

Thus,

$$\frac{Q(|u(s_1)|)}{2} = \frac{1}{B}|u(s_1)|^{\beta+1} - \frac{u^2(s_1)}{2} \geq \frac{1}{2B}|u(s_1)|^{\beta+1}. \tag{3.6}$$

From here, applying Lemma 2.5 with $\bar{s} = s_0$, $s = s_1$, using (3.1) and $\dot{u}(s_1) = 0$, we get

$$\begin{aligned} Q(|u(s_1)|) &= 4d^2 + \frac{8}{\beta+1}\Big(b_0 - b(e^{s_1})\Big)|u(s_1)|^{\beta+1} + \frac{8}{\beta+1}\int_{s_0}^{s_1} b'(e^\sigma)e^\sigma|u(\sigma)|^{\beta+1}d\sigma \\ &\geq 4d^2 - \frac{2b_0}{\beta+1}|u(s_1)|^{\beta+1} \geq 4d^2 - \frac{1}{2B}|u(s_1)|^{\beta+1} \geq 4d^2 - \frac{Q(|u(s_1)|)}{2}. \end{aligned}$$

Thus,

$$Q(|u(s_1)|) \geq \frac{8}{3}d^2. \tag{3.7}$$

Applying Lemma 2.5 with $\bar{s} = s_1$, $s = s_2$, using (3.1), (3.6) and (3.7), we have

$$\begin{aligned} 4\dot{u}^2(s_2) &= Q(|u(s_1)|) - \frac{8}{\beta+1}\Big(b_0 - b(e^{s_1})\Big)|u(s_1)|^{\beta+1} + \frac{8}{\beta+1}\int_{s_1}^{s_2} b'(e^\sigma)e^\sigma|u(\sigma)|^{\beta+1}d\sigma \\ &\geq Q(|u(s_1)|) - \frac{2b_0}{\beta+1}|u(s_1)|^{\beta+1} \\ &\geq Q(|u(s_1)|) - \frac{1}{2B}|u(s_1)|^{\beta+1} \geq \frac{Q(|u(s_1)|)}{2} \geq \frac{4}{3}d^2. \end{aligned}$$

From this and (3.3), we obtain

$$|\dot{u}(s_2)| \geq \frac{d}{\sqrt{3}} > K_3,$$

which contradicts (3.5).

Thus, the solution $u$ satisfying the initial condition (3.2) is defined on $[s_0, \infty)$ and is oscillatory. According to Lemma 2.3, the solution $u$ can be extended to $[0, \infty)$. Moreover, since $s_0$ does not depend on the value $d$, equation (2.2) has infinitely many oscillatory solutions and, in virtue of the transformation (2.1), the same occurs for equation (1.1). □

## 4 Proof of Theorem 1.2

*Proof of Theorem 1.2.* Let $\delta$ be a constant such that

$$|\delta| < \frac{1}{2}\left(\frac{1}{4b_0}\right)^{\frac{1}{\beta-1}}\sqrt{\frac{\beta-1}{2(\beta+1)}},$$

and put

$$\varepsilon = \frac{1}{24}b_0(\beta-1)\left(\frac{2}{\beta+1}\right)^{(\beta+1)/(\beta-1)}.$$

Let $T \geq 1$ be such that

$$\int_T^\infty |b'(t)|\,dt \leq \varepsilon\,, \qquad |b_0 - b(t)| \leq \varepsilon \quad \text{for } t \geq T\,. \tag{4.1}$$

For $s_0 = \log T$, we have

$$\int_{s_0}^\infty |b'(e^\sigma)|e^\sigma d\sigma = \int_T^\infty |b'(t)|\,dt \leq \varepsilon, \qquad |b_0 - b(e^s)| \leq \varepsilon \quad \text{for } s \geq s_0. \tag{4.2}$$

Now, consider the solution $u$ of (2.2) with

$$u(s_0) = A_0\,, \qquad \dot{u}(s_0) = \delta\,, \tag{4.3}$$

where $A_0$ is given by (2.4). By Lemma 2.4 we get

$$Q(u(s_0)) = -\frac{\beta-1}{\beta+1}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} \tag{4.4}$$

and there exists $u_0$, $0 < u_0 < A_0$, such that

$$Q(u_0) = -\frac{Q(u(s_0))}{4} = -\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}}\,. \tag{4.5}$$

We want to prove that the solution $u$ of (2.2) with (4.3) satisfies for $s \geq s_0$

$$0 < u_0 \leq u(s) \leq A\,, \tag{4.6}$$

where $A$ is given in (2.4). Note that (4.6) is satisfied for $s = s_0$ and

$$u_0 < u(s_0) = A_0 < A. \tag{4.7}$$

*Step 1.* We claim that if there exists $s_1 > s_0$ such that

$$u(s_1) = u_0\,, \qquad u(s) > u_0 \quad \text{for } s \in [s_0, s_1)\,, \tag{4.8}$$

then

$$u(s) \leq A \quad \text{on } [s_0, s_1]\,. \tag{4.9}$$

Since $u(s_1) = u_0$, from (4.7) we get $u(s_1) < A$. By contradiction, suppose that there exists $s_2, s_0 < s_2 < s_1$, such that

$$u(s_2) = A\,, \qquad u(s) < A \quad \text{for } s \in [s_0, s_2)\,. \tag{4.10}$$

Using Lemma 2.4, we have

$$Q(u(s_2)) = 0\,. \tag{4.11}$$

According to (4.3) and (4.8), we can use Lemma 2.5 for $\bar{s} = s_0$, $s = s_2$ and this together with (2.5), (4.4) and (4.11) imply

$$\begin{aligned}
4\dot{u}^2(s_2) &= 4\dot{u}^2(s_2) + Q(u(s_2)) \\
&= 4\dot{u}^2(s_2) + Q(u(s_0)) + \frac{8}{\beta+1}\Big(b_0 - b(e^{s_2})\Big)u^{\beta+1}(s_2) \\
&\quad - \frac{8}{\beta+1}\Big(b_0 - b(e^{s_0})\Big)u^{\beta+1}(s_0) + \frac{8}{\beta+1}\int_{s_0}^{s_2} b'(e^\sigma)e^\sigma|u(\sigma)|^{\beta+1}\,d\sigma.
\end{aligned}$$

Thus, we get

$$4\dot{u}^2(s_2) \le 4\delta^2 - \frac{\beta-1}{\beta+1}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} + \frac{8}{\beta+1}\Big|b_0 - b(e^{s_2})\Big| u^{\beta+1}(s_2)$$
$$+ \frac{8}{\beta+1}\Big|b_0 - b(e^{s_0})\Big| A_0^{\beta+1} + \frac{8}{\beta+1}A^{\beta+1}\int_{s_0}^{s_2} |b'(e^{\sigma})|e^{\sigma}\, d\sigma.$$

From this, (2.4), (4.2), and (4.7), we have

$$4\dot{u}^2(s_2) \le 4\delta^2 - \frac{\beta-1}{\beta+1}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} + \frac{24}{\beta+1}\varepsilon A^{\beta+1}. \tag{4.12}$$

Since

$$4\delta^2 < \frac{\beta-1}{2(\beta+1)}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} \tag{4.13}$$

and

$$\frac{24}{\beta+1}\varepsilon A^{\beta+1} = \frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}}, \tag{4.14}$$

the inequality (4.12) implies

$$4\dot{u}^2(s_2) \le -\frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} < 0$$

and this contradiction proves Step 1.

*Step 2.* Now, we prove that

$$u(s) > u_0 > 0 \quad \text{for } s \ge s_0. \tag{4.15}$$

As claimed, (4.15) holds for $s = s_0$. By contradiction, assume that (4.8) is valid and $s_1 > s_0$ exists such that $u(s_1) = u_0$ and $u(s) > u_0$ on $[s_0, s_1)$. Hence, in view of (4.8) and (4.9) we obtain

$$0 < u_0 \le u(s) \le A \quad \text{for } s \in [s_0, s_1]. \tag{4.16}$$

Using this inequality and Lemma 2.5 with $\bar{s} = s_0$ and $s = s_1$, we have

$$4\dot{u}^2(s_1) + Q(u(s_1)) = 4\dot{u}^2(s_0) + Q(u(s_0)) + \frac{8}{\beta+1}\Big(b_0 - b(e^{s_1})\Big)u^{\beta+1}(s_1)$$
$$- \frac{8}{\beta+1}\Big(b_0 - b(e^{s_0})\Big)u^{\beta+1}(s_0) + \frac{8}{\beta+1}\int_{s_0}^{s_1} b'(e^{\sigma})e^{\sigma}u^{\beta+1}(\sigma)\, d\sigma$$
$$\le 4\dot{u}^2(s_0) + Q(u(s_0)) + \frac{8}{\beta+1}\Big|b_0 - b(e^{s_0})\Big| A^{\beta+1}$$
$$+ \frac{8}{\beta+1}\Big|b_0 - b(e^{s_1})\Big| A^{\beta+1} + \frac{8A^{\beta+1}}{\beta+1}\int_{s_0}^{s_1} |b'(e^{\sigma})|e^{\sigma}\, d\sigma\,.$$

From this, (4.2), (4.4) and (4.5) we have

$$4\dot{u}^2(s_1) - \frac{\beta-1}{4(\beta+1)}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} \le 4\delta^2 - \frac{\beta-1}{\beta+1}\left(\frac{1}{4b_0}\right)^{\frac{2}{\beta-1}} + \frac{24}{\beta+1}\varepsilon A^{\beta+1}.$$

Hence, in view of (4.13) and (4.14), we get

$$4\dot{u}^2(s_1) < 0\,,$$

which is a contradiction. This proves the validity of (4.15).

From here, using the transformation (2.1) and Remark 2.8, we obtain that the corresponding solution $x$ of (1.1) is an intermediate solution.

In a similar way, we prove that the second inequality of (4.6) is valid for $s \geq s_0$; the details are left to the reader.

Thus, from the inequality (4.15) and Lemma 2.7, the solution $u$ intersects the function $Z(s)$, given by (2.11), infinitely many times. Using the transformation (2.1), the final statement of Theorem 1.2 follows. □

*Proof of Corollary 1.3.* By using a similar argument to the one presented in [11, Theorem 4.3.], equation (1.1) has infinitely many subdominant solutions. Thus, the assertion follows from Theorems 1.1 and 1.2. □

## 5 Case $b$ nondecreasing

The assumption (1.4) is fulfilled if, in addition, the function $b$ is either nondecreasing and bounded or nonincreasing and bounded away from zero.

If $b$ is nondecreasing, then intermediate solutions $x$ are globally positive, that is $x(t) \neq 0$ on the whole interval $[1, \infty)$. Moreover, any solution with a zero is oscillatory. These properties follow from the following.

**Theorem 5.1.** *Let $b'(t) \geq 0$ for $t \in [1, \infty)$ and $\lim_{t\to\infty} b(t) = b_0$, $b_0 > 0$. Then*

*(i) Equation (1.1) has infinitely many intermediate solutions.*

*(ii) Any eventually positive solution $x$ is globally positive on $[1, \infty)$ and satisfies (2.6) and (2.8).*

*(iii) For any $a \geq 1$ every solution of (1.1) with the initial condition*

$$x(a) = 0 \quad \text{or} \quad |x(a)| > K\sqrt{a} \quad \text{or} \quad |x'(a)| > K_1 a^{-1/2}$$

*where*

$$K = \left(\frac{\beta + 1}{4b(a)}\right)^{1/(\beta-1)}, \qquad K_1 = 2b_0 K^{\beta},$$

*is oscillatory.*

*Proof.* Claim (i) follows from Theorem 1.2.

Claim (ii). Let $u$ be the solution of (2.2), which is obtained from $x$ by the change of variable (2.1). For proving that $x$ is globally positive, it is sufficient to show that $u(s) > 0$ on $[0, \infty)$. By contradiction, suppose that there exists $s_0$ such that

$$u(s_0) = 0\,, \qquad u(s) > 0 \quad \text{on } (s_0, \infty)\,. \tag{5.1}$$

According to Lemma 2.7, we obtain $\liminf_{s\to\infty} u(s) = \bar{u}$, where $\bar{u} \in [0, A_0]$. Moreover, either $\lim_{s\to\infty} u(s) = 0$, or $u$ is an intermediate solution of (2.2).

For these solutions, let $\{s_n\}$ be a sequence such that $\lim_n s_n = \infty$, $s_1 > s_0$,

$$\lim_n u(s_n) = \bar{u}, \qquad \lim_n \dot{u}(s_n) = 0, \tag{5.2}$$

and

$$0 < u(s_n) \le \frac{1}{2}(A + A_0), \qquad n \in \mathbb{N}, \tag{5.3}$$

where $A > A_0$ is given by (2.4). The sequence $\{s_n\}$ may be defined in the following way, according to whether $u$ is either of Type (a) or of Type (b).

Let $u$ be of Type (a). Then $\lim_{s\to\infty} u(s) = 0$ and by Lemma 2.7, we obtain $\dot{u}(s) < 0$ for large $s$. Then any sequence $\{s_n\}$ tending to infinity satisfies (5.2) and (5.3).

Let $u$ be of Type (b). Then by Lemma 2.7, the solution $u$ intersects the function $Z$, for which $\lim_{s\to\infty} Z(s) = A_0$, $Z$ is decreasing and $\lim_{s\to\infty} \dot{Z}(s) = 0$. Thus, $\bar{u} \in [0, A_0]$. Now, consider two cases:

(i) $\bar{u} \in [0, A_0)$,

(ii) $\bar{u} = A_0$.

In the first case the sequence $\{s_n\}$ can be choosen as points at which $u$ has a local minimum. In the second case, if $u$ has a local minimum, then $\{s_n\}$ can be defined as in the first case; if $u$ does not have local minima, i.e., $u$ is nonincreasing to $A_0$, we choose $\{s_n\}$ as

$$\begin{aligned} u(s_n) &= Z(s_n), \\ u(s) &< Z(s) \qquad \text{in a left neighborhood of } s_n. \end{aligned} \tag{5.4}$$

Indeed, the first relation in (5.2) follows from $\lim_{s\to\infty} Z(s) = A_0$. Since $\lim_{s\to\infty} \dot{Z}(s) = 0$, $0 > \dot{u}(s_n) \ge \dot{Z}(s_n)$ and $\lim_n \dot{Z}(s_n) = 0$, the second relation in (5.2) follows. Thus, $\lim_{n\to\infty} \dot{u}(s) = 0$.

From here and Lemma 2.4, we obtain

$$Q(s_n) < 0, \qquad n \in \mathbb{N}. \tag{5.5}$$

By Lemma 2.3 and (5.1) we have

$$\dot{u}(s_0) > 0. \tag{5.6}$$

Thus, applying Lemma 2.5 for $\bar{s} = s_0$ and $s = s_n$, from (5.1) we obtain

$$\begin{aligned} 4\dot{u}^2(s_n) + Q(u(s_n)) &= 4\dot{u}^2(s_0) + \frac{8}{\beta+1}\Big(b_0 - b(e^{s_n})\Big)u^{\beta+1}(s_n) \\ &\quad + \frac{8}{\beta+1}\int_{s_0}^{s_n} b'(e^\sigma)e^\sigma u^{\beta+1}(\sigma)\,d\sigma \ge 4\dot{u}^2(s_0). \end{aligned}$$

Therefore, from (5.2) and (5.6) we get

$$\liminf_{n\to\infty} Q(s_n) \ge 4\liminf_{n\to\infty} \dot{u}^2(s_n) + 4\dot{u}^2(s_0) = 4\dot{u}^2(s_0) > 0,$$

which contradicts (5.5). Hence, $u$ is positive for any $s \ge 0$.

The estimations (2.6), (2.8) follow from Lemma 2.6 and Claim (ii) is proved.

It remains to prove Claim (iii). If $x(a) = 0$, the assertion follows from (ii). Otherwise, using Lemma 2.6 with $T = a$, every nonoscillatory solution of (1.1) satisfies (2.6), (2.8) for $t \ge a$. Therefore, every solution $x$ of (1.1) with the initial condition $|x(a)| > K\sqrt{a}$ or $|x'(a)| > K_1 a^{-1/2}$ must be oscillatory, and the proof is now complete. □

If $b$ is nondecreasing, it would be interesting to give conditions for the existence of intermediate solutions of (1.1) in case $b$ is unbounded. For example, the equation

$$x'' + \frac{15}{64}t^{-11/4}x^3 = 0$$

has an intermediate solution $x(t) = t^{3/8}$, whereby $\gamma = 3$ and $b(t) = t^{1/4}$.

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